Coordinate Plane Project

C. Sormani, MTTI, Lehman College, CUNY

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BACKGROUND: Euclidean Axioms, Half Planes, Unique Perpendicular Lines, Congruent and Similar Triangle Theorems, Parallelograms.

PERPENDICULAR PERPENDICULAR TO PARALLEL THEOREM: If lines L_1 and L_2 are perp and lines L_3 and L_2 are perp, then L_1 and L_3 are parallel.
PROOF: *Give a 2 sentence explanation and sketch on the left.*

PERPENDICULAR PARALLEL TO PERPENDICULAR THEOREM: If lines L_1 and L_2 are perp and lines L_3 and L_2 are parallel, then L_1 and L_3 are perp.
PROOF: *Give a 2 sentence explanation and sketch on the left.*

DEFN: A rectangle is a quadrilateral with four right angles: $ABCD$ such that

$$m(\angle ABC) = m(\angle BCD) = m(\angle CDA) = m(\angle DAB) = 90^\circ.$$

RECTANGLE THEOREM: Opposite sides of a rectangle have the same length.

PROOF: *Prove this by adding in a diagonal and setting up a pair of congruent triangles. Write a two column proof here with a sketch on the left:*

BUILDING A COORDINATE PLANE: Start with a Euclidean Plane and draw a line. Call this line the x axis. Choose a point on the line to be the *origin*. Draw the unique line perpendicular to the x axis through the origin. Call this line the y axis. Assign real numbers to every point on the x axis and the y axis using the ruler axiom with 0 at the origin for each axiom. *Generally one draws the x axis horizontally and the y axis vertically on a sheet of paper with the origin at the center and lays down the rulers so that the positive numbers are on the right end of the x axis and above the origin on the y axis.*

VERTICAL LINES: For each real number a there is a unique line perpendicular to the x axis passing through the x axis at the point assigned the value a. This line is called $x = a$. *What we are saying here is that if I tell you to graph $x = a$, then you can graph it.*

The converse is also true: any line perpendicular to the x axis passes through the x axis at some point assigned some value a, so it is a vertical line $x = a$. *What we are saying here is that if I draw a line perpendicular to the x axis, you can find the real value a and tell me this line is the line $x = a$.*

Note that **vertical lines are parallel** to the y axis. *Why?*

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Given any line parallel to the y axis, it is perpendicular to the x axis, so it is a vertical line and can be described as $x = a$ for some real value a. *Why?*

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HORIZONTAL LINES: For each real number b there is a unique line perpendicular to the y axis passing through the y axis at the point assigned the value b. This line is called $y = b$. *What we are saying here is...*

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The converse is also true: any line perpendicular to the y axis passes through the y axis at some point assigned some value b, so it is a horizontal line $y = b$. *What we are saying here is...*

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Note that **horizontal lines are parallel** to the x axis. *Why?*

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Given any line parallel to the x axis, it is perpendicular to the y axis, so it is a horizontal line and can be described as $y = b$ for some real value b. *Why?*

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COORDINATES GIVE POINTS: Given a pair of real numbers (a, b) there is a unique point P such that P is the intersection point of the lines $x = a$ and $y = b$.

PROOF: *Fill in justifications.*

1) Lines $x = a$ and $y = b$ are perpendicular.

2) They meet at a unique point.

POINTS HAVE COORDINATES: Given any point P, there is a unique pair of real numbers (a, b) such that P is the intersection point of the lines $x = a$ and $y = b$.

PROOF: *Fill in justifications.*

1) Drop the unique perpendicular line from P to the x axis.

2) This line is vertical and can be identified as $x = a$.

3) Drop the unique perpendicular line from P to the y axis.

4) This line is horizontal and can be identified as $y = b$.

5) The lines $x = a$ and $y = b$ meet at P.

THE ORIGIN: The origin has coordinates $(0, 0)$. The point on the x axis corresponding to a is $(a, 0)$. The point on the y axis corresponding to b is $(0, b)$. *Why? Think this over, but don't write it down.*

COORD RECTANGLE THM: Points (a, b), (c, d), (a, d) and (c, b) form a rectangle.

PROOF: *Fill in justifications:*

1) The line segment from (a, b) to (a, d) lies in line $x = a$.

2) The line segment from (a, d) to (c, d) lies in line $y = d$.

3) $x = a$ and $y = d$ are perpendicular.

4) $m(\angle(a, b)(a, d)(c, d)) = 90°$.

Now show $m(\angle(a, d)(c, d)(c, b)) = 90°$:

5)

6)

7)

8)

The rest of the proof follows in the same style and you need not finish it.

