Practice Final, MAT172, Fall 2005

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The answers are given during review sessions.

Part I: Short Answers On the exam there will be ten questions in this part.

1. Multiply $(x+2)$ by $(x-1)$.

   Answer: $(x+2)(x-1) = x^2 - x + 2x - 2 = x^2 + x - 2$.

2. Let $f(x) = 5x^2$. What is $f(x+h)$?

   Answer: $f(x+h) = 5(x+h)^2$.

   A quick check: $x = 2$, $h = 1$, $x + h = 3$ so $f(2+1) = f(3) = 5(3)^2 = 5(9) = 45$ and $5(2+1)^2 = 5(3)^2 = 5(9) = 45$.

3. Suppose the terminal point of θ is $(3/5, -4/5)$, what is $cos(\theta)$?

   Answer: 3/5

4. What is the domain of $\sqrt{x-5}$?

   Answer: $x - 5 \geq 0$ so $x \geq 5$ so the domain is $[5, \infty)$.

5. Write as a single fraction:

   $$\frac{1}{x-2} + \frac{x+3}{x-1}$$

   Answer: Use $(x-2)(x-1)$ as a common denominator and get

   $$\frac{x-1}{(x-2)(x-1)} + \frac{(x-2)(x+3)}{(x-2)(x-1)} = \frac{[(x-1)+(x-2)(x+3)]}{(x-2)(x-1)}.$$

6. Find the vertical asymptotes of

   $$f(x) = \frac{(x-5)}{(x^2+5x+6)}$$

   Answer: Solve $x^2 + 5x + 6 = 0$ either by factoring

   $$x^2 + 5x + 6 = (x+2)(x+3) = 0$$

   or using the quadratic equation and get $x = -2$ and $x = -3$.

7. Simplify $Log_2(8^x4)$.

   Answer: $Log_2(8^x4) = Log_2(8^x) + Log_2(4) = xLog_2(8) + 2 = x3 + 2 = 3x + 2$. Here we've used $Log_2(8) = 3$ because $2^3 = 8$.

8. Find the value of $cos^2(\pi/4)$.

   Answer:
   $$cos^2(\pi/4) = (\, cos(\pi/4)\,)^2 = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{(\sqrt{2})^2}{2^2} = \frac{2}{4} = \frac{1}{2}.$$

9. What is the domain of $Arcsin(x)$?

   Answer: $[-1, 1]$

10. Rewrite in terms of $sin(x)$ and $cos(x)$ and simplify:
   $$1 + sec(\theta) + tan(\theta)$$

   Answer:
   $$\begin{aligned} 1 + sec(\theta) + tan(\theta) &= 1 + \frac{1}{cos(\theta)} + \frac{sin(\theta)}{cos(\theta)} \\ &= \frac{cos(\theta)}{cos(\theta)} + \frac{1}{cos(\theta)} + \frac{sin(\theta)}{cos(\theta)} \\ &= \frac{cos(\theta) + 1 + sin(\theta)}{cos(\theta)} \end{aligned}$$

Part II: Justified True and False

State whether it is true or false and justify. On the exam there will only be ten questions in this part. *To show something is false, you can give an example, but to show something is true you need to work out the formulas. See the first two problems.*

1. A student writes: $1 - sin(t) = cos(t)$

   False, try $t = 30°$ and get $1 - sin(30°) = 1 - (1/2) = 1/2$ but $cos(30°) = \sqrt{3}/2$ is not $1/2$.

2. A student writes: $sin(x + \pi) \ = \ - sin(x)$

   True, use $sin(x+\pi) = cos(\pi)sin(x)+sin(\pi)cos(x) = (-1)sin(x)+0cos(x) = -sin(x)$.

3. A student writes: $cos(2x) = 2cos(x)$

   False. Try $x = \pi/6$. $cos(2\pi/6) = cos(\pi/3) = 1/2$ but $2cos(\pi/6) = 2\sqrt{3}/2 = \sqrt{3}$.

4. The graph of $y = 5sin(2x)$ is a curve which starts at $(0, 5)$ curves downward to $(\pi/4, 0)$ continues down to a minimum at $(\pi/2, -5)$ then goes upward through $(3\pi/4, 0)$ to a second maximum at $(\pi, 5)$ and extends periodically in both directions with period π.

   False, this has the correct amplitude= 5 and period= π, but $sin(0) = 0$. So $y = 5sin(2 \cdot 0) = 0$ not 5 and the curve should start at $(0, 0)$, not $(0, 5)$. In fact the curve described here is $y = 5cos(2x)$. Remember the period is the value of x where $(2x) = 2\pi$.

