MAT 321 Exam II Sample

Professor Sormani

1. (20 points) Let $A = [-2, 1] \times [-5, -3)$. Let $B = B_{(3,4)}(5)$.
   Let $C = \{(x, y) : x^2 + y^2 \geq 100\}$ and let $D = \{(x, y) : y \leq x - 5\}$.

   a) Fold up the bottom of this page and draw all of these sets on one chart shading them lightly if they have area. Use dashed lines if they don't include their edges. Use solid lines otherwise.

   b) Write down the closures of each of these sets:

   $$Cl(A) = \qquad\qquad Cl(B) =$$

   $$Cl(C) = \qquad\qquad Cl(D) =$$

   c) Write down the interiors of each of these sets:

   $$int(A) = \qquad\qquad int(B) =$$

   $$int(C) = \qquad\qquad int(D) =$$

   d) The boundary of a set, X, is defined to be

   $$\partial(X) = Cl(X) \setminus int(X)$$

   . Write down the boundaries of each of these sets:

   $$\partial(A) = \qquad\qquad \partial(B) =$$

   $$\partial(C) = \qquad\qquad \partial(D) =$$

   e) Suppose U is an open set lying in A, is $U \subset int(A)$? Why?

   f) Suppose (a,b) is an accumulation point of D, is b=a-5?

   g) Is $(1, 3) \in \partial A$?

2. (20 pts) **TRUE OR FALSE WITHOUT JUSTIFICATION**

a) If $x \in int(A)$ then there is a radius $r_x > 0$ such that $B_x(r_x) \subset A$.

b) If K is a compact set in $C([0,1])$ then K is closed and bounded.

c) If $K \subset C([0,1])$ is closed and bounded then K is compact.

d) If $x \in Cl(A)$ then there is a sequence of points $x_k \in A$ such that $\lim_{k\to\infty} x_k = x$.

e) If $f : K \to (-\infty, \infty)$ and K is a compact set and f is continuous, then there exists a point $x_0 \in K$ such that $f(x_0) \geq f(x)$ for all $x \in K$.

f) If $f : A \to B$ is a continuous function between metric spaces and U is open in B then $f^{-1}(U)$ is open in A.

g) If x_k is a sequence of points in an interval $[0,1]$, then at least one of the following is true:

Case I: there exists a subsequence which lies in $[0,1/2]$

Case II: there is a subsequence which lies in $[1/2,1]$

h) f is a contraction mapping iff f is Lipschitz with Lipschitz constant < 1.

i) The functions $f(x) = |x|$, $g(x) = x^3$ and $h(x) = \sqrt{x-1}$ are all in $C([0,1])$.

j) The tangent function whose domain is $(-\pi/2, \pi/2)$ has a bounded range.

Other true false problems could concern the Contraction Mapping Principle, open covers and finite subcovers, converging subsequences, completeness and Cauchy sequences.

3. (20 pts) **Proof Formating**

Write the definition of the following statements using for all and there exists notation and then write the first few lines of proofs of the following statements which use those definitions. You do not have to fill in formulas for the values you are "choosing".

a) The set $A = \{(x, y) : y > x\}$ is an open set.

DEFN:

1) Let ....

2) Choose ...

3) We claim:

b) The function $sin : [0, \pi] \to [-1, 1]$ is continuous at $\pi/2$.

DEFN:

1) Let...

2) Choose:

3) We claim:

c) The set $K \subset C([0, 1])$ is compact.

DEFN:

1) Let

d) The set $A = int(B)$.

DEFN:

REWRITE DEFN: All open sets in B, ...

1) Let

2) We claim

e) Returning to (a), try an alternate proof beginning using a theorem about preimages of continuous functions as follows.

State the theorem:

1) Let $f : A = \{(x, y) : y - x > 0\} \to (-\infty, \infty)$ be the function $f(x, y) =$

2) The set $f^{-1}((0, \infty))$ is an open set because:

3) So A is open because:

4. (20 pts) **Lipschitz Functions**

a) Prove the function $f(x) = x^2/10$ where $f : [-2, 3] \to [0, 9/10] \subset [2, 3]$ is Lipschitz.

b) Prove it is a contraction map.

c) What does the Contraction Mapping Principle tell us? Starting at $x_0 = 1$ what is $x_1 = f(x_0)$, $x_2 = f(x_1)$ etc and where does it converge?

d) Is f Lipschitz as a function on the domain $[0, \infty)$?

e) Suppose $x_0 = 100$, what is $x_1 = f(x_0)$ and the rest of this sequence? Does it converge?

f) Is f a contraction mapping from $[0, 10]$ to $[0, 10]$?

5. (20 pts) **Proof**

Prove that if $f : A \to (-\infty, \infty)$ is a continuous function and A is a compact set, then there exists a maximum point $a_0 \in A$ such that $f(a_0) \geq f(a)$ for all $a \in A$. Fill in justifications:

1) Let $b_0 = sup(f(A))$ which exists by

2) There exists $b_i \in f(A)$ such that b_i increase to b_0 by

3) There exists $a_i \in A$ such that $f(a_i) = b_i$ by

4) A subsequence of a_i converge to some point in A by

5) Let a_0 be the limit of that subsequence.

6) We claim a_0 is the maximum point:

Complete the proof of the claim:

Now prove there exists a minimum point: