Circles Project

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BACKGROUND: General Axioms, Half planes, Isosceles Triangles, Perpendicular Lines, Congruent Triangles, Parallel Postulate and Similar Triangles.

DEFN: A **circle** of radius R about a point P in a plane is the collection of all points, X, in the plane such that the distance from X to P is R. That is, $C(P,R) = \{X : d(X,P) = R\}$. Note that a circle is a well defined notion in any metric space. We say P is the **center** and R is the **radius**. If $d(P,Q) < R$ then we say Q is an **interior point** of the circle, and if $d(P,Q) > R$ then Q is an **exterior point** of the circle.

DEFN: The **diameter** of a circle $C(P,R)$ is a line segment $\overline{AB}$ such that $A, B \in C(P,R)$ and the center $P \in \overline{AB}$. The length AB is also called the diameter.

THEOREM: The diameter of a circle $C(P,R)$ has length $2R$.

PROOF: *Complete this proof using the definition of a line segment and the betweenness axioms.*

DEFN: If $A, B \in C(P,R)$ but $P \notin \overline{AB}$ then $\overline{AB}$ is called a **chord**.

THEOREM: A chord of a circle $C(P,R)$ has length $< 2R$ unless it is the diameter.

PROOF: *Complete this proof using axioms and the above theorem.*

DEFN: A line L is said to be a **tangent line** to a circle, if $L \cap C(P,R)$ is a single point, Q. The point Q is called the point of contact and L is said to be tangent to the circle at Q. If $L \cap C(P,R)$ is a pair of points, then L is called a **secant**.

THEOREM: A line, $\overleftrightarrow{AB}$, is tangent to a circle $C(P,R)$ at a point Q if and only if $\overleftrightarrow{AB}$ meets $\overleftrightarrow{PQ}$ perpendicularly at Q.

PROOF PART I (Tangent implies Perpendicular):

1) Assume on the contrary, that $m(\angle AQP) \neq 90°$. Without loss of generality we assume $< 90°$.

Our goal now is to show $\overleftrightarrow{AB}$ *is not tangent by finding a second point* Q' *which is on the circle and on the line* $\overleftrightarrow{AB}$. *We can do this by constructing an isosceles triangle:*

2) Let $\theta = 180 - 2m(\angle AQP)$ and define a ray $\overrightarrow{PC}$ such that $m(\angle QPC) = \theta$ and C is in the same half plane as A. (by the Protractor Postulate)

3) Let $Q' \in \overrightarrow{PC}$ such that $d(P,Q') = R$. (by the Ruler Postulate)

Now complete the proof by first showing that triangle $\Delta PQQ'$ *is an isosceles triangle, then finding* $m(\angle PQQ')$ *and finally showing that* $\overleftrightarrow{BA} = \overleftrightarrow{QQ'}$, *and concluding that* $\overleftrightarrow{AB}$ *is not tangent.*

PROOF PART II: (Perpendicular implies Tangent) *Prove this by contradiction.*

Angles and Arc Measure

DEFN: Recall that an **angle** is the union of two rays: $\angle ABC = \overrightarrow{BA} \cup \overrightarrow{BC}$ and the **interior of an angle** is the region between these rays: $Int(\angle ABC) = H(A, \overleftrightarrow{BC}) \cap H(C, \overleftrightarrow{AB})$ where the half plane $H(P, L)$ is the collection of all points X that are on the same side of the line L as the point P is. When B is the center of a circle then $\angle ABC$ is called a **central angle** of the circle and when B is a point on the circle, then $\angle ABC$ is called an **inscribed angle**.

DEFN: Given a circle $C(P, R)$ and two points A and B on the circle, the **minor arc** $\overset{\frown}{AB}$ is $C(P, R) \cap Int(\angle APB)$. It is called the **subtended arc** of the **central angle** $\angle APB$. The **major arc** is the rest of the circle: $C(P, R) \setminus Int(\angle APB)$. It is sometimes also denoted $\overset{\frown}{AB}$. To distinguish between the two arcs, one can add a third point between the A and the B in the notation.

DEFN: The **measure of the minor arc** is $m(\overset{\frown}{AB}) = m(\angle APB)$. The measure of the major arc is just $360° - m(\overset{\frown}{AB})$ Note that the central angle subtending the arc is used to compute its measure.

ADDITIVITY OF ARC MEASURE THEOREM: If a point C lies on an arc $\overset{\frown}{AB}$ and $\overset{\frown}{CB} \subset \overset{\frown}{AB}$ and $\overset{\frown}{AC} \subset \overset{\frown}{AB}$, then $m(\overset{\frown}{AB}) = m(\overset{\frown}{CB}) + m(\overset{\frown}{AC})$ *The proof of this theorem is a rather tedious process of checking cases and can be found for example in Kay's College Geometry page 197. We will skip the proof but you should draw the statement and convince yourself.*

THEOREM: If $\overline{AB}$ is a diameter of circle $C(P, R)$ and $Q \in C(P, R)$, then the inscribed angle $\angle AQB$ is a right angle.

