Parallel Postulate Project

C. Sormani, MTTI, Lehman College, CUNY

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BACKGROUND: ***All axioms of Absolute Geometry and the Theorems of Congruent Triangles. In this lesson, all points and lines will lie in a common plane.***

DEFINITION:Two lines are *parallel* **if they lie in the same plane but do not intersect.** (draw this)

THEOREM: If two lines are cut by a transversal such that the alternate interior angles are congruent, then the lines are parallel. More precisely:

GIVEN: Two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$ are cut by transversal line, $\overleftrightarrow{BP}$, such that points A and Q are on opposite sides of $\overleftrightarrow{BP}$ and $m(\angle ABP) = m(\angle BPQ)$. (draw)

SHOW: The two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$ are parallel.

INDIRECT PROOF: (add new pictures for each step and justifications)

STATEMENTS JUSTIFICATIONS

1) Assume on the contrary that they are not parallel:
There exists $X \in \overleftrightarrow{AB} \cap \overleftrightarrow{PQ}$.

CASE I: X is on the same side of $\overleftrightarrow{BP}$ as A. (draw)

2) Let M be the midpoint of segment $\overline{PB}$. (draw)

3) Let Y be a point on $\overleftrightarrow{XP}$ with P between X and Y such that the lengths $PY = XB$. (draw)

4) $m(\angle MPY) = m(\angle MBX)$

5) Lengths $BM = PM$.

6) Triangles ΔMBX and ΔMPY are congruent. (draw)

7) $m(\angle XMB) = m(\angle YMP)$.

8) Let Z be a point on $\overleftrightarrow{XM}$ with M between X and Z.

9) $m(\angle XMB) = m(\angle PMZ)$

10) $m(\angle YMP) = m(\angle PMZ)$

11) The rays $\overrightarrow{MY} = \overrightarrow{MZ}$.

12) Lines $\overleftrightarrow{XM}$ and $\overleftrightarrow{XP}$ meet at Y.
CONTRADICTION: two unique lines meet only at one point.

CASE II: X is on the same side of $\overleftrightarrow{BP}$ as Q.
Do this case yourself. It needs its own contradiction. (Extra Credit)

QED

EUCLIDEAN PARALLEL POSTULATE: Given a line $\overleftrightarrow{AB}$ and a point $P \notin \overleftrightarrow{AB}$, then there is a unique line $\overleftrightarrow{PQ}$ parallel to $\overleftrightarrow{AB}$. (draw)

THEOREM: If two parallel lines are cut by a transversal, the alternate interior angles are congruent. More precisely:

GIVEN: Two parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{PQ}$ are cut by transversal line, $\overleftrightarrow{BP}$, such that points A and Q are on opposite sides of $\overleftrightarrow{BP}$. (draw)

SHOW: $m(\angle ABP) = m(\angle BPQ)$. (draw in red on given picture)

INDIRECT PROOF:

1) Assume on the contrary that $m(\angle ABP) \neq m(\angle BPQ)$.

STATEMENTS	JUSTIFICATIONS
2) Without loss of generality we may assume $m(\angle ABP) > m(\angle BPQ)$. (draw)	
3) There is a point Z in the interior of $\angle ABP$ such that $m(\angle ZBP) = m(\angle BPQ)$. (draw)	
4) $\overleftrightarrow{ZB}$ and $\overleftrightarrow{PQ}$ are parallel.	
5) $\overleftrightarrow{ZB} = \overleftrightarrow{AB}$	
6) $m(\angle ABP) = m(\angle BPQ)$.	
CONTRADICTION: Steps 6 and 2.	

QED

THEOREM: The sum of the measures of the interior angles of a triangle is $180°$.

PAPER AND SCISSORS: Make a small triangle and cut out three copies of the triangle. Label corresponding angles A, B and C on all three copies. Line up angles A, B and C at a common point and they appear to add up to $180°$ because it looks like a straight line. Tape the triangles below:

We will use this idea to prove the theorem. It is essential to prove that the three angles do indeed form a straight line. We will prove the theorem imitating this construction first and then give a quick proof which must be taught in a high school geometry class.

