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The Covering Spectrum of Riemannian Manifolds

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The **Covering Spectrum**

$$CovSpec(M)$$

measures the size of holes in a Riemannian Manifold or Length Space.

On a **compact** manifold K

$$CovSpec(K) \subset (1/2)Length(K)$$

where $Length(K)$ is the lengths of smoothly closed geodesics and

$CovSpec(K)$ is determined by the Marked Length Spectrum of K by an iterative process. [S-Wei JDG04]

Related Compact Results:

Croke, Otal, Fathi: M *sect* < 0 Marked Length Spectrum (MLS) determines the manifold

Gornet: $\exists$ Laplace isospectral pairs of nilmanifolds with the same MLS

S-Wei: MLS determines CovSpec

Sunada, S-Wei: Komatsu pairs have the same Laplace and CovSpec.

Sunada, De Smit-Gornet-Sutton: Sunada pairs with same Laplace but different Covering Spectra

Conway-Sloan, De Smit-Gornet-Sutton: Flat tori T^4 with the same Laplace Spec and different CovSpec.

In this talk manifolds are
complete and noncompact.

I will define and discuss:

* The Covering Spectrum of M
* The Shift Spectrum of M
* The Marked Shift Spectrum
* The R Cut-off Spectrum of M
* The Cut-off Spectrum of M
* The Rescaled Spectrum of M
* Applications

Defn: $L(g) = \inf_{x \in M} d_{\tilde{M}}(g\tilde{x}, \tilde{x})$.
On a compact manifold, $L(g)$ is achieved by a closed geodesic.

Defn: The δ cover of M is

$$\tilde{M}^\delta \;=\; \tilde{M}/\pi_1(M,\delta)$$

where $\pi_1(M,\delta) \; = < g \, : \, L(g) < 2\delta >$.

Example 1: A $2\pi \times 4\pi$ flat torus T

$\delta \in (0,\pi]$ implies $\tilde{T}^\delta = \tilde{T}$.

$\delta \in (\pi, 2\pi]$ implies $\tilde{T}^\delta = S^1 \times R$

$\delta \in (2\pi, \infty)$ implies $\tilde{T}^\delta = T$.

Defn: $\delta \in CovSpec(M)$ iff
$\forall\, \delta' > \delta \;\tilde{M}^{\delta'} \neq \tilde{M}^\delta$

Example 1: $CovSpec(T) = \{\pi, 2\pi\}$

Example 2: $M = (-\infty, \infty) \times_f S^1$
where $f(r) = e^{-r}$ has a cusp.

$L(g) = \inf_{x\in M} d_{\tilde{M}}(g\tilde{x}, \tilde{x}) = 0.$

$\pi_1(M,\delta) =< g : L(g) < 2\delta >= \pi_1(M).$

Thus all the δ covers of M are trivial:

$$\tilde{M}^\delta = \tilde{M}/\pi_1(M,\delta) = M$$

$CovSpec(M) = \emptyset$

Example 3: $M = (-\infty, \infty) \times_f S^1$
where $f(r) = e^{-r} + 1$.

For g generating $\pi_1(M)$:

$$L(g) = \inf_{x\in M} d_{\tilde{M}}(g\tilde{x}, \tilde{x}) = 2\pi.$$

So $\tilde{M}^\delta = M$ when $\delta \geq \pi$
$\tilde{M}^\delta = \tilde{M}$ when $\delta < \pi$

$CovSpec(M) = \{\pi\}$

$CovSpec(M)$ not in $(1/2)Length(M)$
because holes need not have geodesics.
Extend $L(M)$?

Defn: The Shift Spectrum:
$\{L(g) : g \in \pi_1(M)\}$
where $L(g) = \inf_{x \in M} d_{\tilde{M}}(g\tilde{x}, \tilde{x})$.

On a compact manifold:
$Shift(K) \subset Length(K)$.
Otherwise: may not be a closed geodesic.

Example 2 with cusp:
$Shift(M) = \{0\}$.
Example 3 asymptotic to a cylinder:
$Shift(M) = \{2\pi\}$

Is $CovSpec(M) \subset (1/2)Shift(M)$?

Is $CovSpec(M) \subset (1/2)Shift(M)$? No!

Example 4: M an infinite solid handlebody with handles of decreasing size and minimal geodesics around them with lengths, L_i, decreasing to $L > 0$.

