Math 226

Assignment 5

November 5–November 13, 2003

1. Consider polar coordinates in the $xy$-plane given by the equations

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x),$$

which can be solved for $x$, $y$ to give

$$x = r \cos \theta, \quad y = r \sin \theta.$$

(a) Find the partial derivatives

$$\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}.$$

(b) If $f(x, y)$ is a differentiable function of $x$ and $y$, find $\frac{\partial f}{\partial r}$ and $\frac{\partial f}{\partial \theta}$.

(c) Show that

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2} \left(\frac{\partial f}{\partial \theta}\right)^2.$$

2. The archeologist Flinders Petrie (1853–1942) visited the Bent Pyramid in Dahshur, Egypt. The equation of the slanted side is

$$z = f(x, y) = \frac{1}{\sqrt{2}} \sqrt{70^2 - (x + y)^2}.$$

Petrie decided to climb to the top of the pyramid following the curve

$$\mathbf{r}(t) = 35 \cos \left(\frac{\pi}{40} t\right) \mathbf{i} + 35 \cos \left(\frac{\pi}{40} t\right) \mathbf{j} + 35 \sqrt{2} \sin \left(\frac{\pi}{40} t\right) \mathbf{k}.$$

At time $t = 10$ Petrie is at the point $A$ described by the position vector

$$\overrightarrow{OA} = \frac{35 \sqrt{2}}{2} \mathbf{i} + \frac{35 \sqrt{2}}{2} \mathbf{j} + 35 \mathbf{k}.$$

(i) What is the direction of fastest ascent at the point $A$?

(j) What is the directional derivative of $f(x, y)$ at the point $A$ in the direction of the unit vector $\frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}}$?

(l) Make a contour map of the function $f(x, y)$. Label on it various level curves. Explain why the level curves look as you have drawn them.

(m) Show on your contour map the gradient vector at $A$. 
(n) The height $z$ of the slanted side of the pyramid can be defined implicitly by the equation
\[ 2z^2 + (x+y)^2 = 70^2. \]
Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ using implicit differentiation and write the equation of the tangent plane at the point $A$.

(o) The slanted side of the pyramid can be considered as the level surface $G(x, y, z) = 70^2$ for the function
\[ G(x, y, z) = 2z^2 + (x+y)^2. \]
Find the equation of the tangent plane to the pyramid at $A$ using the function $G(x, y, z)$.

(p) Find the rate of ascent of Flinders Petrie at time $t = 10$ using the chain rule for multivariable functions.

Solutions

1. 
   (a) $\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2}}$
   \[ \frac{\partial \theta}{\partial x} = \frac{1}{1 + (y/x)^2} \frac{\partial (y/x)}{\partial x} = \frac{1}{1 + (y/x)^2} \frac{-y}{x^2} = -\frac{y}{x^2 + y^2}, \]
   \[ \frac{\partial \theta}{\partial y} = \frac{1}{1 + (y/x)^2} \frac{\partial (y/x)}{\partial y} = \frac{1}{1 + (y/x)^2} \frac{1}{x} = \frac{x}{x^2 + y^2}. \]
   \[ \frac{\partial x}{\partial r} = \cos \theta, \quad \frac{\partial y}{\partial r} = \sin \theta, \quad \frac{\partial x}{\partial \theta} = -r \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta. \]
   (b) $\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} \sin \theta$
   \[ = \left( \frac{\partial f}{\partial x} \right)^2 \cos^2 \theta + \left( \frac{\partial f}{\partial y} \right)^2 \sin^2 \theta + 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \cos \theta \sin \theta \]
   \[ + \left( \frac{\partial f}{\partial x} \right)^2 \sin^2 \theta + \left( \frac{\partial f}{\partial y} \right)^2 \cos^2 \theta - 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} \cos \theta \sin \theta \]
   \[ = \left( \frac{\partial f}{\partial x} \right)^2 (\cos^2 \theta + \sin^2 \theta) + \left( \frac{\partial f}{\partial y} \right)^2 (\sin^2 \theta + \cos^2 \theta) = \left( \frac{\partial f}{\partial x} \right)^2 + \left( \frac{\partial f}{\partial y} \right)^2. \]

