Area Project

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MAT631, Fall 2009, Project X

BACKGROUND: Congruent and Similar triangles, Squares and Parallelograms, Coordinate Transformations and Images.
In this project we introduce the notion of area with four new axioms:
AREAS OF UNIT SQUARES: The area of a square of side length 1 is defined to be 1.
DOMINANCE PRINCIPLE: If one set $S_1 \subset S_2$ then $Area(S_1) \leq Area(S_2)$.
ADDITION PRINCIPLE: If we have two disjoint sets S_1 and S_2 then $Area(S_1 \cup S_2) = Area(S_1) + Area(S_2)$.
ISOMETRY PRINCIPLE: If $F : E^2 \to E^2$ is an isometry, then $Area(S) = Area(F(S))$.
THEOREM: The area of a square of side length s is s^2. *Add pictures:*
PROOF (when $s = n$ where n is a natural number):
Divide the square into n^2 smaller squares all of which are isometric to a unit square. Then add up the areas using the addition principal. QED for $s = n$
PROOF (when $s = 1/n$ where n is a natural number):
Take n^2 isometric copies of the given square arranged in a large unit square. Each copy has the same area A and the sum of all the squares has area $n^2 A = 1$, so $A = 1/n^2 = s^2$. QED for $s = 1/n$.
PROOF (when s is rational, $s = p/q$): Fill the square with p^2 copies of isometric tiny squares of side length $1/q$. Each tiny square has area $1/q^2$ by the above case. Finally sum up all the small areas to get the area of the given square $p^2(1/q^2) = p^2/q^2 = s^2$. QED for s rational.
PROOF (when s is irrational) One approximates the square with squares of rational side length both outside and inside and then takes the limit applying the squeeze theorem from calculus to get s^2. QED
AREA OF A RECTANGLE: The area of a rectangle of side length a and b is $a \cdot b$. *Prove this when $a = 1/2$ and $b = 7/3$ including a precise sketch*:

AREA OF A RIGHT TRIANGLE: The area of a right triangle ΔABC with right angle at C is $AC \cdot BC/2$.

Proof: *Fill in justifications:*

1) Let M be the midpoint of AB.
2) The reflection through M, r_M is an isometry.
3) $r_M(A) = B$ and $r_M(B) = A$.
4) Let $C' = r_M(C)$, then $Area(\Delta ABC) = Area(\Delta BAC')$
5) $ACBC'$ form a rectangle.
6) The rectangle's area is $AC \cdot BC$.
7) The area of the triangles add up the the rectangle.
8) The area of the triangle is $AC \cdot BC/2$.

AREA OF A GENERAL TRIANGLE: If ΔABC is any triangle we can choose one side to be the base. Suppose this side is AB. Next we drop an altitude to the base to find a point $P \in \overleftrightarrow{AB}$ such that $\overleftrightarrow{CP}$ is perpendicular to $\overleftrightarrow{AB}$. The length CP is called the height. The area of the triangle is then (1/2) base times height: $(1/2)AB \cdot CP$.

Prove this by adding up the areas of the right triangles when $P \in \overline{AB}$ right here. (Prove the other cases on the back of the last page):

AREA OF A PARALLELOGRAM: If $ABCD$ form a parallelogram, and we say AB is the base, then we drop an altitude to the base to find a point $P \in \overleftrightarrow{AB}$ such that $\overleftrightarrow{CP}$ is perpendicular to $\overleftrightarrow{AB}$. The length CP is called the height. The area of the parallelogram is then (1/2) base times height: $(1/2)AB \cdot CP$.

Prove this:

AREA OF A REGULAR POLYGON: A regular polygon with n sides can be broken up into n isosceles triangles each with height h and base b, so the area of the polygon is $n(1/2)bh$.

DEFINING Π: The definition of π is one half the circumference of a unit circle. This can be used to find the area of a unit circle.

