Metric Spaces: Rectifiability

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BACKGROUND: Metric spaces, balls, open sets, continuity

DEFN: A path \( \gamma : [0, 1] \to X \) is rectifiable iff

\[
\sup \left\{ \sum_{j=1}^{N} d(\gamma(t_{j-1}), \gamma(t_j)) : 0 = t + 0 < t_1 < \cdots < t_{N-1} < t_N = 1 \right\} < \infty.
\]

The value of the supremum is then called the length of \( \gamma \).

PROBLEM 1: Prove that if \( \{s_1, s_2, \ldots, s_{N'-1}\} \subset \{t_1, \ldots, t_{N-1}\} \) then

\[
\sum_{j=1}^{N'} d(\gamma(s_{j-1}), \gamma(s_j)) \leq \sum_{j=1}^{N} d(\gamma(t_{j-1}), \gamma(t_j)).
\]

PROBLEM 2: Suppose that for all \( t \in [0, 1] \) we have

\[
d(\gamma(0), \gamma(t)) + d(\gamma(t), \gamma(1)) = d(\gamma(0), \gamma(1))
\]

(a) Prove that for all \( t_1 < t_2 < t_3 \):

\[
d(\gamma(t_1), \gamma(t_2)) + d(\gamma(t_2), \gamma(t_3)) = d(\gamma(t_1), \gamma(t_3))
\]

(b) Prove that \( \gamma \) is rectifiable and \( L(\gamma) = d(\gamma(0), \gamma(1)) \).

PROBLEM 3: Prove that line segments in \( \mathbb{R}^2 \) are rectifiable:

\[
\gamma(t) := (at + a_0, bt + b_0).
\]

and that the length is the distance between the endpoints.

PROBLEM 4: Prove that smooth curves in \( \mathbb{R}^2 \) are rectifiable

\[
\gamma(t) := (\gamma_1(t), \gamma_2(t)).
\]

using the fact that for all \( t_j < t_{j+1} \) there exists \( s_j \in (t_j, t_{j+1}) \) such that

\[
f'(s_j) = (f(t_{j+1}) - f(t_j))/(t_{j+1} - t_j).
\]

and that the length satisfies the arclength formula.