
PERPENDICULAR PERPENDICULAR TO PARALLEL THEOREM: If lines $L_1$ and $L_2$ are perp and lines $L_3$ and $L_2$ are perp, then $L_1$ and $L_3$ are parallel.
PROOF: Give a 2 sentence explanation and sketch on the left.

PERPENDICULAR PARALLEL TO PERPENDICULAR THEOREM: If lines $L_1$ and $L_2$ are perp and lines $L_3$ and $L_2$ are parallel, then $L_1$ and $L_3$ are perp.
PROOF: Give a 2 sentence explanation and sketch on the left.

DEFN: A rectangle is a quadrilateral with four right angles: $ABCD$ such that $m(\angle ABC) = m(\angle BCD) = m(\angle CDA) = m(\angle DAB) = 90^\circ$.

RECTANGLE THEOREM: Opposite sides of a rectangle have the same length.
PROOF: Prove this by adding in a diagonal and setting up a pair of congruent triangles. Write a two column proof here with a sketch on the left:
BUILDING A COORDINATE PLANE: Start with a Euclidean Plane and draw a line. Call this line the $x$ axis. Choose a point on the line to be the origin. Draw the unique line perpendicular to the $x$ axis through the origin. Call this line the $y$ axis. Assign real numbers to every point on the $x$ axis and the $y$ axis using the ruler axiom with 0 at the origin for each axiom. Generally one draws the $x$ axis horizontally and the $y$ axis vertically on a sheet of paper with the origin at the center and lays down the rulers so that the positive numbers are on the right end of the $x$ axis and above the origin on the $y$ axis.

VERTICAL LINES: For each real number $a$ there is a unique line perpendicular to the $x$ axis passing through the $x$ axis at the point assigned the value $a$. This line is called $x = a$. What we are saying here is that if I tell you to graph $x = a$, then you can graph it.

The converse is also true: any line perpendicular to the $x$ axis passes through the $x$ axis at some point assigned some value $a$, so it is a vertical line $x = a$. What we are saying here is that if I draw a line perpendicular to the $x$ axis, you can find the real value $a$ and tell me this line is the line $x = a$.

Note that vertical lines are parallel to the $y$ axis. Why?

Given any line parallel to the $y$ axis, it is perpendicular to the $x$ axis, so it is a vertical line and can be described as $x = a$ for some real value $a$. Why?

HORIZONTAL LINES: For each real number $b$ there is a unique line perpendicular to the $y$ axis passing through the $y$ axis at the point assigned the value $b$. This line is called $y = b$. What we are saying here is...

The converse is also true: any line perpendicular to the $y$ axis passes through the $y$ axis at some point assigned some value $b$, so it is a horizontal line $y = b$. What we are saying here is...

Note that horizontal lines are parallel to the $x$ axis. Why?

Given any line parallel to the $x$ axis, it is perpendicular to the $y$ axis, so it is a horizontal line and can be described as $y = b$ for some real value $b$. Why?
COORDINATES GIVE POINTS: Given a pair of real numbers \((a, b)\) there is a unique point \(P\) such that \(P\) is the intersection point of the lines \(x = a\) and \(y = b\).

PROOF: Fill in justifications.

1) Lines \(x = a\) and \(y = b\) are perpendicular.
2) They meet at a unique point.

POINTS HAVE COORDINATES: Given any point \(P\), there is a unique pair of real numbers \((a, b)\) such that \(P\) is the intersection point of the lines \(x = a\) and \(y = b\).

PROOF: Fill in justifications.

1) Drop the unique perpendicular line from \(P\) to the \(x\) axis.
2) This line is vertical and can be identified as \(x = a\).
3) Drop the unique perpendicular line from \(P\) to the \(y\) axis.
4) This line is horizontal and can be identified as \(y = b\).
5) The lines \(x = a\) and \(y = b\) meet at \(P\).

THE ORIGIN: The origin has coordinates \((0, 0)\). The point on the \(x\) axis corresponding to \(a\) is \((a, 0)\). The point on the \(y\) axis corresponding to \(b\) is \((0, b)\). Why? Think this over, but don’t write it down.

COORD RECTANGLE THM: Points \((a, b)\), \((c, d)\), \((a, d)\) and \((c, b)\) form a rectangle.

PROOF: Fill in justifications:

1) The line segment from \((a, b)\) to \((a, d)\) lies in line \(x = a\).
2) The line segment from \((a, d)\) to \((c, d)\) lies in line \(y = d\).
3) \(x = a\) and \(y = d\) are perpendicular.
4) \(m(\angle(a,b)(a,d)(c,d)) = 90^\circ\).

Now show \(m(\angle(a,d)(c,d)(c,b)) = 90^\circ:\)

5) 
6) 
7) 
8)

The rest of the proof follows in the same style and you need not finish it.