AREA INSIDE A CIRCLE: A unit circle can be placed between two regular polygons so its area is between their areas. The inside polygon is inscribed the outside one is circumscribed. As we take n to infinity, we see the heights approach the radius r. Also notice that nb approaches the circumference, which is 2π. So the areas of the regular polygons both inside and outside are approaching $2\pi r(1/2)r = \pi r^2$.

AREAS OF OTHER SHAPES: The areas of other shapes are found by approximating with many tiny squares, summing their areas and taking limits. This is made clear using integration and calculus. If F is an isometry then $Area(F(S)) = Area(S)$ because we can map all the small squares approximating S to small squares approximating $F(S)$. *Draw this:*

SKEW TRANSFORMATIONS PRESERVE AREA: Recall that a skew transformation along the x axis has the formula $F(x, y) = (x + yk, y)$. We claim this preserves area: $Area(S) = Area(F(S))$. To see this first observe that the skew along the x axis of a square whose base is parallel to the x axis is a parallelogram whose base and height are the same. *Draw this:*

So if you approximate a set S with many small squares, you can approximate $F(S)$ with many small parallelograms. The total areas will be the same. *Draw this:*

DILATIONS BY r SCALE AREA BY r^2: Recall a dilation by scale $r > 0$ is a map $dil_r : E^2 \to E^2$ such that $dil_r(P_1, P_2) = (rP_1, rP_2)$. and that distances scale by r under a dilation: $d(dil_r(P), dil_r(Q)) = rd(P, Q)$. Dilations preserve angles and so they take squares to squares. If we dilate a square by dil_r then the square's area goes from s^2 to $(rs)^2 = r^2s^2$. The new area is r^2 times the old area. *Draw this right here:*

If you dilate a shape filled with squares by dil_r then all the squares have their areas scale by r^2 and so the total area scales by r^2 and so does the limit. Thus any shape's area scales by r^2. *Draw this:*

AREAS OF CIRCLES: The area of a circle of radius R is πR^2.
PROOF: *Fill in justifications.*
1. $Area(C(P, R)) = Area(T_{-P}(C(P, R))$
2. $\qquad = Area(C(O, R))$
3. $\qquad = Area(dil_R(C(O, 1))$
4. $\qquad = R^2 Area(C(O, 1))$
5. $\qquad = R^2\pi.$

STRETCHING COORDINATES DIFFERENTLY: Suppose one has the transformation $F(x, y) = (ax, by)$ where $a, b > 0$.
What happens to squares?

How do their areas change?

Explain why $Area(F(S)) = abArea(S)$*?*

ELLIPSES: An ellipse is the image of a circle under a map like F:

$$F\{(x, y) : x^2 + y^2 = 1\} = \{(ax, by) : x^2 + y^2 = 1\} = \{(X, Y) : (X/A)^2 + (Y/B)^2 = 1\}.$$

What is the area of an ellipse?

VOLUME EXTRA CREDIT: [STRONGLY RECOMMENDED FOR HIGH SCHOOL TEACHERS] *Use the same basic axioms to build up a theory of volume. Explain the volume of a cube and a rectangular block. Suppose one divides a cube into 6 isometric square pyramids each of whose base is a side, what is the volume of a pyramid as a formula depending on base and height? Suppose one defines a map* $F(x, y, z) = (ax, by, cz)$*, how does this change volume? What is the volume of a square cylinder of arbitrary height and base? What if the base is a right triangle? an isosceles triangle? a regular polygon? a circle? What is the volume of a cone? What happens if you apply a skew transformation which takes* (x, y, z) *to* $(x + kz, y, z)$*? Can you derive all the formulas for volume on the NYS Geometry Regents?*
3D DILATION EXTRA CREDIT: [FOR THOSE WHO COMPLETE THE ABOVE FIRST] Dilations can also be defined in three dimensions $dil_r(x, y, z) = (rx, ry, rz)$. In the above extra credit project explain why dilations in three dimensions scale distances by r, areas by r^2 and volumes by r^3.