Coordinate Transformations Project

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BACKGROUND: General Axioms, Congruent and Similar Triangles, Coordinate Plane, Distances Between Points in the Plane, Lines Through the Origin.
DEFINITION: A transformation is a map which takes the plane to the plane which is one-to-one and onto: $F : E^2 \to E^2$. If we use coordinates, then we can write $P = (p_1, p_2)$ and write

$$F(p_1, p_2) = (F_1(p_1, p_2), F_2(p_1, p_2)).$$

THE SHIFT TRANSFORMATION: Given a point $Q = (q_1, q_2)$ we can define a unique shift transformation which takes the origin to Q as follows:

$$T_Q(p_1, p_2) := (p_1 + q_1, p_2 + q_2).$$

For example, if $Q = (2, 3)$ then the shift transformation is $T_{(2,3)}(p_1, p_2) := (p_1 + 2, p_2 + 3)$.
DEFINITION: An isometry is a transformation which preserves distances:

$$d(A, B) = d(F(A), F(B))$$

THE SHIFT TRANSFORMATION IS AN ISOMETRY: *Prove this by first writing $d(A, B)$ using the coordinates $A = (A_1, A_2)$ and $B = (B_1, B_2)$. Then finding the formula for $T_Q(A)$ and $T_Q(B)$ and computing the distance between these points. This takes some computation. Two columns are not necessary:*

DEFINITION: Given a map $F : E^2 \to E^2$, the inverse of F is a map $F^{-1} : E^2 \to E^2$ such that $F(F^{-1}(P)) = P$ and $F^{-1}(F(P)) = P$.
INVERSE OF A SHIFT: The inverse of $T_{(Q_1,Q_2)}$ is $T_{(-Q_1,-Q_2)}$. *Prove this here:*

DEFN AND FORMULA FOR A CIRCLE: A circle $C(P,R)$ about center point P of radius R, is the collection of all points, Q, in the plane such that $d(P,Q)=R$. That is

$$C(P,R)=\{Q:\ d(P,Q)=R\}=\{(Q_1,Q_2):(Q_1-P_1)^2+(Q_2-P_2)^2=R^2\}.$$

THE IMAGE OF A TRANSFORMATION: Given a transformation, F, and a set of points S, the image of S under F is that set

$$F(S)=\{F(P):\ P\in S\}=\{Q:\ \exists P\in S \text{ s.t. } F(P)=Q\}$$

For example, if S is the circle $C((0,0),3)=\{P:d(P,(0,0))=3\}=\{(p_1,p_2):p_1^2+p_2^2=9\}$ then

$$T_{(2,4)}(S)=\{T_{(2,3)}(p_1,p_2):p_1^2+p_2^2=9\}=\{(p_1+2,p_2+3):p_1^2+p_2^2=9\}$$

and to simplify we can write $q_1=p_1+2$ and $q_2=p_2+3$ so $p_1=q_1-2$ and $p_2=q_2-3$, so

$$T_{(2,4)}(S)=\{(p_1+2,p_2+3):p_1^2+p_2^2=9\}=\{(q_1,q_2):(q_1-2)^2+(q_2-3)^2=9\}.$$

Interestingly the image of this circle under this shift is a circle.
Can you prove this is always true for any shift and any circle? What about any isometry? Try this on the back of the last page for extra credit.
THE IMAGE OF A LINE UNDER AN ISOMETRY IS A LINE: *this was proven in the isometry project.*
LINES THROUGH THE ORIGIN: Recall that we have proven a line through the origin and a point (x_1,y_1) has the following set notation:

$$\{(x,y):y=mx\} \text{ where } m=y_1/x_1 \text{ is called the slope}$$

and when $x_1=0$ it is $\{(x,y):x=0\}$ which is the y axis. We've proven two lines through the origin are perpendicular then their slopes are negative reciprocals.
THE SHIFT OF A LINE IS A LINE: *This follows immediately from the fact that a shift is an isometry.*
FINDING THE FORMULA OF ANY LINE: Suppose we have a line L which crosses the y axis at $(0,b)$ and the x axis at $(a,0)\neq(0,0)$ then the formula for the line is

$$\{(x,y):y=mx+b\} \text{ where } m=-b/a.$$

All other lines are either parallel to the x axis or to the y axis or pass through the origin.
PROOF: *fill in the justifications*
1) $T_{(0,-b)}(L)$ is a line through the origin.
2) $T_{(0,-b)}(L)$ is a line through $(a,-b)$.
3) $T_{(0,-b)}(L)=\{(x,y):y=mx\}$ where $m=-b/a$.
4) $L=T_{(0,b)}(T_{(0,-b)}(L))$
5) $\quad =T_{(0,b)}(\{(x,y):y=mx\})$.
6) $\quad =\{(x+0,y+b):y=mx\}$
7) Let $X=x+0$ and $Y=y+b$ so $x=X$ and $y=Y-b$
so $L=\{(X,Y):Y-b=mX\}=\{(X,Y):Y=mX+b\}$.
SLOPES OF PARALLEL LINES: *Prove two lines are parallel iff they have the same slope. Prove two lines are parallel iff there is a shift which takes one to the other.*
SLOPES OF PERPENDICULAR LINES: *Prove two lines are perpendicular iff their slopes are negative reciprocals.*