2. (i) The direction of the fastest ascent at $A$ is the gradient vector: $\nabla f(A) = -\frac{1}{\sqrt{2}} \mathbf{i} - \frac{1}{\sqrt{2}} \mathbf{j}$
   (j) $D_v f(A) = \nabla f(A) \cdot \mathbf{v} = \left( -\frac{1}{\sqrt{2}} \mathbf{i} - \frac{1}{\sqrt{2}} \mathbf{j} \right) \cdot \left( \frac{\mathbf{i} - \mathbf{j}}{\sqrt{2}} \right) = -\frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} + \left( -\frac{1}{\sqrt{2}} \right) \left( -\frac{1}{\sqrt{2}} \right) = 0.$
The fact that the directional derivative is 0 means that \( \mathbf{v} \) has the direction of the tangent line to the level curve through the point \( A \). As we will see in (l) the contour lines are lines of the form \( x + y = k \), with slope \(-1\) parallel to the vector \( \mathbf{v} \).

(l) Level curve at height \( z = m \) means

\[
\frac{1}{\sqrt{2}} \sqrt{70^2 - (x+y)^2} = m \iff 70^2 - (x+y)^2 = 2m^2 \iff (x+y)^2 = 70^2 - 2m^2 \iff x+y = \pm \sqrt{70^2 - 2m^2}.
\]

These are the equations of two lines with slope \(-1\). The intercepts are equal to \( \pm \sqrt{70^2 - 2m^2} \). They are closer to the origin for larger \( m \) and further away for smaller \( m \). We get level curves only when \( 70^2 - 2m^2 \geq 0 \), i.e., \( 0 \leq m \leq 70/\sqrt{2} \). The level curves are closer for small \( m \), since the pyramid is steeper at low height and flattens out on top. If you have MAPLE, you can download the contour map from the course web site under Bent Pyramid.

(m) The gradient vector is perpendicular to the level curve (=line with slope \(-1\) through \( A \)) and has direction the direction of increasing height i.e. towards the origin.

(n) We differentiate in \( x \) treating \( y \) as constant and \( z \) as depending on \( x \):

\[
4z \frac{\partial z}{\partial x} + 2(x+y) = 0 \implies \frac{\partial z}{\partial x} = -\frac{2(x+y)}{4z} = -\frac{x+y}{z}.
\]

Similarly

\[
4z \frac{\partial z}{\partial y} + 2(x+y) = 0 \implies \frac{\partial z}{\partial y} = -\frac{2(x+y)}{4z} = -\frac{x+y}{z}.
\]

We evaluate the partial derivatives at the point \( A \):

\[
\frac{\partial z}{\partial x} = -\frac{35\sqrt{2}/2 + 35\sqrt{2}/2}{35} = -\frac{35\sqrt{2}}{35} = -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}} = \frac{\partial z}{\partial y}.
\]

The equation of the tangent plane is \( z-35 = -1/\sqrt{2}(x-35\sqrt{2}/2) -1/\sqrt{2}(y-35\sqrt{2}/2) \).

(o) The tangent plane is tangent to the level surfaces of \( G(x,y,z) \), i.e. perpendicular to the gradient of \( G(x,y,z) \).

\[
\nabla G(x,y,z) = \frac{\partial G}{\partial x} \mathbf{i} + \frac{\partial G}{\partial y} \mathbf{j} + \frac{\partial G}{\partial z} \mathbf{k} = 2(x+y)i + 2(x+y)j + 4zk.
\]

At \( A \) we get \( \nabla G(A) = 2(35\sqrt{2}/2 + 35\sqrt{2}/2)i + 2(35\sqrt{2}/2 + 35\sqrt{2}/2)j + 4 \cdot 35k = 70\sqrt{2}i + 70\sqrt{2}j + 140k \). With this normal vector the equation of tangent plane becomes

\[
70\sqrt{2}(x-35\sqrt{2}/2) + 70\sqrt{2}(y-35\sqrt{2}/2) + 140(z-35) = 0.
\]

By dividing by 140 and moving the terms with \( x \) and \( y \) on the other side of the equation we get the previous form of the equation of the tangent plane.

(p) \( \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = -\frac{x+y}{z} \frac{35\pi}{40}(-\sin(\pi t/4)) - \frac{x+y}{z} \frac{35\pi}{40}(-\sin(\pi t/4)) \).

At \( A \) we get

\[
\frac{dz}{dt} = -\frac{1}{\sqrt{2}} \frac{35\pi}{40}(-\sin(\pi 10/4)) - \frac{1}{\sqrt{2}} \frac{35\pi}{40}(-\sin(\pi 10/4)) = -\frac{1}{\sqrt{2}} \frac{35\pi}{40}(-1/\sqrt{2})2 = \frac{35\pi}{40}.
\]