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Weyl's Quadtree Algorithm for the Unsymmetric Eigenvalue Problem

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Abstract—Wyel's quadtree algorithm was proposed in 1924 for approximating polynomial zeros. We extend it to the unsymmetric eigenvalue problem; the extension relies on the heuristic proximity test based on the known correlation between the reciprocal of the norm of matrix inverse and the distance from the matrix to the closest singular matrix.

We propose a new algorithm for approximating the eigenvalues of an unsymmetric matrix A. The algorithm can be viewed as 2-dimensional bisection or as a variant of Weyl's algorithm, originally devised for approximating polynomial zeros [1] and also well known in computational geometry under the name of quadtree algorithm.

In principle, one may directly apply Weyl's algorithm to approximating the eigenvalues of A. This would essentially amount to recursive application of a subalgorithm that would approximate from below the distance $d(z)$ from the origin to the nearest eigenvalue of $A - zI$, where z is a current approximation to an eigenvalue of A. Such a subalgorithm should output z if $d(z) \leq t$ for a fixed tolerance t.

We propose an alternative approach where approximating $d(z)$ is replaced by approximating the distance $\hat{d}(z)$ from $A - zI$ to the nearest singular matrix (so that $\hat{d}(z) \leq d(z)$), with the goal of outputting z such that $\hat{d}(z) \leq t$ for a fixed tolerance t. The computation of $\hat{d}(z)$ relies on the following well-known equation [2,3], which holds for any fixed matrix B and any fixed operator norm (that is, matrix norm subordinate to some vector norm):

$$\frac{1}{\|B^{-1}\|} = \min_S \|B - S\|, \tag{1}$$

where the minimum is over all singular matrices S. Due to this equation, we may set

$$\hat{d}(z) = \frac{1}{\|(A - zI)^{-1}\|}, \tag{2}$$

for a fixed matrix norm, and then estimate $\hat{d}(z)$ by means of some well-known techniques, [4, p. 128–132]. In the next specific implementation of this approach (in the form of Algorithm 1), we assume any fixed operator (matrix) norm (in particular, the row norm $\|\cdot\|_\infty$ seems to be a good choice) and let $S(x, r)$ denote the square on the complex plane with center x and 4 vertices, $x + (-1)^{(1+2g)/4} r$, $g = 0, 1, 2, 3$.

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ALGORITHM 1.

Input: an $n \times n$ matrix A and a positive scalar t (error tolerance).

Output: a natural N and a set of N complex values $\hat{\lambda}_1, \dots, \hat{\lambda}_N$ such that

(a) every $\hat{\lambda}_g$ is an eigenvalue of some matrix A_g satisfying $\|A_g - A\| \le t$,

(b) if $\hat{\lambda}$ is an eigenvalue of a matrix $\hat{A}$ satisfying $\|\hat{A} - A\| \le t$, then $|\hat{\lambda} - \hat{\lambda}_g| \le t$ for some g, $1 \le g \le N$, and

(c) under the assumptions (a) and (b), the computed value N is within the factor 4 from its minimum.

Computation

State 0: Initialization. Compute a square $S(x_0^{(0)}, r_0)$ containing all the eigenvalues of A. (For instance, one may set $x_0^{(0)} = 0, r_0 = \|A\|_\infty$.) Call $S(x_0^{(0)}, r_0)$ a *suspect square*. Compute $H = \lceil \log_2(r_0/t) \rceil$. Set $N_0 = 1$.

Stage h, h = 1, . . . , H. Partition each suspect square $S(x_i^{(h-1)}, r_{h-1})$ into 4 congruent subsquares $S(z_{4i+g}^{(h)}, r_h)$, $z_{4i+g} = x_i^{(h-1)} + (-1)^{(1+2g)/4} r_h$, $r_h = r_{h-1}/2 = r_0/2^h$, $g = 0, 1, 2, 3$; $i = 1, \dots, N_{h-1}$. For every pair (g, i), compute a lower bound $\hat{d}_{4i+g}^{(h)}$ on $\hat{d}(z_{4i+g}^{(h)})$, see (2). If $\hat{d}_{4i+g}^{(h)} > r_h$, discard the square $S(z_{4i+g}^{(h)}, r_h)$. Otherwise, call this square suspect. Having performed such computation for all the suspect squares $S(x_i^{(h-1)}, r_{h-1}), i = 1, \dots, N_{h-1}$, discard them. Then renumber the centers $z_{4i+g}^{(h)}$ of all the remaining suspect squares $S(z_{4i+g}^{(h)}, r_h)$ and denote these centers $x_i^{(h)}$, $i = 1, \dots, N_h$. If $h = H$, stop and output $N = N_h, \hat{\lambda}_i = x_i^{(h)}$, $i = 1, \dots, N$. Otherwise, go to stage $h + 1$.

Comments

(1) Correctness of the algorithm follows from (1) and from the observation that the diameter of every suspect square decreases by the factor of 2 in each stage.

(2) If at every stage h of Algorithm 1 we had computed lower bounds on $d(z_{4i+g}^{(h)})$ (within relative errors of 10%, say) and had used them instead of $\hat{d}_{4i+g}^{(h)}$, for all i and g, then every eigenvalue of A could have generated at most 4 suspect squares for each h, so that we would have had $N_h \le 4n$. We have actually chosen to approximate $\hat{d}_{4i+g}^{(h)}$ (rather than $d(z_{4i+g}^{(h)})$) in order to simplify the computations; but for matrices A with well-conditioned eigenspaces, the values $\hat{d}_{4i+g}^{(h)}$ and $d(z_{4i+g}^{(h)})$ lie close to each other, and then we should have $N_h = O(n)$ for all h.

REFERENCES

1. P. Henrici, *Applied and Computational Complex Analysis*, Wiley, New York, (1974).
2. C. Eckart and G. Young, A principal axis transformation for non-hermitian matrices, *Bull. Amer. Math. Soc., New Ser.* **45**, 118–121 (1939).
3. W. Kahan, Numerical linear algebra, *Can. Math. Bull.* **9**, 757–801 (1966).
4. G.H. Golub and C.F. Van Loan, *Matrix Computations*, Johns Hopkins University Press, Baltimore, MD, (1989).