Similar Triangles Lesson and Project

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BACKGROUND: Euclidean geometry axioms including the parallel postulate and the SSS, SAS, ASA, Vertical Angle, Alternate Interior Angles and Parallelogram Theorems.

WARMUP: *Draw triangle ΔABC with vertex A on top and base BC horizontal. Now draw points $E \in \overline{AB}$ and $F \in \overline{AC}$ such that $\overleftrightarrow{EF}$ is parallel to $\overleftrightarrow{BC}$.*

GIVEN: $\overleftrightarrow{EF}$ is parallel to $\overleftrightarrow{BC}$. SHOW: $m(\angle AEF) = m(\angle ABC)$:

PROOF: *(hint: draw the line $\overleftrightarrow{EF}$ then mark congruent angles on your diagram)*

CONVERSE:

GIVEN: $m(\angle AEF) = m(\angle ABC)$. SHOW: $\overleftrightarrow{EF}$ is parallel to $\overleftrightarrow{BC}$.

PROOF: *Fill in justifications*

1) Assume on the contrary that $\overleftrightarrow{EF}$ is not parallel to $\overleftrightarrow{BC}$.

2) Let L be a line parallel to $\overleftrightarrow{BC}$ passing through E.

3) Let P be the point where L intersects $\overline{AC}$

4) $m(\angle AEP) = m(\angle ABC)$

5) $m(\angle AEF) = m(\angle ABC)$

6) $m(\angle AEF) = m(\angle AEP)$

7) $\overrightarrow{EF} = \overrightarrow{EP}$

8) $\overleftrightarrow{EF} = \overleftrightarrow{EP}$

9) $\overleftrightarrow{EF}$ is parallel to $\overleftrightarrow{BC}$

CONTRADICTION.

QED

SIDE SPLITTING THEOREM: Keeping in mind the same diagram as on the first page, where we have a triangle ΔABC and points $E \in \overline{AB}$ and $F \in \overline{AC}$, we will now prove:

$$\overleftrightarrow{EF} \text{ is parallel to } \overleftrightarrow{BC} \qquad \text{IFF} \qquad \frac{AE}{AB} = \frac{AF}{AC}$$

This is a difficult theorem to prove when the ratios are irrational. The proof for irrational ratios can be proven by taking limits or using Archimedes Principal (see Kay's *College Geometry*).

Here we only prove the Side Splitting Theorem when the ratios are rational.

GIVEN: There are natural numbers p, q such that $\frac{AE}{AB} = \frac{AF}{AC} = \frac{p}{q}$. SHOW: $\overleftrightarrow{EF}$ is parallel to $\overleftrightarrow{BC}$.

1) First we can divide $\overline{AB}$ and $\overline{AC}$ into q equal length subsegments so that we have $B_0 = A$ then B_1 then B_2 and so on up to $B_q = B$ on $\overline{AB}$ and we have $C_0 = A$ then C_1 then C_2 and so on up to $C_q = C$ on $\overline{AC}$. *(draw)*

2) Notice that since $\frac{AE}{AB} = \frac{AF}{AC} = \frac{p}{q}$, we have $E = B_p$ and $F = C_p$.

3) We now draw many small parallelograms. To do this we first draw lines through each B_i parallel to $\overleftrightarrow{AC}$ and then we draw lines through each C_j parallel to $\overleftrightarrow{BC}$ by applying the parallel postulate. *(add to drawing)*

4) By the Parallelogram Theorem, the sides of the parallelograms parallel to $\overleftrightarrow{AB}$ all have length AB/q and the sides parallel to $\overleftrightarrow{AC}$ all have length AC/q.

5) Notice that all the parallelograms also have two angles whose measure is equal to $m(\angle BAC)$ by the Alternate Interior Angles Theorem combined with Vertical Angles Theorem. *(add angles)*

6) If we draw in the diagonals of each of these parallelograms we get a collection of congruent triangles by the SAS Theorem. Some are upside right and some are upside down. *(add diagonals)*

7) The diagonals which are line segments come together colinearly because the sum of the angles of a triangle is 180°. Here we have used corresponding angles on congruent triangles and linear pairs. In fact the line segments $\overline{BC}$ and $\overline{EF}$ are made of many of these diagonal segments.

8) $m(\angle ABC) = m(\angle AEF)$ by corresponding angles on congruent triangles.

9) $\overleftrightarrow{BC}$ is parallel to $\overleftrightarrow{EF}$, by vertical angles and alternate interior angles theorems.

QED

GIVEN: $\overleftrightarrow{EF}$ is parallel to $\overleftrightarrow{BC}$. SHOW: There are natural numbers p, q such that $\frac{AE}{AB} = \frac{AF}{AC} = \frac{p}{q}$.
Prove this by contradiction imitating the proof by contradiction on the first page.

