Sequences Project

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Write down your social security number here:
At various times in this project you will be asked to use a digit from your social security number as a constant in the problem. You can work together, but don't forget that your problems are different.

1. Let S_i= the i^{th} derivative of $e^{(\frac{1}{2}y)}$ at y=0. Write down the first five terms of this sequence.

   Write down a formula for the nth term of this sequence.

   Write down a formula for the nth derivative of the function.

   Prove this is the correct formula for the derivative using induction. That is, check that your formula is correct when n=1. Then assume that the formula is correct for the nth case and show that it is correct for the n+1 case by taking a derivative.

Now that you have proven your formula for the nth derivative, all you have to do to check your formula for S_n is substitute y=0. Is your formula correct?

Does this sequence converge? Why? Where?

2. Sometimes sequences are defined using difference equations. That is, you are given a number for y_1 and a formula for the differences between all the subsequent terms. Here is an example:

$$y_{i+1} - y_i = \frac{1}{4000} y_i (1000 - y_i)$$

$$y_1 = 2$$

This equation is a model for population growth in an environment that can only support 1000 people. y_i stands for the number of people in the year i.

Write down the first five terms of this sequence:

This model would not make sense if there were ever more than 1000 people. We can prove that there are never more than 1000 people by assuming that there are more than 1000 people and looking at the last year, n, that there were less than a thousand people. That means that there is some number n such that $y_n < 1000$ and $y_{n+1} \geq 1000$. This tells us that $y_{n+1} - y_n \geq 1000 - y_n$. Use the difference equation and the fact the $y_n < 1000$ to prove that this is impossible:

How did we get this contradiction? There are only two ways to get contradictions in mathematics: either by making a mistake or by making a false assumption. We made a false assumption when we assumed that there was a time when we had more than 1000 people. Thus y_i is always less that 1000. We have shown that the sequence is bounded above.

We can also show that the sequence is bounded below by 0. Prove this by induction. Use the difference equation and the upper bound of 1000.

The first five terms of the sequence are increasing. Use your upper and lower bounds to prove that the sequence is always increasing. Remember, you want to show that $y_{i+1} - y_i$ is bigger than 0.

Does the sequence converge?
We will examine where it converges in the last problem.

3. Suppose we have a unit halfshere and we want to approximate its length (which is obviously π!). We can divide the semicircle evenly into i pieces and draw straight lines between the endpoints of the pieces. The length of this jagged line approaches the length of the halfcircle as i goes to ∞. Each succesive approximation can be thought of as an element of a sequence, l_i, which converges to π.

Each l_i is the sum of the lengths of i line segments. Each line segment has a length that can be computed using the following formula:

$$c^2 = a^2 + b^2 - 2abcos\theta$$

Here a and b are 1 because they are radial lines. Find θ. Be careful, it depends on i.

What is the length of a line segment?

What is l_i?

It is surprising that this ugly sequence converges to π without knowing the geometry behind it. However, it can be proved to converge to π using l'hopital's rule. Do it:

4. We will now learn a trick to find the limit of a sequence once we know that it has a limit. Suppose the sequence can be written in the following form:

$$S_1 = S$$

$$S_{i+1} = f(S_i)$$

where f is a given function. The sequence in problem 2 can be written this way with $f(x) = x + \frac{x}{4000}(1000 - x)$. Another example would be $f(x) = x^2$.
The advantage of writing the sequence this way is that we can now take the limit on both sides of the equation:

$$lim S_{i+1} = lim f(S_i)$$

Now if we know the limit of the sequence is L, then so is the limit of the shifted sequence and our limits give us:

$$L = f(L)$$

If we solve this equation we know all possible limits L of the equation, given any initial element S_1.

Find all possible limits L in the case of problem 2.

Which limit corresponds to the situation where $y_1 = 2$? Why? Note that we have already proved that this limit exists because the sequence was increasing and bounded above by 1000.

What value of y_1 would give us a limit of 0?

Now look at the case where $f(x) = x^2$. What are the possible L? Examine the

situation with the following values of S_1. Write down the first few terms and give the limit in each case with an explanation.

$S_1 = 0$

$S_1 = 1$

$S_1 = -1$

$S_1 = \frac{1}{2}$

$S_1 = 3$