Then $\tilde{M}^\delta$ opens all the handles with length $L_i \geq 2\delta$. So

$$
\begin{aligned}
CovSpec(M) &= \\
&= \{\delta : \forall \delta' > \delta\ \tilde{M}^{\delta'} \neq \tilde{M}^{\delta}\} \\
&= \{L_1/2, L_2/2, L_3/2, ...\} \cup \{L/2\}
\end{aligned}
$$

Thm (S-Wei):

$$CovSpec(M) \subset Cl_{lower}((1/2)Shift(M))$$

where $Cl_{lower}(A)$ includes all the limits of decreasing sequences in A.

$CovSpec(M) \subset Cl_{lower}((1/2)Shift(M))$

Example 5: A solid handlebody with L_i running through the rationals has $Shift(M)$ equal to the rationals and $CovSpec(M) = (0, \infty)$.

Defn: The Marked Shift Spectrum

$$L : \pi_1(M) \to [0, \infty)$$

where $L(g) = \inf_{x \in M} d_{\tilde{M}}(g\tilde{x}, \tilde{x})$.

On a compact manifold, this is the well known Marked Length Spectrum.

Thm: The Marked Shift Spectrum determines the Covering Spectrum of a complete manifold (or length space).

The proof is not constructive.

Summarizing: K compact, M complete:

$CovSpec(K) \subset (1/2) Length(K)$

The Marked Length Spectrum of K determines $CovSpec(K)$ constructively.

$CovSpec(M) \subset Cl_{lower}((1/2) Shift(M))$

The Marked Shift Spectrum of M determines $CovSpec(M)$.

But the covering spectrum no longer detects geodesics. Nor does it have the nice convergence properties.

The Cut-off Covering Spectra will...

Recall Defn: The δ cover of M is

$$\tilde{M}^\delta = \tilde{M}/\pi_1(M,\delta)$$

where $\pi_1(M,\delta) = < g : L(g) < 2\delta >$.

This is *simplified* for M with $\tilde{M}$.

In general $\tilde{M}^\delta$ is the *Spanier cover* defined using the covering of M by $B_q(\delta)$ so that curves $\alpha\beta\alpha^{-1}$ where
the loop $\beta \subset B_q(\delta)$
lift as closed curves to $\tilde{M}^\delta$.

Lemma (S-Wei): When $\tilde{M}$ exists, then $g = [\alpha\beta\alpha^{-1}]$ where $\beta \subset B_q(\delta)$ are generated by g with $L(g) < 2\delta$.

When $\tilde{K}$ does not exist we still have
$CovSpec(K) \subset L(K)$.

Towards the cut-off spectra:

Defn: The R *cut-off* δ *cover* of M, $\tilde{M}^{\delta,R}_{cut}$, is the Spanier cover of M defined so that curves $\alpha\beta\alpha^{-1}$ where
$\beta \subset B_q(\delta)$ or $\beta \subset M \setminus \bar{B}_p(\delta)$
lift as closed curves.

Example 6: If M is an infinite solid handlebody then $\tilde{M}^{\delta,R}_{cut}$ is the cover which opens all handles except those with $L_i \geq 2\delta$ and those lying outside $\bar{B}_p(R)$.

Example 7: M a cylinder, then all the $\tilde{M}^{\delta,R}_{cut}$ are just M because all loops are freely homotopic to a loop lying outside $\bar{B}_p(R)$.

Recall Defn:
The R *cut-off delta cover* of M, $\tilde{M}^{\delta,R}_{cut}$, is the Spanier cover of M defined so that curves $\alpha\beta\alpha^{-1}$ where $\beta \subset B_q(\delta)$ or $\beta \subset M \setminus \bar{B}_p(\delta)$ lift as closed curves.

Defn:
The R *cut-off covering spectrum* of M,
$CovSpec^R_{cut}(M) =$

$$= \{\delta : \forall \delta' > \delta \; \tilde{M}^{\delta',R}_{cut} \neq \tilde{M}^{\delta,R}_{cut}\}$$

Recall Example 6:
M an infinite handlebody with k handles of length L_i lying inside $\bar{B}_p(R)$ has

$$CovSpec^R_{cut}(M) = \{L_1/2, L_2/2, ...L_k/2\}$$

Recall Example 7:
M is a cylinder. All the $\tilde{M}^{\delta,R}_{cut} = M$ because all loops are freely homotopic to a loop lying outside $\bar{B}_p(R)$. So

$$CovSpec^R_{cut}(M) = \emptyset.$$

Defn: The *cut-off delta cover* of M,

$\tilde{M}^{\delta}_{cut} = \lim_{R\to\infty} \tilde{M}^{\delta,R}_{cut}$.

