Towards Renormalize Limit Measure

Defn 7.4.10: X & Y compact metric spaces

$\varepsilon, \delta > 0$ X & Y are $\underline{\varepsilon\delta \text{ approx}}$ of each other if $\exists$ finite εnets $\{x_i : i = 1..N\} \subseteq X$ and $\{y_i : i = 1..N\} \subset Y$ such that $|d_X(x_i, x_j) - d_Y(y_i, y_j)| < \delta \quad \forall i,j \in 1..N$.

One can say $X_j \to X$ in the GH sense if $\forall \varepsilon > 0$ $\exists M_\varepsilon \in \mathbb{N}$ such that X_j is an ε-ε approximation of X for all $j \geq M_\varepsilon$.

We will learn many equivalent definitions later.

Defn 7.4.13: A class $\mathcal{X}$ of compact metric spaces is <u>uniformly</u> <u>totally</u> <u>bounded</u> if

(1) $\exists D > 0$ s.t. $\operatorname{diam} X \leq D$ $\forall X \in \mathcal{X}$

(2) $\forall \varepsilon > 0$ $\exists N = N(\varepsilon) \in \mathbb{N}$ such that every $X \in \mathcal{X}$ has an ε net with a most N points.

Exer 7.4.14: Prove (2) $\Rightarrow$ (1) if all $X \in \mathcal{X}$ are length spaces.

Theorem 7.4.15 GROMOV'S COMPACTNESS THEOREM

Any uniformly totally bounded class $\mathcal{X}$ of compact metric spaces is precompact in the Gromov Hausdorff topology.

(2)

Exer^A: Prove that flat tori $S^1_{1/j} \times S^1$ where S^1_ε has diameter ε, converge in the GH sense to S^1. Explicitly construct the ε nets and $N(\varepsilon)$

Exer^B: Prove the picture in Figure 7.4 gives a sequence of M_j converging to S^2. Assume diameter of handle in M_j is less than $1/j$

One of the most important theorems in Riemannian Geometry is the Bishop-Gromov Volume Comparison Thm.

Thm: If Ricci $\geq (n-1)H$ and M^n is a complete Riemannian Manifold then

$$\star \quad \frac{Vol(B_p(r))}{Vol(B_p(R))} \geq \frac{V_H(r)}{V_H(R)} \qquad 0<r<R$$

where $V_H(\rho) = $ Volume of $B_p(\rho)$ in M^n_H

where M^n_H is the n dimensional space form of constant curvature H

~~[illegible]~~ $M^n_1 = S^n \quad M^n_0 = \mathbb{E}^n \quad M^n_{-1} = \mathbb{H}^n$

Exer^C: Prove that if $p, q \in M^n$ $\operatorname{diam}(M^n) \le D$ and $\text{Ricci} \ge 0$ then

$$\operatorname{Vol}(B_p(r)) \ge \frac{r^n}{D^n} \operatorname{Vol}(B_q(D)) \ge \frac{r^n}{D^n} \operatorname{Vol}(M^n).$$

Then find the maximum number of disjoint balls of radius r in M^n.

Then prove $\mathcal{H} = \left\{ M^n : \begin{array}{c} M^n \text{ compact Riemannian} \\ \text{manifold with } \operatorname{diam}(M^n) \le D \\ \text{and } \text{Ricci} \ge 0 \end{array} \right\}$ is precompact!

Exer^D: Let $M_j^2 = \left\{ (x,y,z) : \frac{x^2}{(\frac{1}{j})^2} + y^2 + z^2 = 1 \right\}$ be ellipsoids. They have positive Ricci curvature. Describe the limit space. Be careful it is not just $\{ (0,y,z) : y^2+z^2 \le 1 \}$

One way to describe it would be to define a metric $d : S^2 \times S^2 \to \mathbb{R}$ on $S^2 = \{ (x,y,z) : x^2+y^2+z^2 = 1 \}$

$$= \left\{ (\varphi, \theta) : \begin{array}{l} \varphi \in [0,\pi] \\ \theta \in [0, 2\pi] \end{array} \right\} \Bigg| \begin{array}{l} (0,\theta_1) \sim (0,\theta_2) \\ (\pi,\theta_1) \sim (\pi,\theta_2) \\ (\varphi,0) \sim (\varphi,2\pi). \end{array}$$

Exer^E: Show tori in Exer^A have ✵
This is a collapsing sequence

Since volume is so important in applying Gromov's Compactness Theorem, we often study metric spaces with measures. Sometimes the measure is the Hausdorff measure, μ_n, but other times it is any measure, μ, which is positive and finite on balls.

<u>Defn</u>: μ is a "doubling measure" if there is a constant $C>0$ such that $\mu(B_p(2r)) \le C\mu(B_p(r))$ $\forall r>0$.

<u>Exer^F</u>: Prove volume on a Complete manifold with Ricci ≥ 0 is a doubling measure

<u>ExerG^*</u>: If $\mathcal{X}_c$ is a collection of metric measure spaces (X, d, μ) with doubling measure of constant C, then $\mathcal{X}_c$ is precompact in GH sense.

The difficulty is that even though $(X_i, d_i, \mu_i) \in \mathcal{X}_c$ and (X_i, d_i) converges to some metric space (X_∞, d_∞) the μ_i could have no limit

<u>Exer H</u>: In the collapsing tori, (Exer A) show $\forall p_j \in S^1_{1/j} \times S^1$ $\forall r > 0$ $\mu_j(S^1_{1/j} \times S^1) =$ usual area has a limit 0!

<u>Cheeger Colding Renormalized Limit Measures</u>:

Suppose μ_i obey doubling.

Let $\bar{\mu}_i(S) = \dfrac{\mu_i(S)}{\mu_i(B_{p_i}(1))}$ for some sequence $p_i \in M_i$

<u>Exer I</u>: Show $\bar{\mu}_i(B_{p_i}(2^k)) \in [1, C^k]$

Show $\bar{\mu}_i(B_q(r)) \in [a, b]$ where a and b depend only on $r, C, \mathrm{diam}(M_i)$.

So we can chose subsequences and get $\bar{\mu}_{i_j}(B_{q_i}(r)) \longrightarrow \bar{\mu}_\infty(B_{q_\infty}(r))$

(Note $q_i \in X_i \to q_\infty \in X_\infty$ means q_i is near a net point corresponding to a net point near q_∞ in X_∞)

Setting up a countable dense collection of points in a limit space X_∞ of $X_i \in \mathcal{X}_c$, and a countable dense collection of radii $r \in [0, D]$, where $D \geq \mathrm{diam}(X_i)$, Cheeger and Colding choose a subsequence such that $(X_i, d_i, \bar{\mu}_i)$ converge in the metric measure set to $(X_\infty, d_\infty, \bar{\mu}_\infty)$ where $(X_i, d_i) \xrightarrow{GH} (X_\infty, d_\infty)$

and $\bar{\mu}_\infty(B_p(r)) = \lim_{i \to \infty} \bar{\mu}_i(B_{p_i}(r_i))$

and $\bar{\mu}_\infty$ is a doubling measure

Now $\bar{\mu}_\infty$ is called the Renormalized Limit Measure.