DISTANCE BETWEEN POINTS THEOREM: The distance between points is

\[
d((a, b), (c, d)) = \sqrt{(c - a)^2 + (d - b)^2}
\]

PROOF: First apply the ruler axiom to find the distance between \((a, 0)\) and \((c, 0)\), then the rectangle theorem to find \(d((a, b), (c, b))\). Then do the same for \((c, b)\) and \((c, d)\). Apply the Pythagorean Theorem justifying the right angle. Write a two column proof on the back of the previous page.
HALFPLANES AND QUADRANTS

UPPER HALF PLANE DEFN: The upper half plane is the half plane of all points on the same side of the $x$ axis ($y = 0$) as the point $(0, 1): H(y = 0, (0, 1))$.

UPPER HALF PLANE THEOREM: $(x_0, y_0)$ is in the upper half plane iff $y_0 > 0$.
PROOF: Extra Credit. Requires betweenness axioms and two directions.

LOWER HALF PLANE DEFN AND THEOREM: The lower half plane is $H(y = 0, (0, -1))$ which is equal to the set $\{(x_0, y_0) : y_0 < 0\}$.

RIGHT HALF PLANE DEFN AND THEOREM: The right half plane is $H(x = 0, (1, 0))$ which is equal to the set $\{(x_0, y_0) : x_0 > 0\}$.

LEFT HALF PLANE DEFN AND THEOREM: The left half plane is $H(x = 0, (-1, 0))$ which is equal to the set $\{(x_0, y_0) : x_0 < 0\}$.

QUADRANT I DEFN:
Upper-right Quadrant or Quadrant I is $H(x = 0, (1, 0)) \cap H(y = 0, (0, 1))$

QUADRANT I THM:

$$H(x = 0, (1, 0)) \cap H(y = 0, (0, 1)) = \{(x_0, y_0) : x_0 > 0, y_0 > 0\}.$$ 

PROOF: 
PART I: Given $(x_0, y_0) \in H(x = 0, (1, 0)) \cap H(y = 0, (0, 1))$. Show: $x_0 > 0$ and $y_0 > 0$.
1) 
2) 
3) 
4) 
PART II: Given: $x_0 > 0$ and $y_0 > 0$. Show: $(x_0, y_0) \in H(x = 0, (1, 0)) \cap H(y = 0, (0, 1))$.
1) 
2) 
3) 
4) 

QUADRANT II DEFN AND THM: Quadrant II or the Upper Left Quadrant is $H(x = 0, (-1, 0)) \cap H(y = 0, (0, 1)) = \{(x_0, y_0) : x_0 > 0, y_0 < 0\}$.

QUADRANT III DEFN AND THM: Quadrant III or the Lower Left Quadrant is $H(x = 0, (-1, 0)) \cap H(y = 0, (0, -1)) = \{(x_0, y_0) : x_0 < 0, y_0 < 0\}$.

QUADRANT IV DEFN AND THM: Quadrant IV or the Lower Right Quadrant is

QUADRANTS AND ANGLES: If $(x_0, y_0)$ and $(x_1, y_1)$ are in the same quadrant then $m(\angle(x_0, y_0)(0, 0)(x_1, y_1)) < 90^\circ$.
PROOF: If they are in quadrant I then they are in the interior of $\angle(0, 1)(0, 0)(1, 0)$, so $m(\angle(x_0, y_0)(0, 0)(x_1, y_1)) < m(\angle(0, 1)(0, 0)(1, 0)) = 90^\circ$. The other quadrants have the same idea. Draw four small pictures here:
LINES IN THE COORDINATE PLANE

VERTICAL LINES: Suppose \( L \) is the vertical line \( x = a \) then
\[
(x_0, y_0) \text{ is on the line } L \iff x_0 = a.
\]

PROOF: PART I: Given: \( L \) is the vertical line \( x = a \) and \((x_0, y_0) \in L \). Show: \( x_0 = a \).
1) The point \((x_0, y_0)\) is on the vertical line \( x = x_0 \) by Coordinates give Points Thm.
2) \( L \) is the unique vertical line through \((x_0, y_0)\) by Unique Perpendicular Lines theorem.
3) \( x_0 = a \) by steps (1) and (2). QED

PROOF: PART II: Given: \( L \) is the vertical line \( x = a \) and \( x_0 = a \). Show: \((x_0, y_0) \in L \).
1) \((x_0, y_0)\) is on the line \( x = x_0 \) by Coordinates give Points Theorem.
2) \( L \) is the line \( x = x_0 \) by given and \( a = x_0 \)
3) \((x_0, y_0)\) in \( L \) by steps (1) and (2). QED

HORIZONTAL LINES: Suppose \( L \) is the horizontal line \( y = b \) then
\[
(x_0, y_0) \text{ is on the line } L \iff y_0 = b.
\]

PROOF: Prove this on the back of the last pages imitating the above proof.

LINES THROUGH THE ORIGIN: Suppose \( L \) is a line which passes through the origin and a point \((a, b)\) with \( a, b > 0 \), then \((x_0, y_0)\) is on the line \( L \) iff \( y_0 = mx_0 \) where \( m = b/a \).

