Concurrent Lines Lesson and Project

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Goal: ***To review the concurrence of the angle bisectors, medians, perpendicular bisectors and altitudes of triangles.***

Warmup Problems:

1. Three lines are said to be concurrent if they meet at a common point. Prove that the lines $y = 2x$, $y = 3x$ and $x = 0$ are concurrent by graphing, finding their common point and verifying this point lies on all three lines algebraically.

2. Recall that the distance between a point P and a line l, denoted $d(P, l)$, is the length of the segment PQ where Q is on l and PQ is perpendicular to l. Prove that the circle about P of radius $R = d(P, l)$ meets l exactly at one point and that point is Q. *Hint: assume on the contrary that the circle meets the line at two points, form an isosceles triangle and reach a contradiction.* We say that the line is ***tangent*** to the circle and call Q the ***point of tangency***.

3. Draw a triangle ΔABC and construct the angle bisector, l_A of the angle $\angle BAC$ using a compass and straight edge. Prove that for all $P \in l_A$, we have $d(P, AB) = d(P, AC)$.

4. Prove the converse: if X is a point such that $d(X, AB) = d(X, AC)$, then X is on the angle bisector of $\angle BAC$.

PART I: In the last two warmup problems we proved that:

The angle bisector of $\angle BAC$ ***is the set*** $\{X : d(X, AB) = d(X, AC)\}$.

This special property of angle bisectors will enable us to prove that the three angle bisectors of a triangle meet at a common point and that, furthermore, this common point is the center of the inscribed circle. We now prove each of these facts:

The angle bisectors of a triangle are concurrent.

Rephrased mathematically we say: Let ΔABC be a triangle and l_A, l_B and l_C be the angle bisectors of angles A, B and C respectively. Let p be the point where l_A and l_B meet. Show $p \in l_C$.

Proof:
(1) $d(p, AB) = d(p, AC)$ by the special property of angle bisectors and $p \in l_A$.
(2) $d(p, BC) = d(p, BA)$ by the special property of angle bisectors and $p \in l_B$.
(3) $d(p, AB) = d(p, BC)$ by transitivity and steps (1) and (2).
(4) $p \in l_C$ by the special property of angle bisectors and step (3). QED

If we set $r = d(p, AB)$ then the circle of radius r about p, $C(p, r)$, is inscribed in the triangle ΔABC. The point p is called the ***incenter*** and the radius r is called the ***inradius***, and $C(p, r)$ is called the ***inscribed circle***.

The inscribed circle is the only circle which is tangent to each of the three sides of the triangle.

We know the three sides of ΔABC are tangent to the inscribed circle by the fact that radius is the distance to each side (see the warmup problems). How can we show it is the only circle with this property? **Problem 1**: We suppose we have another circle $C(p', r')$ which is tangent to all three sides and prove $p = p'$ and $r = r'$.

PART II: Recall that the ***perpendicular bisector***, l, of a line segment, AB, is a line which is perpendicular to AB and passes through its midpoint.

Problem 2: Show that if P lies on the perpendicular bisector of AB, then $PA = PB$.

Problem 3: Show that if $PA = PB$ then P lies on the perpendicular bisector of AB.

Problem 4: Construct the perpendicular bisector of AB using a compass a straightedge.

Problem 5: Use this special property of perpendicular bisectors to write a four line proof that ***the perpendicular bisectors of the sides of a triangle are concurrent***. The point where they meet is called the ***circumcenter***.

Problem 6: Draw a circle around the circumradius which passes through one of the vertices of the triangle. Show that it passes through all the vertices. This is called the ***circumscribed circle*** and its radius is called the ***circumradius***.

Problem 7: Prove the circumscribed circle is unique.

Problem 8: Draw two triangles, one with a circumcenter that lies within the triangle and one with a circumcenter which lies outside of the triangle.

PART III: Recall that the ***median*** of a triangle is a line from a vertex to the midpoint of the opposite side. If A' is the midpoint of BC then the median AA' divides ΔABC into two triangles $\Delta ABA'$ and $\Delta ACA'$ of equal area. This can be seen because the area of a triangle is 1/2 base times height and both triangles have the same height, $d(A, BC)$, and the same base length $BA' = A'C$.

Problem 9: Prove that if a point A'' lies on BC and $Area(\Delta ABA'') = Area(\Delta ACA'')$ then A'' is the midpoint of BC.

The medians of a triangle are concurrent.

Problem 10: Prove this statement by first assuming the medians AA' and BB' meet at a point p and defining C'' to be the point where CP meets AB. The idea is to show C" is the midpoint of AB. Notice six small triangles have been formed and start studying the areas of these triangles using the fact that medians divide areas as described above. Eventually one can show all the areas are the same and apply Problem 9 to show C" is the midpoint. The point where the medians meet is called the ***centroid*** and is the balancing point of the triangle.

PART IV: Recall that the ***altitude*** at vertex A of a triangle ΔABC is a line through the vertex A which meets the line BC perpendicularly.

The altitudes of a triangle are concurrent meeting at a point called the orthocenter.

Extra Credit: Prove that the altitudes are concurrent.

A good resource for these problems and related problems is Coxeter and Greitzer **Geometry Revisited** Sections 1.2-1.4.