DISTANCE BETWEEN POINTS THEOREM: The distance between points is

$$d((a, b), (c, d)) = \sqrt{(c - a)^2 + (d - b)^2}$$

PROOF: *First apply the ruler axiom to find the distance between* $(a, 0)$ *and* $(c, 0)$*, then the rectangle theorem to find* $d((a, b), (c, b))$*. Then do the same for* (c, b) *and* (c, d)*. Apply the Pythagorean Theorem justifying the right angle. Write a two column proof on the back of the previous page.*

HALFPLANES AND QUADRANTS

UPPER HALF PLANE DEFN: The upper half plane is the half plane of all points on the same side of the x axis $(y = 0)$ as the point $(0, 1)$: $H(y = 0, (0, 1))$.
UPPER HALF PLANE THEOREM: (x_0, y_0) is in the upper half plane iff $y_0 > 0$.
PROOF: *Extra Credit. Requires betweenness axioms and two directions.*
LOWER HALF PLANE DEFN AND THEOREM: The lower half plane is $H(y = 0, (0, -1))$ which is equal to the set $\{(x_0, y_0) : y_0 < 0\}$.
RIGHT HALF PLANE DEFN AND THEOREM: The right half plane is $H(x = 0, (1, 0))$ which is equal to the set $\{(x_0, y_0) : x_0 > 0\}$.
LEFT HALF PLANE DEFN AND THEOREM: The left half plane is

QUADRANT I DEFN:
Upper-right Quadrant or Quadrant I is $H(x = 0, (1, 0)) \cap H(y = 0, (0, 1))$

QUADRANT I THM:

$$H(x = 0, (1, 0)) \cap H(y = 0, (0, 1)) = \{(x_0, y_0) :\ x_0 > 0,\ y_0 > 0\}.$$

PROOF:
PART I: Given $(x_0, y_0) \in H(x = 0, (1, 0)) \cap H(y = 0, (0, 1))$. Show: $x_0 > 0$ and $y_0 > 0$.
1)
2)
3)
4)
PART II: Given: $x_0 > 0$ and $y_0 > 0$. Show: $(x_0, y_0) \in H(x = 0, (1, 0)) \cap H(y = 0, (0, 1))$.
1)
2)
3)
4)

QUADRANT II DEFN AND THM: Quadrant II or the Upper Left Quadrant is

$$H(x = 0, (-1, 0)) \cap H(y = 0, (0, 1)) = \{(x_0, y_0) :\ x_0 > 0,\ y_0 < 0\}.$$

QUADRANT III DEFN AND THM: Quadrant III or the Lower Left Quadrant is

$$H(x = 0, (-1, 0)) \cap H(y = 0, (0, -1)) = \{(x_0, y_0) :\ x_0 < 0,\ y_0 < 0\}.$$

QUADRANT IV DEFN AND THM: Quadrant IV or the Lower Right Quadrant is

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QUADRANTS AND ANGLES: If (x_0, y_0) and (x_1, y_1) are in the same quadrant then $m(\angle(x_0, y_0)(0, 0)(x_1, y_1)) < 90°$.
PROOF: If they are in quadrant I then they are in the interior of $\angle(0, 1)(0, 0)(1, 0)$, so $m(\angle(x_0, y_0)(0, 0)(x_1, y_1)) < m(\angle(0, 1)(0, 0)(1, 0)) = 90°$. The other quadrants have the same idea. *Draw four small pictures here:*

LINES IN THE COORDINATE PLANE

VERTICAL LINES: Suppose L is the vertical line $x = a$ then

(x_0, y_0) is on the line L iff $x_0 = a$.

PROOF: PART I: Given: L is the vertical line $x = a$ and $(x_0, y_0) \in L$. Show: $x_0 = a$.
1) The point (x_0, y_0) is on the vertical line $x = x_0$ by Coordinates give Points Thm.
2) L is the unique vertical line through (x_0, y_0) by Unique Perpendicular Lines theorem.
3) $x_0 = a$ by steps (1) and (2). QED

PROOF: PART II: Given: L is the vertical line $x = a$ and $x_0 = a$. Show: $(x_0, y_0) \in L$.
1) (x_0, y_0) is on the line $x = x_0$ by Coordinates give Points Theorem.
2) L is the line $x = x_0$ by given and $a = x_0$
3) (x_0, y_0) in on L by steps (1) and (2). QED

HORIZONTAL LINES: Suppose L is the horizontal line $y = b$ then

(x_0, y_0) is on the line L iff $y_0 = b$.

PROOF: *Prove this on the back of the last pages imitating the above proof.*

LINES THROUGH THE ORIGIN: Suppose L is a line which passes through the origin and a point (a, b) with $a, b > 0$, then (x_0, y_0) is on the line L iff $y_0 = mx_0$ where $m = b/a$.

