Metric Spaces: Limits and Closures

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BACKGROUND: Project 1: Metric Spaces: An Introduction, Project 2: Open Sets
In this problem set each problem has hints appearing in the back.
DEFN: A sequence of points \( x_j \in X \) converges to a point \( x \in X \) iff
\[ \forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N} \text{ such that } d(x_j, x) < \epsilon \forall j \geq N_\epsilon. \]
We say that \( x \) is the limit of the sequence \( x_j \) and write \( \lim_{j \to \infty} x_j = x \).

PROBLEM 1: Prove a sequence of points \( x_j \in X \) converges to a point \( x \in X \) iff
\[ \forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N} \text{ such that } x_j \in B_x(\epsilon) \forall j \geq N_\epsilon. \]

PROBLEM 2: Prove that if \( \lim_{j \to \infty} x_j = x \) and \( x \in U \) where \( U \) is an open set then eventually \( x_j \in U \).

PROBLEM 3: Prove that if \( x_j \notin B_p(r) \forall j \in \mathbb{N} \) and \( \lim_{j \to \infty} x_j = x \) then \( x \notin B_p(r) \).

PROBLEM 4: Prove (or disprove that) if \( x_j \in B_p(r) \forall j \in \mathbb{N} \) and \( \lim_{j \to \infty} x_j = x \) then \( x \in B_p(r) \).

PROBLEM 5: Prove (or disprove that) if \( x_j \in B_p(r) \forall j \in \mathbb{N} \) and \( \lim_{j \to \infty} x_j = x \) then \( d(x, p) \leq r \).

DEFN: Given a set \( A \subset X \), the closure of \( A \), denoted \( Cl(A) \) or \( \overline{A} \) is the set
\[ Cl(A) = \{ x : \exists a_j \in A \text{ such that } \lim_{j \to \infty} a_j = x \}. \]

PROBLEM 6: Prove (or disprove that)
\[ Cl(B_p(r)) = \{ x : d(x, p) \leq r \}. \]

DEFN: A closed set, \( K \), is a set such that \( K = Cl(K) \).

PROBLEM 7: Prove that if \( K \) is closed then \( X \setminus K \) is open.

PROBLEM 8: Prove that if \( U \) is open then \( X \setminus U \) is closed.

PROBLEM 9: Prove that \( K \) is closed iff \( X \setminus K \) is open. This is often stated as the definition of a closed set.

PROBLEM 10a: Given a set \( A \subset B \) then \( Cl(A) \subset Cl(B) \).

PROBLEM 10b: Given a set \( A \) contained in a closed set \( K \), then \( Cl(A) \subset K \).

PROBLEM 10c: Given a set \( A \) the closure of \( A \) is the intersection of all closed sets containing \( A \). This is often stated as the definition of a closed set.
**HINTS:** Read only one at a time! Also look over the hints after completing a proof. If you did a proof differently, show it to me, you may well be right.

Hint 0: (All problems) Draw a picture and keep drawing pictures.

Hint 1: (1) This is just definitions but don’t forget to do both directions. (2) Write out the definition of open set defining an $r_x > 0$. (3) Write out the definitions to get a couple inequalities resulting from the two givens. (4) Remember from calculus that $\lim_{j \to \infty} 2 - \frac{1}{j} = 2$. (5) Now if it reaches the edge, that is fine, so prove it. (6) You need to prove each set in the other set (so two parts). (7) You want to find an $r_x > 0$ for each $x \in X \setminus K$. (8) Try this by contradiction, $Cl(X \setminus U) \neq X \setminus U$, which means there exists $x \in Cl(X \setminus U)$ such that $x \notin X \setminus U$. This means $x \in U$. (9) This is problems 8 and 9 put together. (10) Let $x \in Cl(A)$ and follow the definitions of closure and subset and closure. This gives (a).

Hint 2: (1) no more needed. This is two lines. (2) Use problem 1. (3) Choose $N_\epsilon = N_{r_x}$ where $B_x(r_x) \in U$. (4) This is false, since $x_j$ could approach the edge. (5) Set up triangle inequalities and do this similarly to problem 3. (6) Show $r_x = \inf \{d(x,y) : y \in K\}$.

Hint 3: (2) eventually means $\exists N$ such that $x_j \in U \forall j \geq N$. So choose the right $N$ using the given $N_\epsilon$. (3) Choose a sequence of points in Euclidean space. (4) Choose $N$ so that $d(x,p) \geq r - \epsilon$ from the triangle inequality. (5) Choose $N$ so that $d(x,p) \geq r - \epsilon$ from the triangle inequality. (6) Look at hint 1 to help construct a sequence to disprove the statement. (7) What about discrete space? (8) If $r_x = 0$, then show there exists $y_j \in K$ such that $d(y_j, x) \to 0$ by the definition of infimum. (9) Use Problem 2. (10) Next show the intersection of all closed sets containing $A$ is a subset of each closed set containing $A$ and the definition of intersection.

Hint 4: (2) Choose $N = N_{r_x}$ where $B_x(r_x) \in U$. (3) But this is true for all $\epsilon > 0$ and neither $x$ nor $p$ depends on $\epsilon$. So $d(x,p) \geq r - \epsilon$ from the triangle inequality. (4) Choose a sequence of points in Euclidean space. (5) Enough. (6) What about discrete space? (7) If $r_x = 0$, then show there exists $y_j \in K$ such that $d(y_j, x) \to 0$ by the definition of infimum. (8) Use Problem 2. (10) Next show the intersection of all closed sets containing $A$ is a subset of any particular closed set containing $A$ by definition of intersection.

Hint 5: (2) $x_j \in B_x(r_x) \subset U$. Done. (3) If the above doesn’t feel right, try a proof by contradiction, assuming $d(x,p) < r$. (4) Enough. (6) What about $r = 1$ in discrete space? What are the two sets? (7) Once you have shown $r_x > 0$, complete the proof showing $B_x(r_x) \subset X \setminus K$. (8) That should be a contradiction. (10) One particular such closed set is $Cl(A)$. 