Metric Spaces: Continuity and Connectedness

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BACKGROUND: Metric spaces, balls, open sets, continuity, connected sets

PROBLEM 1: Prove that the continuous image of a connected set is connected. That is, if \( f : A \subset X \rightarrow Y \) is continuous on \( A \) and \( A \) is a connected set, prove \( f(A) \) is connected.

PROBLEM 2: If \( f : X \rightarrow Y \) is continuous on \( X \) and a set \( K \subset Y \) is connected, is \( f^{-1}(K) \) connected? Prove or provide a counter example being careful to prove all the facts you claim in your counter example.

DEFN: A path \( \gamma : [0, 1] \rightarrow X \) is a continuous function on \([0, 1]\).

DEFN: A set \( A \subset X \) is pathwise connected if every pair of points, \( x, y \in A \) has a path \( \gamma : [0, 1] \rightarrow A \) such that \( \gamma(0) = x \) and \( \gamma(1) = y \).

PROBLEM 3: Prove that \((a, b) \subset \mathbb{R} \) is pathwise connected.

PROBLEM 4: Prove that \( B_0(R) \subset \mathbb{R}^N \) is pathwise connected.

PROBLEM 5: Prove that the continuous image of a path connected set is path connected. That is, if \( f : A \subset X \rightarrow Y \) is continuous on \( A \) and \( A \) is a pathwise connected set, prove \( f(A) \) is pathwise connected.

PROBLEM 6: If \( f : X \rightarrow Y \) is continuous on \( X \) and a set \( K \subset Y \) is pathwise connected, is \( f^{-1}(K) \) pathwise connected? Prove or provide a counter example being careful to prove all the facts you claim in your counter example.

PROBLEM 7: Prove that a path connected set in a discrete metric space contains only one point.

PROBLEM 8: Suppose a set \( A \) is not connected and open sets \( U \) and \( V \) satisfy (i)-(iv) of the definition of not connected. Suppose \( \gamma : [0, 1] \rightarrow X \) is a path such that \( \gamma(0) \in U \) and \( \gamma(1) \in V \). Prove this implies \([0, 1]\) is not connected and thus gives a contradiction.

PROBLEM 9: Prove that a path connected set is connected.

PROBLEM 10: Prove that

\[
A = \{(0, y) : y \in [-1, 1]\} \cap \{(x, \sin(1/x)) : x \in (0, 1/\pi)\} \subset \mathbb{R}^2
\]

is connected but not path connected.