PROOF: *Add justifications, fill in blanks and draw pictures.*

1) $PQ = PB = PA = R$

2) ____________ and ____________ are isosceles triangles.

3) $\theta = m(\angle$____________$) = m(\angle$____________$)$

and $\varphi = m(\angle$____________$) = m(\angle$____________$)$.

4) $\theta + \theta +$ ____________ $= 180°$, So $\theta =$

and $\varphi + \varphi +$ ____________ $= 180°$, So $\varphi =$

5) $m(\angle APQ) + m(\angle QPB) = 180°$

6) $m(\angle AQB) = \theta + \varphi =$ ____________.

THEOREM: If $Q \in C(P, R)$ and $Q \notin \overset{\frown}{AB}$, then $m(\angle AQB) = m(\overset{\frown}{AB})/2$.

PROOF: *Prove this by finding isosceles triangles.*

COROLLARY: If $Q, Q' \in C(P, R)$ and $Q, Q' \notin \overset{\frown}{AB}$, then $m(\angle AQ'B) = m(\angle AQB)$.

PROOF: *This very important corollary follows immediately applying the above theorem twice.*

Using the theorems in these two pages and this corollary combined with congruent triangle, isosceles triangle and similar triangle theorems one can rederive all the chord-arc theorems in Euclidean Geometry without having to memorize them. On the next page we list all the chord arc theorems on the NYS Regents.

Rederiving Theorems

The following theorems are all on the NYS Geometry high school Regents Exam and can be quite overwhelming if the students try to memorize them. It should also be noted that memorizing them does not prepare a student for college level mathematics. The skill being tested here is the ability to rederive the theorems and figure out relationships given all the Euclidean geometry taught up to this date.

You and your students must know and memorize the theorems on the first two pages of this project and be familiar with how to work with those theorems and the theorems of congruent and similar triangles and perpendicular bisectors and isosceles triangles. It is essential to become quick at marking corresponding angles and lengths and drawing consequences to answer multiple choice questions and to apply geometry in future coursework.

Your students may also be asked to prove one of these theorems in a two column proof. As teachers, you need to recognize any correct proofs.

PERPENDICULAR BISECTORS OF CHORDS: If $\overline{AB}$ is a chord of of circle $C(P, R)$ and D is the midpoint, then the perpendicular bisector $\overleftrightarrow{DQ}$ passes through the center, P. *Prove this in a few lines by showing $\overleftrightarrow{PD}$ is a perpendicular bisector.*

PARALLEL CHORDS THEOREM: If $\overline{AC}$ and $\overline{BD}$ are chords of circle $C(P, R)$, that are parallel, then $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ have the same measure. *Give a 2-3 line proof.*

TANGENT-TANGENT LAW: If $\overleftrightarrow{AB}$ is tangent to circle $C(P, R)$ at B and $\overleftrightarrow{AC}$ is tangent to circle $C(P, R)$ at C, then $AB = AC$. *Find two congruent triangles and justify. (Very short).*

SECANT-SECANT LAW: If two secants meet at an exterior point then certain lengths are proportional. *Label the secants $\overleftrightarrow{AB}$ with $A, B \in C(P, R)$ and $\overleftrightarrow{CD}$ with $C, D \in C(P, R)$ and assume they meet at an exterior point Q. To show some lengths are proportional one needs to show some triangles are similar, and thus we need to investigate angles: Compare $\angle(ABC)$ to $\angle(ADC)$ using the major and minor arcs $\overset{\frown}{AC}$. So what can one say about $\angle QDA$? Which triangles are similar?*

CHORD-CHORD LAW: If $\overline{AC}$ and $\overline{BD}$ are chords of circle $C(P, R)$, and meet at a point Q, then ΔAQB is similar to ΔDQC and so we have certain proportional lengths. *Justify the similarity using the corollary and write out the proportional lengths. Be very careful because any angle at Q is neither a center angle nor an inscribed angle..*

TANGENT-SECANT ANGLE THEOREM: The angle between a tangent line and a secant line passing through its point of tangency can be found using the arc created by the secant. *Let the line $\overleftrightarrow{AC}$ be tangent to $C(P, R)$ at A and secant $\overleftrightarrow{AB}$ have $B \in C(P, R)$. How is $m(\angle BAC)$ related to the measure of $\overset{\frown}{AB}$? Hint: draw $\overline{AP}$ and $\overline{BP}$.*

