Isometry Project

C. Sormani, MTTI, Lehman College, CUNY

MAT631, Fall 2009, Project IX, Part II

BACKGROUND: Euclidean Axioms, Midpoints, Perpendicular Bisectors, Congruent Triangles, Symmetry, Maps, Bijections.

DEFN: A Euclidean **isometry** is a map, $F : E^2 \to E^2$, which takes every point in the Euclidean plane to another point in the Euclidean plane such that distances are preserved:

$$d(F(P), F(Q)) = d(P, Q).$$

DEFN: Given a line L, there is a map $F_L : E^2 \to E^2$ called reflection across the line L. The map is defined by the following rule: given a point P, drop the unique perpendicular line from P to L, then find a point P' on that line such that L is the perpendicular bisector (so P' and P are symmetric across the line L). Define $F_L(P) = P'$.

LINE REFLECTIONS ARE ISOMETRIES THEOREM: F_L is an isometry.

PROOF: *Prove this using results from the symmetry project. You are given all the facts from the defn of F_L and you are proving the equation in the defn of isometry.*

DEFN: Given a point X, there is a map $F_X : E^2 \to E^2$ called reflection through the point X. The map is defined by the following rule: given a point P, draw line $\overleftrightarrow{PQ}$, then find a point P' on that line such that X is the midpoint of PP' (so P and P' are symmetric through the point X). Define $F_X(P) = P'$.

POINT REFLECTIONS ARE ISOMETRIES THEOREM: F_X is an isometry.

PROOF: *Prove this using results from the symmetry project.*

THEOREM: If $F : E^2 \to E^2$ and $H : E^2 \to E^2$ are both isometries, then $F \circ H$ is also an isometry. $F \circ H(x) = F(H(x))$.

PROOF: $d(F(H(P)), F(H(Q))) = d(H(P), H(Q)) = d(P, Q)$. *QED*

EXPLORATION 1: Suppose we have two perpendicular lines L_1 and L_2 which meet at a point X, compare the isometry $F_{L_1} \circ F_{L_2}$ to the isometry F_X. Are they the same?

EXPLORATION 2: Suppose we have two points X_1 and X_2, what can you observe about the isometry $F_{X_1} \circ F_{X_2}$? How far are points moved by this isometry?

EXPLORATION 3: Suppose we have two parallel lines L_1 and L_2, what can you observe about the isometry $F_{L_1} \circ F_{L_2}$? How far are points moved by this isometry?

EXTRA CREDIT: Prove your observations in the three explorations.

PROPERTIES OF ISOMETRIES

THEOREM: An isometry is one-to-one (injective). That is, if $F : E^2 \to E^2$ is an isometry and $X \neq Y$, then $F(X) \neq F(Y)$.

PROOF:

1) $d(X, Y) > 0$	1) Distance Axiom D-2
2) $d(X, Y) = d(F(X), F(Y))$	2) Defn of Isometry
3) $d(F(X), F(Y)) > 0$	3) Substituting (2) into (1)
4) $F(X) \neq F(Y)$	4) Distance Axiom D-2 QED

THEOREM: Suppose F is an isometry and $X' = F(X)$, $Y' = F(Y)$ and $Z' = F(Z)$. If X, Y, Z are colinear, then X', Y', Z' are colinear. If Y is between X and Z, then Y' is between X' and Z'.

PROOF: *Recall that three distinct colinear points always have one between the other two.*

1) $d(X, Y) + d(Y, Z) = d(X, Z)$	1) Defn of Betweeness
2) $d(X, Y) = d(X', Y'),\ d(Y, Z) = d(Y', Z'),\ d(X, Z) = d(X', Z')$	2) Defn of Isometry
3) $d(X', Y') + d(Y', Z') = d(X', Z')$	3) Substituting (2) into (1)
4) Y' is between X' and Z'	4) Defn of Betweeness. QED

THEOREM: If $F : E^2 \to E^2$ is an isometry and $X' = F(X)$, $Y' = F(Y)$ and $Z' = F(Z)$, then ΔXYZ and $\Delta X'Y'Z'$ are congruent.

PROOF:

1)

2)

3)

4)

THEOREM: If $F : E^2 \to E^2$ is an isometry and $X' = F(X)$, $Y' = F(Y)$ and $Z' = F(Z)$, then $m(\angle XYZ) = m(\angle X'Y'Z')$.

PROOF:

1)

2)

THEOREM: An isometry is onto (surjective). That is, if $F : E^2 \to E^2$ is an isometry and Q is any point, then there is a point P such that $F(P) = Q$.

PROOF IDEA: *We are given F and Q and we need to find P. So let X and Y are points with images $F(X) = X'$ and $F(Y) = Y'$. Then ΔXYP should be congruent to $\Delta X'Y'Q$. This can help us find P. In fact there are only two possible points for P. Add a third point Z and its image $Z' = F(Z)$ and you should be able to find P precisely:*

Exercise: suppose that $X = (0, 0)$, $Y = (0, 1)$ and $Z = (1, 0)$ and I tell you that $F(X) = (0, 5)$, $F(Y) = (0, 6)$ and $F(Z) = (1, 5)$. Now take $Q = (3, 4)$. What is P? Explain clearly why.

Now complete the proof that an isometry is surjective.