5. A student is solving a max/min problem and writes:

   $$5(\,(x + 3)^2 \ - \ 9 \ + \ 2\,) = 5(x + 3)^2 - 7$$

   False, it equals $5(\,(x + 3)^2 \ - \ 7\,) = 5(x + 3)^2 - 35$.

6. A student is working on the following problem: *If you have fifty dollars in the bank with 3 percent interest compounded annually, how much do you have in 2 years?*

   The student writes:

   $$50(1 \ + \ .03)(1 \ + \ .03) = 50(1.03)^2$$

   Then the student has begun the problem correctly.

   True, 50 is the principal and .03 is the rate, and the time is 2 years.

   On the exam, we might compound quarterly or continuously.

7. The student writes $Arcsin(sin(150°)) = 30°$.

   True, $sin(150°) = 1/2$ and $Arcsin(1/2) = 30°$ and the answers to $Arcsin$ are always in $[-90°, 90°]$.

8. The student writes $\sqrt{x^2} = x$.

   False, it is $|x|$ because x might be negative and the value of the square root function is never negative. For example taking $x = -3$ we have $\sqrt{(-3)^2} = \sqrt{9} = 3$ not -3.

9. The domain of 2^x is $(0, \infty)$.

   False, the domain is $(-\infty, \infty)$, exponential functions are defined for negative numbers. For example: $2^{-3} = 1/2^3 = 1/8$.

10. When completing the square a student writes $x^2 + 6x = (x + 3)^2 - 9$.

    True, half of 6 is 3, so you add and subtract $3^2 = 9$, to get

    $$x^2 + 6x + 9 - 9 = (x + 3)^2 - 9$$

    Or

    True, just check the completed square is correct:

    $(x + 3)^2 - 9 = (x + 3)(x + 3) - 9 = x^2 + 3x + 3x + 3^2 - 9 = x^2 + 6x + 9 - 9 = x^2 + 6x.$

11. The student writes $x^2 - 9$ divided by $x + 3$ is $x - 3$.

    True, just check $(x + 3)(x - 3) = x^2 - 3x + 3x + 3(-3) = x^2 + 0x - 9 = x^2 - 9$.

12. The student writes $Ln(5e^x) = Ln(5) + Ln(e^x) = Ln(5) + x$.

    True, because $Ln(ab) = Ln(a) + Ln(b)$ and $Ln(x)$ is the inverse of e^x.

13. The student writes $sin^{-1}(x) = sin(1/x)$.

    False, $sin^{-1}(x) = Arcsin(x)$ is the inverse function of $sin(x)$. It's value is an angle $\theta = Arcsin(x)$ such that $sin(\theta) = x$. The answer to $sin(1/x)$ is not an angle. Note that

    $$sin(x^{-1}) = sin(1/x)$$

    The important point is that if one writes $f^{-1}(x)$ where $f(x)$ is a function like $sin(x), cos(x), tan(x)$, then we mean an inverse of the function and if one writes a^{-1} where a is a variable like $x, t, s, \theta, u, v, y, z...$ or a number, then $a^{-1} = 1/a$.

    The most tricky situation is when you take $(f(a))^{-1}$, this means find the value of $f(a)$, which is some unknown number, and then take it's -1 power, so we get

    $$(f(a))^{-1} = \frac{1}{f(a)} \text{ and } (sin(x))^{-1} = \frac{1}{sin(x)}.$$

Part III: Long Answers

1. Let $y = f(x)$ be the line of slope 2 which passes through $(1, 4)$ and $y = g(x)$ be the line of slope -0.5 which passes though $(4, 0)$.

   a) What is the formula for $f(x)$?

   b) What is the formula for $g(x)$?

   c) Plot both lines neatly on graph paper.

   d) Where do they meet?

   e) Verify your answer to (d) using algebra.

   f) What is the angle between the lines? Justify.

   Answers:

   a) The point slope formula says: $y-4 = 2(x-1)$ So $y = 2(x-1)+4$ so $f(x) = 2(x-1)+4$.

   b) Similarly $g(x) = -0.5(x-4) + 0$.

   c)

   d) They appear to meet at $(0, 2)$.

   e) Check $f(0) = 2$ and $g(0) = 2$.

   f) The angle between the lines looks like a right angle. Justification: Can say the slopes are negative reciprocals: $-0.5 = -1/2$ or you can draw a triangle and check using the law of cosines, or you can do the following:

2. Let $f(x) = x^2 + 8x$ and $g(x) = x^2 + 8x + 20$.

a) $f(0) =?$

b) What is the y intercept of f? of g?

c) Factor f.

d) What are the roots of f?

e) Complete the square for f.

f) What is the vertex of f? Find both the coordinates.

g) Plot f

h) Plot g

i) What is the vertex of g?

j) What are the roots of g? Check whether your answer from the graph agrees with the algebraic solution using the quadratic formula.

