Metric Spaces: Continuity

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BACKGROUND: Metric spaces, balls, open sets,

DEFN: Given two metric spaces, \( X \) and \( Y \), a function \( f : A \subset X \rightarrow Y \) is continuous at a point \( a \in A \) if for all \( \epsilon > 0 \) there exists \( \delta_\epsilon > 0 \) such that \( d(x, a) < \delta_\epsilon \) implies \( d(f(x), f(a)) < \epsilon \). We say \( f \) is continuous on \( A \) if it is continuous at \( a \) for all \( a \in A \).

PROBLEM 1: For \( X \) and \( Y \) both the same space \( \mathbb{R} \) with the metric \( d(x, y) = |x - y| \), prove that

(a) \( f(x) = 5x + 8 \) is continuous at any \( a \in \mathbb{R} \).
(b) \( f(x) = |x| \) is continuous at any \( a \in \mathbb{R} \).
(c) \( f(x) = \tan(x) \) is continuous at any \( a \in (-\pi/2, \pi/2) \).

PROBLEM 2: Prove that if \( X, Y \), and \( Z \) are metric spaces and \( f : X \rightarrow Y \) is continuous at \( a \) and \( h : Y \rightarrow Z \) is continuous at \( f(a) \) then \( h \circ f : X \rightarrow Z \) is continuous at \( a \).

PROBLEM 3: Prove that if \( f : A \subset X \rightarrow Y \) is continuous on \( A \) and \( a_j \in A \) is a sequence converging to \( a \in A \), then \( f(a_j) \) converges to \( f(a) \).

PROBLEM 4: Prove that if \( X \) is the discrete metric space and \( Y \) is any metric space and \( f : X \rightarrow Y \) is any function, then \( f \) is continuous on \( X \).

DEFN: Given a function \( f : A \subset X \rightarrow Y \), the image of a set \( K \subset X \) is

\[
 f(K) = \{ y \in Y : \exists x \in K \text{ such that } y = f(x) \}.
\]

The preimage of a set \( U \subset Y \) is \( f^{-1}(U) = \{ x \in X : f(x) \in U \} \). Note that the preimage is defined even if \( f \) has not inverse.

PROBLEM 5: Prove that \( f \) is continuous at \( a \) iff for all \( \epsilon > 0 \) there exists \( \delta_\epsilon > 0 \) such that \( f(B_a(\delta_\epsilon)) \subset B_{f(a)}(\epsilon) \).

PROBLEM 6: Prove that \( f \) is continuous at \( a \) iff for all \( \epsilon > 0 \) there exists \( \delta_\epsilon > 0 \) such that \( f(B_a(\delta_\epsilon)) \subset B_{f(a)}(\epsilon) \).

PROBLEM 7: Prove that if \( f : A \subset X \rightarrow Y \) is continuous on \( A \) and \( U \subset Y \) is open then \( f^{-1}(U) \) is open.

PROBLEM 8: Find an example of \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that a set \( V \subset \mathbb{R} \) is open but \( f(V) \) is not open.

PROBLEM 9: Find an example of \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that a set \( A \subset \mathbb{R} \) is closed but \( f(A) \) is not closed.

PROBLEM 10: Find an example of \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that a set \( A \subset \mathbb{R} \) is bounded but \( f(A) \) is not bounded.