PROOF: PART I: Given: L through $(0, 0)$ and (a, b) and $(x_0, y_0) \in L$. Show: $y_0/b = x_0/a$.
Six cases: I. (x_0, y_0) in Quad I. II. (x_0, y_0) in Quad II. III. (x_0, y_0) in Quad III.
IV. (x_0, y_0) in Quad IV. V. (x_0, y_0) on x axis. VI. (x_0, y_0) on y axis.

CASE I: (x_0, y_0) in Quad I. *Fill in justifications and draw.*

0) $x_0 > 0$ and $y_0 > 0$
1) $m(\angle(x_0, y_0)(x_0, 0)(0, 0)) = m(\angle(a, b)(a, 0)(0, 0))$
2) $\angle(x_0, y_0)(0, 0)(x_0, 0) = \angle(a, b)(0, 0)(a, 0)$
3) $m(\angle(x_0, y_0)(0, 0)(x_0, 0)) = m(\angle(a, b)(0, 0)(a, 0))$
4) $\Delta(x_0, y_0)(0, 0)(x_0, 0))$ is similar to $\Delta(a, b)(0, 0)(a, 0))$
5) $\frac{d((x_0,y_0),(x_0,0))}{d((x_0,y_0),(y_0,0))} = \frac{d((a,b),(a,0))}{d((a,b),(0,b))}$
6) $\frac{|y_0|}{|x_0|} = \frac{|b|}{|a|}$.
7) $y_0 = (b/a)x_0$.

CASE II: (x_0, y_0) in Quad II .
1) (x_0, y_0) and (a, b) are in the upper half plane because $y_0 > 0$ and $b > 0$.
2) The line segment between them is in the upper half plane by defn half plane.
3) (x_0, y_0) is in the left half plane because $x_0 < 0$.
4) (a, b) is in the right half plane because $b > 0$.
5) The line segment between them crosses the y axis because they are on opposite half planes of $x = 0$.
6) It crosses at a point $(0, c)$ with $c > 0$ by steps 2 and 5.
7) This contradicts that it should cross at $(0, 0)$ by given, so Case II does not happen.

CASE III-VI: *Do these on the back of the previous page. IV-VI do not happen..*

PROOF: PART II: Given: L a line through $(0,0)$ and (a,b) with $a>0, b>0$, and $y_0/b = x_0/a$. Show: $(x_0,y_0)\in L$.

Six cases: I. (x_0,y_0) in Quad I. II. (x_0,y_0) in Quad II. III. (x_0,y_0) in Quad III.
IV. (x_0,y_0) in Quad IV. V. (x_0,y_0) on x axis. VI. (x_0,y_0) on y axis.
On the back of the prior page:
Prove Case I by showing $\Delta(x_0,y_0)(0,0)(x_0,0))$ *is similar to* $\Delta(a,b)(0,0)(a,0))$ *and then proving the ray from* $(0,0)$ *to* (x_0,y_0) *is the ray from* $(0,0)$ *to* (a,b) *and thus* $(x_0,y_0)\in L$.
Prove Case II does not happen because $x_0/y_0>0$.
Extra Credit: complete the rest of the cases.

FORMULA FOR LINES THROUGH THE ORIGIN: The above theorem can also be proven for a line, L through $(0,0)$ and any other point (a,b) not on the x or y axis. We say such lines have the formula $y=mx$ because a point (x_0,y_0) is on the line iff $y_0=mx_0$.

PERPENDICULAR LINES THROUGH THE ORIGIN: Two lines through the origin with formulas $y=m_1x$ and $y=m_2x$ where $m_i\neq 0$ are perpendicular iff $m_1m_2=-1$.

PROOF: PART I: Given $y=m_1x$ and $y=m_2x$ are perpendicular. Show: $m_1m_2=-1$.

Without loss of generality assume $m_1<m_2$.

1) $(0,0)$ and $(1,m_1)$ are on the line $y=m_1x$ because they satisfy the formula.

2) $(0,0)$ and $(1,m_2)$ are on the line $y=m_2x$ because they satisfy the formula.

3) $m(\angle(1,m_1)(0,0)(1,m_2))=90°$ by the definition of perpendicular lines.

4) $m_2>0>m_1$, otherwise $(1,m_1)$ and $(1,m_2)$ would be in the same quadrant and the Quadrant Angle Theorem would contradict step 3.

Complete the proof by showing the triangles $\Delta(0,m_1)(0,0)(1,m_1)$ *and* $\Delta(0,m_2)(0,0)(1,m_2)$ *are similar and then finding proportional sides*:

PROOF: PART II: *Prove on the back of this page including given/show statements.*

FORMULAS FOR OTHER LINES: To find the formulas that describe other lines we first prove a certain map from the plane to the plane is an isometry. So we will do this in the future.