Combining the theorem on the first page with this theorem, we now know that

$$m(\angle ABC) = m(\angle AEF) \qquad \text{IFF} \qquad \overleftrightarrow{EF} \text{ is parallel to } \overleftrightarrow{BC} \qquad \text{IFF} \qquad \frac{AE}{AB} = \frac{AF}{AC}$$

DEFN: We say triangles ΔABC and $\Delta A'B'C'$ are similar if they have congruent angles:

$$m(\angle ABC) = m(\angle A'B'C') \qquad m(\angle BCA) = m(\angle B'C'A') \qquad m(\angle CAB) = m(\angle C'A'B')$$

and proportional sides:

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

AAA THEOREM: If two triangles ΔABC and $\Delta A'B'C'$ have congruent angles:

$$m(\angle ABC) = m(\angle A'B'C') \qquad m(\angle BCA) = m(\angle B'C'A') \qquad m(\angle CAB) = m(\angle C'A'B')$$

then they are similar. (The converse is obvious).

PROOF: We only need to show $\frac{A'B'}{AB} = \frac{A'C'}{AC}$.

Three possibilities: Case I: $AB = A'B'$ Case II: $AB > A'B'$ Case III: $AB < A'B'$

CASE I: *Show triangles are congruent and conclude the ratios of all three sides are proportional.*

CASE II: $AB > A'B'$ *Fill in justifications.*

1) Draw a point $E \in \overline{AB}$ such that $AE = A'B'$
and a point $F \in \overline{AC}$ such that $AF = A'C'$.

2) Triangles $\Delta A'B'C'$ and ΔAEF are congruent.

3) $m(\angle AEF) = m(\angle A'B'C')$

4) $m(\angle AEF) = m(\angle ABC)$

5) $\frac{AE}{AB} = \frac{AF}{AC}$

6) $\frac{A'B'}{AB} = \frac{A'C'}{AC}$

CASE III: $AB < A'B'$ follows immediately as in Case II.

QED

PROPORTIONAL SIDES THEOREM: If two triangles ΔABC and $\Delta A'B'C'$ have proportional sides:

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

then they are similar.

PROOF: *Prove this yourselves using the same cases as in the last proof and the same idea as in step 1 of Case II above. The rest of the proof will work differently of course.*

MEAN PROPORTION THEOREM AND PYTHAGOREAN THEOREM:
If ΔABC has $m(\angle BCA) = 90°$ and $\overline{CP}$ is the altitude to $\overline{AB}$ (that is $\overline{CP}$ is perpendicular to $\overline{AB}$ with $P \in \overline{AB}$) then

$$CP^2 = AP \cdot PB \qquad AB \cdot CP = AC \cdot BC \qquad AB^2 = AC^2 + BC^2.$$

Prove these two theorems by first proving that ΔABC is similar to ΔPBA which is also similar to ΔPAC. When drawing the diagrams for the proofs, it is best to make length AC about twice the length AB so that it is easy to see which sides and angles correspond. This proof of Pythagoras as well as other excellent proofs may be found on Wikipedia.

CONVERSE OF PYTHAGORAS' THEOREM: If ΔABC is a triangle such that $AB^2 = AC^2 + BC^2$ then $m(\angle BCA) = 90°$.

Prove this by constructing another triangle $\Delta A'B'C'$ such that $m(\angle B'C'A') = 90°$, $AC = A'C'$ and $BC = B'C'$. Show that this triangle is congruent to ΔABC using Pythagorus' Theorem and SSS.

SOHCAHTOA THEOREM: Given two right triangles ΔABC and $\Delta A'B'C'$ where $m(\angle BCA) = m(\angle B'C'A') = 90°$ and $m(\angle ABC) = m(\angle A'B'C') = \theta$ then these triangles are similar so

$$\frac{AC}{AB} = \frac{A'C'}{A'B'} = \frac{opp}{hyp} \qquad \frac{BC}{AB} = \frac{B'C'}{A'B'} = \frac{adj}{hyp} \qquad \frac{AC}{BC} = \frac{A'C'}{B'C'} = \frac{opp}{adj}$$

are all functions of the angle θ. So we call them

$$\sin(\theta) = \frac{opp}{hyp} \qquad \cos(\theta) = \frac{adj}{hyp} \qquad \tan(\theta) = \frac{opp}{adg}.$$

In trigonometry these functions are defined for all angles using a unit circle and one can prove the formulas given here agree with that definition for angles $\theta \in (0, 90°)$.

THE LAW OF SINES: Given any triangle ΔABC then

$$\frac{\sin(\alpha)}{a} = \frac{\sin(\beta)}{b} = \frac{\sin(\gamma)}{c}$$

where $a = BC$, $b = AC$, $c = AB$ and $\alpha = m(\angle BAC)$, $\beta = m(\angle ABC)$, $\gamma = m(\angle ACB)$. *Prove this by dropping an altitude from C and forming right triangles. These right triangles are not similar but you can write out the formula for $\sin(\alpha)/sin(\beta)$ using the definition of sine and then simplifying. There are two cases: one where the altitude drops inside the triangles and one where it drops outside.*

THE LAW OF COSINES: Given any triangle ΔABC then

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

where $a = BC$, $b = AC$, $c = AB$ and $\alpha = m(\angle BAC)$, $\beta = m(\angle ABC)$, $\gamma = m(\angle ACB)$. *Prove this by dropping an altitude from A and forming right triangles. These right triangles are not similar but you can write out the formula for $\cos(\gamma)$ using one of the right triangles, and then find the formula for $2ab\cos(\gamma)$. You can also use Pythagoras' Theorem to find a formula for c^2 using another triangle.*