Approximating Improper Integrals

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due Nov. 1, 1994

In Calculus I you were taught to approximate definate integrals using the midpoint method and other techniques. To approximate an integral you divided it into a certain number of rectanges and then added up the areas of the rectangles. The heights of the rectangles were either the value of the integrand at the midpoint or the right endpoint depending on the method you were using. The width was found by dividing the difference between the endpoints of integration by the number of rectangles.

Now we are studying improper integrals. There is no way to divide the area under the curve into a finite nuimber of rectangles. So instead we can try to approximate the integral by taking infinitely many rectangles of the same length. Now we know that not all improper integrals converge so we expect that sometimes the approximations might be infinitely large as well. If we are careful to make sure that all the rectangles fit under the graph, then the sum of all their areas will be smaller than the integral. If all the rectangles are above then the sum of all their areas will be larger than the integral.

1. Graph the following integral and draw approximating rectangles of width 1 using the right endpoint method. Notice that all the rectangles are under the curve.

$$\int_1^\infty 2^{-x}\,dx$$

2. Write down the areas of the first five rectangles.

3. Check that the sum of these areas is less than:

$$\int_1^6 2^{-x}\,dx$$

by working out the integral. Hint:

$$2^{-x} = e^{(-ln2\,x)}$$

Don't forget that ln2 is just a constant.

4. Write a formula for the area of the ith rectangle.

Write down a formula for the sum of the areas of the first n rectangles using summation notation.

5. Draw a picture of the first five rectangles piled on top of each other.

   Do you notice any pattern?

   If you put in as many rectangles as you want, would they still be less than 1/2 high?

6. What is the value of the indefinite integral in step 1?

   Is it less than a half?

   What does this say about the sum of the areas of the first n rectangles?

Now look at the integral:

$$\int_1^\infty \frac{1}{x}\,dx$$

1. Prove that this integral does not converge by evaluating it.

2. Draw a graph of the integrand and draw a rectangle approximation with each rectangle exactly 1 unit wide, using the left endpoint method. Each rectangle should be above the graph.

3. Write down the area of the first 4 rectangles:

4. Write down the area of the kth rectangle:

   Write down the sum of the areas of the first n rectangles using summation notation:

5. The sum of the first n rectangles is bigger than:

$$\int_1^{n+1} \frac{1}{x}\, dx$$

because the rectangles are above the curve.
Compute this integral:

6. In the first half of this project there was a special number that was bigger than the sum of the first n rectangles no matter how big n was. Do you think such a number exists in this case?

Suppose that such a number did exist. Let us call it M. Then:

$$M \geq S(n) \geq \int_1^{n+1} \frac{1}{x}\, dx$$

for all n. Using your answer from the last problem, show that M cannot possibly exist: