BACKGROUND: Euclidean Axioms, Congruent Triangles and Similar Triangles.

DEFN: An altitude of a triangle $\triangle ABC$ dropped from $C$ is the unique line perpendicular to $\overrightarrow{AB}$ which passes through $C$. We say that we drop an altitude from $C$ to a point $X \in \overrightarrow{AB}$.

ALTITUDE LEMMA: If $\triangle ABC$ is a triangle with $\angle ACB = 90^\circ$ and we drop an altitude from $C$ to $X \in \overrightarrow{AB}$ then $X \in \overrightarrow{AB}$.

PROOF: Fill in justifications
1) Assume on the contrary that $X$ is not on segment $\overrightarrow{AB}$.
2) Without loss of generality, we can say $B$ is between $A$ and $X$.
3) $m(\angle ACX) = m(\angle ACB) + m(\angle BCX)$
4) $m(\angle ACX) > 90^\circ$
5) $m(\angle CXA) = 90^\circ$
6) $m(\angle ACX) + m(\angle CXA) + m(\angle XAC) > 180^\circ$

Contradiction

QED

PYTHAGOREAN THEOREM: If $\angle ACB = 90^\circ$, then $AB^2 = AC^2 + BC^2$.

PROOF: Fill in justifications and mark matching angles on a diagram.
1) Drop an altitude from $C$ to $X \in \overrightarrow{AB}$.
2) $\triangle ACX \sim \triangle ABC$
3) $\triangle BCX \sim \triangle ABC$
4) $AC/AB = AX/AC$
5) $BC/AB = BX/BC$
6) $AB = AX + XB$
7) $AC^2 + BC^2 = AX \cdot AB + AB \cdot BX = AB(AX + BX) = AB^2$

QED

MEAN PROPORTION THEOREM: If $\triangle ABC$ is a triangle with $\angle ACB = 90^\circ$ and we drop an altitude from $C$ to $X \in \overrightarrow{AB}$, then $CX^2 = AX \cdot BX$.

Prove this theorem starting by showing that $\triangle ACX$ and $\triangle BCX$ can be shown to be similar taking a careful choice of vertices. Then show $CX/AX = BX/CX$ and complete the proof using algebra.
CONSTRUCTING CONGRUENT TRIANGLES:

SAS CONSTRUCTION: Given two positive real numbers \( r \) and \( s \) and an angle \( \theta \in (0^\circ, 180^\circ) \), there exists a triangle \( \triangle XYZ \) such that \( XZ = r \), \( YZ = s \) and \( m(\angle XZY) = \theta \).

PROOF: Prove this using the segment construction theorem and the angle construction theorem.
1)
2)
3)

QED

PYTHAGOREAN CONVERSE: If \( AB^2 = AC^2 + BC^2 \) then \( \angle ACB = 90^\circ \).

PROOF: Fill in justifications and draw a diagram.
1) Draw triangle \( \triangle XYZ \) such that \( XZ = AC \), \( YZ = BC \) and \( m(\angle XZY) = 90^\circ \).
2) \( XY^2 = XZ^2 + YZ^2 \)
3) \( XY^2 = XZ^2 + YZ^2 = AC^2 + BC^2 = AB^2 \)
4) \( \triangle ABC \) is congruent to \( \triangle XYZ \).
5) \( m(\angle ACB) = m(\angle AZY) = 90^\circ \).

QED

SSS CONSTRUCTION THEOREM: Given \( \triangle ABC \), then there exists a triangle \( \triangle XYZ \) such that \( XY = AB \), \( YZ = BC \) and \( XZ = AC \).

PROOF: Fill in justifications.
1) Draw triangle \( \triangle XYZ \) such that \( XZ = AC \), \( YZ = BC \) and \( m(\angle XZY) = m(\angle ACB) \).
2) \( \triangle ABC \) is congruent to \( \triangle XYZ \).
3) \( XY = AB \), \( YZ = BC \) and \( XZ = AC \).

QED

WARNING: The following statement is FALSE: Given \( r > 0 \), \( s > 0 \), \( t > 0 \), there exists a triangle \( \triangle XYZ \) such that \( XY = r \), \( YZ = s \) and \( XZ = t \). Find a counter example.

ASA CONSTRUCTION THEOREM: Given \( \triangle ABC \), then there exists a triangle \( \triangle XYZ \) such that \( XY = AB \), \( m(\angle ZXY) = m(\angle CAB) \) and \( m(\angle ZYX) = m(\angle CBA) \).

PROOF: Prove this theorem.
1)
2)
3)

QED

WARNING: The following statement is FALSE: Given \( r > 0 \), \( \theta \in (0^\circ, 180^\circ) \), and \( \varphi \in (0^\circ, 180^\circ) \), there exists a triangle \( \triangle XYZ \) such that \( XY = r \), \( m(\angle ZXY) = \theta \) and \( m(\angle ZYX) = \varphi \). Find a counter example.
Determining Lengths and Angles

At this point we turn to the practical problem of determining lengths and angles for ten given triangles. Each will have vertices $A$, $B$ and $C$ with a point $D$ lying on the line segment $AB$. For each problem draw a new triangle and label the given information in pen, then add to the diagram anything you determine in pencil (including values of sides, matching angles, values of angles and so on). Justify all the information you write using congruent triangles, similar triangles, and any theorems you may know. Write clearly and use sentences. Each problem has a question, your answer may be

- Yes/No (with an explanation) or
- a precise number (with an explanation) or
- undetermined (with a description of the multiple solutions) or
- the triangle is impossible (with an explanation as to why).

Naturally you can apply the Pythagorean Theorem and the Mean Proportion Theorem as well as Similar and Congruent Triangle Theorems. The first six problems should be easy applications of these theorems but be careful to apply them correctly. For the last few problems (7-10) the difficulty level increases and you may need to use reasoning similar to the proof of the Converse of Pythagoras’ Theorem, comparing the given triangle to one of the earlier triangles and proving they are congruent. Be explicit as to which triangle you compare it to and explain how you know they are congruent.

1. If $AC = 15$, $CB = 20$ $m(\angle ACB) = 90^\circ$ and $m(\angle ADC) = 90^\circ$, what is $CD$?

2. If $AD = 9$, $DB = 16$, $m(\angle ACB) = 90^\circ$, what is $CD$?

3. If $\triangle ADC \sim \triangle CDB$, what is $m(\angle ADC)$?

4. If $m(\angle ADC) = 90^\circ$, $m(\angle ACB) = 90^\circ$ is $\triangle ADC \sim \triangle CDB$?

5. If $AC = 15$, $CB = 20$, $AB = 25$ and $AD = 9$, what is $m(\angle ADC)$? Is $\triangle ADC \sim \triangle CDB$?

6. If $AC = 15$, $CB = 20$, $AB = 25$ and $m(\angle ADC) = 90^\circ$, is $\triangle ADC \sim \triangle CDB$? What is $CD$?

7. If $AC = 15$, $CB = 20$, $AB = 25$ and $CD = 12$, what is $AD$? What is $DB$?

8. If $AC = 15$, $CB = 20$, $AB = 25$ and $CD = 11$, what is $AD$? What is $DB$?

9. If $AC = 15$, $CB = 20$, $AB = 25$ and $CD = 13$, what is $AD$? What is $DB$?

10. If $AC = 15$, $CB = 20$, $AD = 4$ and $DB = 21$, what is $CD$?

This part of the project has been designed so all students can complete 1-6 easily within an hour and then start the more challenging problems completing them perhaps for homework.