Working With Triangles Project

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MAT631, Fall 2009, Project VIII

BACKGROUND: Euclidean Axioms, Congruent Triangles and Similar Triangles.

DEFN: An altitude of a triangle ΔABC dropped from C is the unique line perpendicular to $\overleftrightarrow{AB}$ which passes through C. We say that we drop an altitude from C to a point $X \in \overleftrightarrow{AB}$.

ALTITUDE LEMMA: If ΔABC is a triangle with $\angle ACB = 90°$ and we drop an altitude from C to $X \in \overleftrightarrow{AB}$ then $X \in \overline{AB}$.

PROOF: *Fill in justifications*
1) Assume on the contrary that X is not on segment $\overline{AB}$.
2) Without loss of generality, we can say B is between A and X.
3) $m(\angle ACX) = m(\angle ACB) + m(\angle BCX)$
4) $m(\angle ACX) > 90°$
5) $m(\angle CXA) = 90°$
6) $m(\angle ACX) + m(\angle CXA) + m(\angle XAC) > 180°$

Contradiction QED

PYTHAGOREAN THEOREM: If $\angle ACB = 90°$, then $AB^2 = AC^2 + BC^2$.

PROOF: *Fill in justifications and mark matching angles on a diagram.*
1) Drop an altitude from C to $X \in \overline{AB}$.
2) $\Delta ACX \sim \Delta ABC$
3) $\Delta BCX \sim \Delta ABC$
4) $AC/AB = AX/AC$
5) $BC/AB = BX/BC$
6) $AB = AX + XB$
7) $AC^2 + BC^2 = AX \cdot AB + AB \cdot BX = AB(AX + BX) = AB^2$

QED

MEAN PROPORTION THEOREM: If ΔABC is a triangle with $\angle ACB = 90°$ and we drop an altitude from C to $X \in \overleftrightarrow{AB}$, then $CX^2 = AX \cdot BX$.

Prove this theorem starting by showing that ΔACX and ΔBCX can be shown to be similar taking a careful choice of vertices. Then show $CX/AX = BX/CX$ and complete the proof using algebra.

CONSTRUCTING CONGRUENT TRIANGLES:

SAS CONSTRUCTION: Given two positive real numbers r and s and an angle $\theta \in (0°, 180°)$, there exists a triangle ΔXYZ such that $XZ = r$, $YZ = s$ and $m(\angle XZY) = \theta$.

PROOF: *Prove this using the segment construction theorem and the angle construction theorem.*
1)
2)
3)

QED

PYTHAGOREAN CONVERSE: If $AB^2 = AC^2 + BC^2$ then $\angle ACB = 90°$.

PROOF: *Fill in justifications and draw a diagram.*
1) Draw triangle ΔXYZ such that
$XZ = AC$, $YZ = BC$ and $m(\angle XZY) = 90°$.
2) $XY^2 = XZ^2 + YZ^2$
3) $XY^2 = XZ^2 + YZ^2 = AC^2 + BC^2 = AB^2$
4) ΔABC is congruent to ΔXYZ.
5) $m(\angle ACB) = m(\angle AZY) = 90°$.

QED

SSS CONSTRUCTION THEOREM: Given ΔABC, then there exists a triangle ΔXYZ such that $XY = AB$, $YZ = BC$ and $XZ = AC$.

PROOF: *Fill in justifications.*
1) Draw triangle ΔXYZ such that $XZ = AC$,
$YZ = BC$ and $m(\angle XZY) = m(\angle ACB)$.
2) ΔABC is congruent to ΔXYZ.
3) $XY = AB$, $YZ = BC$ and $XZ = AC$.

QED

WARNING: The following statement is FALSE: Given $r > 0, s > 0, t > 0$, there exists a triangle ΔXYZ such that $XY = r$, $YZ = s$ and $XZ = t$. *Find a counter example.*

ASA CONSTRUCTION THEOREM: Given ΔABC, then there exists a triangle ΔXYZ such that $XY = AB$, $m(\angle ZXY) = m(\angle CAB)$ and $m(\angle ZYX) = m(\angle CBA)$.

PROOF: *Prove this theorem.*
1)
.
2)
3)

QED

WARNING: The following statement is FALSE: Given $r > 0$, $\theta \in (0°, 180°)$, and $\varphi \in (0°, 180°)$, there exists a triangle ΔXYZ such that $XY = r$, $m(\angle ZXY) = \theta$ and $m(\angle ZYX) = \varphi$. *Find a counter example.*

Determining Lengths and Angles

At this point we turn to the practical problem of determining lengths and angles for ten given triangles. Each will have vertices A, B and C with a point D lying on the line segment AB. For each problem draw a new triangle and label the given information in pen, then add to the diagram anything you determine in pencil (including values of sides, matching angles, values of angles and so on). Justify all the information you write using congruent triangles, similar triangles, and any theorems you may know. Write clearly and use sentences. Each problem has a question, your answer may be

- *Yes/No* (with an explanation) or
- a *precise number* (with an explanation) or
- *undetermined* (with a description of the multiple solutions) or
- *the triangle is impossible* (with an explanation as to why).

Naturally you can apply the Pythagorean Theorem and the Mean Proportion Theorem as well as Similar and Congruent Triangle Theorems. The first six problems should be easy applications of these theorems but be careful to apply them correctly. For the last few problems (7-10) the difficulty level increases and you may need to use reasoning similar to the proof of the Converse of Pythagoras' Theorem, comparing the given triangle to one of the earlier triangles and proving they are congruent. Be explicit as to which triangle you compare it to and explain how you know they are congruent.

1. If $AC = 15$, $CB = 20$ $m(\angle ACB) = 90°$ and $m(\angle ADC) = 90°$, what is CD?
2. If $AD = 9$, $DB = 16$, $m(\angle ACB) = 90°$, what is CD?
3. If $\Delta ADC \sim \Delta CDB$, what is $m(\angle ADC)$?
4. If $m(\angle ADC) = 90°$, $m(\angle ACB) = 90°$ is $\Delta ADC \sim \Delta CDB$?
5. If $AC = 15$, $CB = 20$, $AB = 25$ and $AD = 9$, what is $m(\angle ADC)$? Is $\Delta ADC \sim \Delta CDB$?
6. If $AC = 15$, $CB = 20$, $AB = 25$ and $m(\angle ADC) = 90°$, is $\Delta ADC \sim \Delta CDB$? What is CD?
7. If $AC = 15$, $CB = 20$, $AB = 25$ and $CD = 12$, what is AD? What is DB?
8. If $AC = 15$, $CB = 20$, $AB = 25$ and $CD = 11$, what is AD? What is DB?
9. If $AC = 15$, $CB = 20$, $AB = 25$ and $CD = 13$, what is AD? What is DB?
10. If $AC = 15$, $CB = 20$, $AD = 4$ and $DB = 21$, what is CD?

This part of the project has been designed so all students can complete 1-6 easily within an hour and then start the more challenging problems completing them perhaps for homework.