GIVEN: Triangle ΔABC.
SHOW: $m(\angle ABC) + m(\angle BCA) + m(\angle CAB) = 180°$.

PROOF IMITATING THREE TRIANGLE CONSTRUCTION:

Fill in the missing justifications and draw pictures for each step.

STATEMENTS	JUSTIFICATIONS
1) Let C' be the point such that $m(\angle C'AB) = m(\angle CBA)$ and length $C'A = CB$. (draw)	1) Protractor Postulate and Segment Construction Theorem.
2) $\Delta C'AB$ and ΔCBA are congruent. (draw)	
3) Let B' be the point such that $m(\angle CAB') = m(\angle ACB)$ and length $B'A = CB$. (draw)	3) Protractor Postulate and Segment Construction Theorem.
4) $\Delta CAB'$ and ΔACB are congruent. (draw)	
5) $\overleftrightarrow{AC'}$ is parallel to $\overleftrightarrow{BC}$.	
6) $\overleftrightarrow{AB'}$ is parallel to $\overleftrightarrow{BC}$.	
7) $\overleftrightarrow{AB'} = \overleftrightarrow{AC'}$.	
8) B' A and C' are colinear.	
9) $m(\angle C'AB) + m(\angle CAB') + m(\angle CAB) = 180°$. (draw)	
10) $m(\angle ABC) + m(\angle BCA) + m(\angle CAB) = 180°$. (draw)	

QED

The above is a long proof and you will notice we never really needed to show the triangles were congruent. The important steps involved the parallel lines. So why bother with triangles at all? See below for an elegant short proof perfect for high school:

ELEGANT PROOF:

Fill in the missing justifications and draw pictures for each step.

STATEMENTS	JUSTIFICATIONS
1) There is a unique line, L, through A parallel to line $\overleftrightarrow{BC}$. (draw)	
2) Let P and Q be points on L such that P and C are on opposite sides of $\overleftrightarrow{AB}$ and Q and B are on opposite sides of $\overleftrightarrow{AC}$. (draw)	2) Ruler and Half Plane Axioms
3) $m(\angle PAB) = m(\angle ABC)$ (draw)	
4) $m(\angle QAC) = m(\angle BCA)$ (draw)	
5) $m(\angle PAB) + m(\angle QAC) + m(\angle CAB) = 180°$. (draw)	
6) $m(\angle ABC) + m(\angle BCA) + m(\angle CAB) = 180°$. (draw)	

QED

Draw a picture and write the given and show for each of the following definitions and theorems. Then prove as many as you can. Prove them using only the axioms, the congruent triangles theorems, the linear pair theorem, and the theorems proven today. Prove each theorem independantly, as if you were doing it in a separate lesson.

DEFINITION: A Parallelogram is a quadrilateral whose opposite sides are parallel. We label the vertices counterclockwise. (draw)

THEOREM 1: If $ABCD$ is a parallelogram and $\overline{AC}$ is a diagonal, then triangles ΔABC and ΔCDA are congruent.

THEOREM 2: If $ABCD$ is a parallelogram then the opposite sides have the same length: $AB = CD$ and $BC = AD$.

THEOREM 3: If $ABCD$ is a quadrilateral and the opposite sides have the same length, $AB = CD$ and $BC = AD$, then $ABCD$ is a parallelogram.

THEOREM 4: If $ABCD$ is a parallelogram and the diagonals meet at a point X, then X is the midpoint of the diagonals.

DEFINITION: A rhombus is a parallelogram with four equal sides.

THEOREM 5: If $ABCD$ is a parallelogram and the diagonals meet perpendicularly, then $ABCD$ is a rhombus.

THEOREM 6: If $ABCD$ is a parallelogram, the diagonals meet perpendicularly and the diagonals have the same length, then $ABCD$ is a square.