PROOF: PART I: Given: \( L \) through \((0, 0)\) and \((a, b)\) and \((x_0, y_0) \in L \). Show: \( y_0/b = x_0/a \).

Six cases: I. \((x_0, y_0)\) in Quad I. II. \((x_0, y_0)\) in Quad II. III. \((x_0, y_0)\) in Quad III.
IV. \((x_0, y_0)\) in Quad IV. V. \((x_0, y_0)\) on \( x \) axis. VI. \((x_0, y_0)\) on \( y \) axis.

CASE I: \((x_0, y_0)\) in Quad I. Fill in justifications and draw.
0) \( x_0 > 0 \) and \( y_0 > 0 \)
1) \( m(\angle(x_0, y_0)(0, 0)) = m(\angle(a, b)(0, 0)) \)
2) \( \angle(x_0, y_0)(0, 0) = \angle(a, b)(0, 0) \)
3) \( m(\angle(x_0, y_0)(0, 0)) = m(\angle(a, b)(0, 0)) \)
4) \( \Delta(x_0, y_0)(0, 0) \) similar to \( \Delta(a, b)(0, 0)(a, 0) \)
5) \[
\frac{\frac{y_0}{x_0}}{\frac{y}{x}} = \frac{b}{a}.
\]
6) \( y_0 = (b/a)x_0 \).

CASE II: \((x_0, y_0)\) in Quad II.
1) \((x_0, y_0)\) and \((a, b)\) are in the upper half plane because \( y_0 > 0 \) and \( b > 0 \).
2) The line segment between them is in the upper half plane by defn half plane.
3) \((x_0, y_0)\) is in the left half plane because \( x_0 < 0 \).
4) \((a, b)\) is in the right half plane because \( b > 0 \).
5) The line segment between them crosses the \( y \) axis because they are on opposite half planes of \( x = 0 \).
6) It crosses at a point \((0, c)\) with \( c > 0 \) by steps 2 and 5.
7) This contradicts that it should cross at \((0, 0)\) by given, so Case II does not happen.

CASE III-VI: Do these on the back of the previous page. IV-VI do not happen.
PROOF: PART II: Given: \( L \) a line through \((0, 0)\) and \((a, b)\) with \(a > 0, b > 0\), and \(y_0/b = x_0/a\). Show: \((x_0, y_0) \in L\).

Six cases: I. \((x_0, y_0)\) in Quad I. II. \((x_0, y_0)\) in Quad II. III. \((x_0, y_0)\) in Quad III. IV. \((x_0, y_0)\) in Quad IV. V. \((x_0, y_0)\) on x axis. VI. \((x_0, y_0)\) on y axis.

On the back of the prior page:
Prove Case I by showing \(\Delta(x_0, y_0)(0, 0)(x_0, 0)\) is similar to \(\Delta(a, b)(0, 0)(a, 0)\) and then proving the ray from \((0, 0)\) to \((x_0, y_0)\) is the ray from \((0, 0)\) to \((a, b)\) and thus \((x_0, y_0) \in L\).
Extra Credit: complete the rest of the cases.

FORMULA FOR LINES THROUGH THE ORIGIN: The above theorem can also be proven for a line, \( L \) through \((0, 0)\) and any other point \((a, b)\) not on the x or y axis. We say such lines have the formula \(y = mx\) because a point \((x_0, y_0)\) is on the line iff \(y_0 = mx_0\).

PERPENDICULAR LINES THROUGH THE ORIGIN: Two lines through the origin with formulas \(y = m_1x\) and \(y = m_2x\) where \(m_i \neq 0\) are perpendicular iff \(m_1m_2 = -1\).

PROOF: PART I: Given \(y = m_1x\) and \(y = m_2x\) are perpendicular. Show: \(m_1m_2 = -1\).

Without loss of generality assume \(m_1 < m_2\).
1) \((0, 0)\) and \((1, m_1)\) are on the line \(y = m_1x\) because they satisfy the formula.
2) \((0, 0)\) and \((1, m_2)\) are on the line \(y = m_2x\) because they satisfy the formula.
3) \(m(\angle(1, m_1)(0, 0)(1, m_2)) = 90^\circ\) by the definition of perpendicular lines.
4) \(m_2 > 0 > m_1\), otherwise \((1, m_1)\) and \((1, m_2)\) would be in the same quadrant and the Quadrant Angle Theorem would contradict step 3.

Complete the proof by showing the triangles \(\Delta(0, m_1)(0, 0)(1, m_1)\) and \(\Delta(0, m_2)(0, 0)(1, m_2)\) are similar and then finding proportional sides:

PROOF: PART II: Prove on the back of this page including given/show statements.
FORMULAS FOR OTHER LINES: To find the formulas that describe other lines we first prove a certain map from the plane to the plane is an isometry. So we will do this in the future.