MIDPOINT DEFN: A point Z is the midpoint of $\overline{PQ}$ iff $d(P,Q) = 2d(P,Z) = 2d(Q,Z)$.
COORDINATE FORMULA: Z is the midpoint of $\overline{PQ}$ iff $Z_1 = (P_1 + Q_1)/2$ and $Z_2 = (P_2 + Q_2)/2$. *Prove this on the back of the last page.*
ISOMETRIES AND MIDPOINTS: If $F : E^2 \to E^2$ is an isometry and Z is a midpoint of $\overline{PQ}$ then $F(Z)$ is a midpoint of $\overline{F(P)F(Q)}$. *Prove on back of last page.*
REFLECTION THROUGH A POINT: A reflection through a point Z is a transformation $r_Z : E^2 \to E^2$ such that a point P is taken to a point Q such that Z is the midpoint of $\overline{PQ}$. We proved this was an isometry in the isometry project.
DERIVING THE FORMULA FOR A POINT REFLECTION:
1) *First verify the formula for* $r_{(0,0)}(P_1, P_2) = (-P_1, -P_2)$ *using the formula midpoint here:*

2) Note that if Z is the midpoint of $\overline{PQ}$ then the origin is the midpoint of $\overline{T_{-Z}(P), T_{-Z}(Q)}$. *Why?*

3) Thus $T_{-Z}(Q) = r_{(0,0)}(T_{-Z}(P))$. *Why?*

4) So taking T_Z of both sides of the equation we get $Q = T_Z(r_{(0,0)}(T_{-Z}(P))$ because T_Z is the inverse of T_{-Z}.
5) This means $r_Z = T_Z \circ r_{(0.0)} \circ T_{-Z}$. Thus the formula for

$r_{(Z_1,Z_2)}(P_1, P_2) =$

REFLECTING ACROSS A LINE: A reflection across a line L is a transformation $r_L : E^2 \to E^2$ such that a point P is taken to a point Q such that L is the perpendicular bisector of PQ. *We proved in the isometry project that this is an isometry.*
Find the formula

$r_{x=0}(P_1, P_2) =$

Find the formula

$r_{x=y}(P_1, P_2) =$

Explain in words and pictures why $r_{y=mx+b} = T_{0,b} \circ r_{y=mx} \circ T_{0,-b}$.

Extra credit: Find the formula for $r_{y=mx}$.

DEFINING A DILATION: *A dilation by scale $r > 0$ is a map $dil_r : E^2 \to E^2$ such that $dil_r(P_1, P_2) = (rP_1, rP_2)$.*
DILATING DISTANCES: $d(dil_r(P), dil_r(Q)) = rd(P, Q)$*:*
PROOF: *Fill in here with a few equations:*

DILATING CIRCLES: $dil_r(C(P, R)) =$ *Find a formula and prove it on the back of the last page.*
DEFINING A SIMILTUDE: A similtude is a transformation, $F : E^2 \to E^2$ such that if $X' = F(X)$, $Y' = F(Y)$ and $Z' = F(Z)$ then ΔXYZ is similar to $\Delta X'Y'Z'$. In particular similtudes preserve angles!
ISOMETRIES ARE SIMILTUDES: Recall that we proved isometries map triangles to congruent triangles. Thus the triangles are also similar.
Is a dilation a similtude? Prove or give a counter example on the back of the last page.
DILATIONS OF SQUARES ARE SQUARES: *Prove this here:*

DEFINING A SKEW OR SHEAR: A skew along the x axis is a transformation of the form $F(P_1, P_2) = (P_1 + kP_2, P_2)$.
Draw the image of a skew of a square with vertices $(0, 0)$ $(0, 1)$ $(1, 0)$ *and* $(1, 1)$*:*

Show $F(\{(x, y) : y = y_1\}) = \{(x, y) : y = y_1\}$ *so horizontal lines are preserved.*

What is $F(\{(x, y) : y = mx + b\})$*? Conclude that if L is a line* $F(L)$ *is a line.*

SKEWING PARALLEL LINES: If L_1 and L_2 are parallel lines then $F(L_1)$ and $F(L_2)$ are parallel lines. *Prove here:*

Is a skew an isometry? Prove or give a counter example on the back.
Is a skew a similtude? Prove or give a counter example on the back.