TANGENT-SECANT LAW: If a tangent line and a secant line meet at an exterior point, then certain lengths are proportional. *Label the secant $\overleftrightarrow{AB}$ with $A, B \in C(P, R)$ and the tangent line $\overleftrightarrow{ED}$ with $D \in C(P, R)$ and assume they meet at an exterior point Q. To show some lengths are proportional one needs to show some triangles are similar, and thus we need to investigate angles again. Mark the right angle and any angles which are congruent using the corollary on the last page, find two similar triangles and complete the proof.*

Arclengths and Circumference

Recall that the measure of an arc is given in degrees and is equal to the central angle subtending the arc. It is not the arclength. Here we define the arclength and the circumference and derive their formulas.

DEFN: The length of a straight line segment is the distance between the endpoints. The length of a path made up out of line segments is the sum of the lengths of the segments.

REGULAR N-GON: Take a circle $C(P,R)$ and place n points on the circle $P_1, P_2, ...P_n$ evenly spaced so that $m(\angle P_i P P_{i+1}) = 360°/n$. Then $d(P_i, P_{i+1})$ is the same for every i, so this is a regular n-gon. *Why?*. The perimeter of the regular n-gon is the length around the edge:

$$S(n) = \sum_{j=1}^{n} d(P_{j-1}, P_j) = d(P_1, P_2) + d(P_2, P_3) + ... + d(P_{n-1}, P_n).$$

How does this length depend on R? That is, if $S(n,R)$ is the perimeter of an n-gon inscribed in a circle of radius R, what is $S(n,R)/S(n,r)$? Does $S(n,R)$ depend on the center of the circle?

REGULAR HEXAGON: Take a circle $C(P,R)$ and place 6 points on the circle $P_1, P_2, ...P_6$ evenly spaced so that $m(\angle P_i P P_{i+1}) = 360°/6$. *What is $d(P_i, P_{i+1})$ and what is* $S(6,R)$?

DEFN: The length of a smooth curve like the circumference of a circle or an arc, is found by marking n evenly spaced points P_j and then measuring the sum of their lengths, $S(n)$, and then taking the limit as n approaches infinity. (There is also a notion of a rectifiable curve which is defined using infimums).

CIRCUMFERENCE OF A CIRCLE: When we take the limit as n approaches infinity of the $S(n,R)$ for the regular n-gon we get the formula for the circumference of the circle, $Circum(P,R)$. *Does $Circum(P,R)$ depend on P? What is $Circum(P,R)/Circum(P,r)$?*

DEFINING PI: The real number π is defined as the ratio of the circumference over the diameter of a circle: $\pi := Circum(n,R)/(2R)$. *Verify that the definition of π does not depend on the radius of the circle: $Circum(P,R)/(2R) = Circum(P,r)/(2r)$.* The circumference of a circle is then $2\pi R$.

ESTIMATING PI: One can estimate π by taking larger and larger values of n and explicitly computing $S(n,R)$. Archimedes calulated this is by taking $n = 4, 8, 16, 32, ...$ doubling so that the formula for each $S(2n,R)$ can be computed using the prior formula $S(n,R)$.

DERIVING ARCLENGTH: The arclength of an arc in a circle can also be computed using limits. Suppose $\widehat{AB}$ is an arc in $C(P,R)$ and $m(\angle APB)/360°$ is a rational number p/q. Then we can draw an inscribed $q-gon$ in the circle with p sides having endpoints in the arc $\widehat{AB}$. The sum of the p sides approximating the arc is $(p/q)S(q,R)$. (*draw this*). We can draw an inscribed $2q-gon$ in the circle with $2p$ sides having endpoints in the arc $\widehat{AB}$. The sum of the p sides approximating the arc is $((2p)/(2q))S(2q,R)$.(*draw this*). In fact we can draw an inscribed $(mq)-gon$ in the circle with (mp) sides having endpoints in the arc $\widehat{AB}$. The sum of the mp sides approximating the arc is $((mp)/(mq))S(mq,R) = (p/q)S(mq,R)$. So taking the number of sides to infinity, we see that the arclength is $(p/q)2\pi R$ which is $(m(\angle APB)/360°)2\pi R$.

ARCLENGTH: The arclength of a minor arc $\widehat{AB} \in C(P,R)$ is $(m(\angle APB)/360°)2\pi R$.

ARCLENGTH PROBLEMS:

(1) What is the arclength of an arc with inscribed angle θ in a circle of radius R?

(2) What is the arclength between a tangent and secant that meet at angle θ?

(3) What is the arclength of $\widehat{AB} \subset C(P,R)$ when the distance from AB to P is $r < R$?