Answers:

a) f(0)=0

b) y intercept of f is 0 and of g is $g(0) = 20$.

c) $f(x) = x(x + 8)$

d) The roots are 0 and -8.

e) $f(x) = x^2 + 2(4x) + 4^2 - 4^2 = (x + 4)^2 - 16$

f) The vertex of f is where $(x + 4) = 0$, so $x = -4$ and $y = f(-4) = -16$ So the vertex is $(-4, -16)$.

g) and h)

i) The vertex of g is 20 places above the vertex of f, so it is $(-4, 4)$.

j) What are the roots of g? From the graph, g never crosses the x axis because its minimum is above the x axis, so g has no roots. Checking this algebraicly:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-8 \pm \sqrt{(-8)^2 - 4(1)(20)}}{2(1)} = \frac{-8 \pm \sqrt{64 - 80}}{2}$$

There is a negative number in the square root so g is has no real roots.

3. Let $f(x) = (x^2 - 9)/(x^2 + 8x + 16)$.

a) When is $f(x) = 0$?

b) When is $f(x)$ undefined?

c) What are the vertical asymptotes of $f(x)$?

d) Which of the following graphs could be $y = f(x)$? Explain your choice carefully!!!!

e) Label your chosen graph with all zeroes and vertical asymptotes.

Answers:

a) Solve numerator $x^2 - 9 = 0$ and get $x = 3, -3$.

b) Solve denominator: $x^2 + 8x + 16 = 0$ and get $x = -4$.

c) Vertical asymptote is $x = -4$.

d) and e)

4. Find the domain, range and inverse of $f(x) = (e^x + 3)^2$.

Answer:

Domain is everything.

The range is $(9, \infty)$ because $e^x > 0$ so $e^x + 3 > 3$ so $(e^x + 3)^2 > 3^2 = 9$. So the answers are > 9.

So the inverse's domain is $(9, \infty)$ and its range is everything.

Solving for the inverse:

$$\begin{aligned}
y &= (e^x + 3)^2 && (1)\\
\sqrt{y} &= e^x + 3 \text{ this is OK because } y > 9 > 0 && (2)\\
\sqrt{y} - 3 &= e^x && (3)\\
Ln(\sqrt{y} - 3) &= x && (4)\\
\text{done solving for x so switch:} & && (5)\\
Ln(\sqrt{x} - 3) &= y && (6)
\end{aligned}$$

So $f^{-1}(x) = Ln(\sqrt{x} - 3)$.

Check your answer by concatenating f and f^{-1}:

$$\begin{aligned}
f(f^{-1}(x)) &= f(Ln(\sqrt{x} - 3)) = (e^{(Ln(\sqrt{x}-3))} + 3)^2\\
&= ((\sqrt{x} - 3) + 3)^2 = (\sqrt{x})^2 = x \text{ because } x > 9.\\
f^{-1}(f(x)) = f^{-1}((e^x + 3)^2) &= Ln(\sqrt{(e^x + 3)^2} - 3)\\
&= Ln(e^x + 3 - 3) \text{ because } e^x + 3 > 0\\
&= Ln(e^x) = x.
\end{aligned}$$

5. a) Divide $f(x) = 3x^3 - 9x^2 + 4x - 12$ by $x - 3$ and then factor $f(x)$.

Answer: You will get $3x^2 + 0x + 4$ with no remainder. So $f(x) = (x-3)(3x^2+4)$ which does not factor further because $3x^2 + 4$ has no roots.

b) Divide $f(x) = 3x^3 - 9x^2 - 4x + 12$ by $x - 3$ and then factor $f(x)$.

Answer: You will get $3x^2 + 0x - 4$ with no remainder. So

$$f(x) = (x-3)(3x^2-4) = 3(x-3)(x^2-(4/3)) = 3(x-3)(x+\sqrt{4/3})(x-\sqrt{4/3}).$$

6. Find $sin(arccos(x/2))$ assuming x is in quadrant I.

Answer: We want to find $sin(\theta)$ where $\theta = arccos(x/2)$, which means $cos(\theta) = x/2$. So we draw a right triangle with an angle θ, hyp= 2 and adj= x:

We want $sin(\theta)$, which is opp/adj, so we need to find opp. But we can use Pythogorus' Formula, to solve for opp and get $\sqrt{4-x^2}$. So

$$sin(\theta) = \frac{\sqrt{4-x^2}}{2}$$

and since θ is in quadrant I we don't need to adjust the sign.

7. When an icecream man charges a dollar for an icecream cone he sells 1000 cones a day. For each additional 5 cents, he sells 200 less cones. Write the formula for a function that can be used to figure out his maximum profit. Do not solve the problem.

   Answer: This depends on your method for doing this problem. There is more than one solution. Feel free to email me your answer and I will check it.