Thm (S-Wei): It exists and is unique.

Defn: The *cut-off covering spectrum* $CovSpec_{cut}(M) =$

$= \{\delta : \forall \delta' > \delta \, \tilde{M}^{\delta'} \neq \tilde{M}^{\delta}\}$

Recall Example 6:
M an infinite solid handlebody with handles of length L_i, $\tilde{M}^{\delta}_{cut}$ unravels all handles with $L_i \geq 2\delta$, so $\tilde{M}^{\delta}_{cut} = \tilde{M}^{\delta}$ and $CovSpec_{cut}(M) = \{L_1/2, L_2/2, ...\}$.

Recall Example 7:
M is a cylinder. All the $\tilde{M}^{\delta,R} = M$. So $\tilde{M}^{\delta}_{cut} = M$ and $CovSpec_{cut}(M) = \emptyset$.

Thm: If M^n has the loops to infinity property then $CovSpec_{cut}(M) = \emptyset$.

Thm (S-Wei): For any complete M

$$CovSpec^R_{cut}(M) \subset CovSpec_{cut}(M)$$

Thm (S-Wei): For any complete M

$$CovSpec_{cut}(M) =$$
$$= Cl_{lower}(\bigcup_{R>0} CovSpec^R_{cut}(M))$$

Thm (S-Wei): Locally compact M

$$CovSpec^R_{cut}(M) \subset Length(M).$$

$$CovSpec_{cut}(M) = Cl_{lower}(Length(M))$$

The cut-off covering spectrum does not detect holes that:
** disappear into cusps* [Ex 2] *or*
** are asymptotic to a cylinders* [Ex 3].
It detects geodesics so it is a natural object of study for spectral theorists.

Soul Thm (Cheeger-Gromoll): If M has *sect* ≥ 0 then M has a compact totally geodesic submanifold, S, and M is a normal bundle over S.

Thm (Sharafutdinov): There is a distance nonincreasing deformation retraction from M to the soul S.

Cor (S-Wei): $CovSpec(M) \subset L(M)$ and is determined by the MLS.

Thm (S): If N has $Ricci \geq 0$ then N either has the *loops to infinity property* or it is a flat normal bundle over a compact totally geodesic soul, S^k.

Cor (S-Wei): $CovSpec_{cut}(N^n) = \emptyset$ or it is a flat normal bundle...

Summarizing our paper:

K compact, M locally compact:

$CovSpec(K) \subset (1/2)Length(K)$

$CovSpec^R_{cut}(M) \subset (1/2)Length(M)$

$CovSpec_{cut}(M) \subset Cl_{lower}((1/2)Length(M))$

Gromov-Hausdorff Convergence:

If K_i simply connected and $K_i \to K$
then $CovSpec(K) = \emptyset$.

Note M_i a sequence of stretching spheres
converging to M a cylinder
then $CovSpec(M) \neq \emptyset$.

If M_i simply connected and $M_i \to M$
then $CovSpec_{cut}(M) = \emptyset$.

The Rescaled Spectra:

Recall Defn: The *covering spectrum* was defined using lengths of $g \in \pi_1(M)$:

$$L(g) = \inf_{x\in M} d_{\tilde{M}}(g\tilde{x}, \tilde{x})$$

and groups $\pi_1(\delta) =< g \,:\, L(g) < 2\delta \,>$.

So $CovSpec(M) = \{\delta : \pi_1(\delta') \neq \pi_1(\delta)\, \forall \delta' > \delta\}$

Defn: The *rescaled covering spectrum* is defined the same way using the *rescaled length*

$$L_{rs}(g) = \inf_{x\in M} \frac{d_{\tilde{M}}(g\tilde{x},\tilde{x})}{d(x,p)}$$

$CovSpec_{rs}(M)$ is scale invariant and always ≤ 1.

Defn: The *infinite rescaled covering spectrum* is defined using the *infinite rescaled length*

$$L^{\infty}_{rs}(g) = \lim_{R\to\infty} \inf_{x\in M\setminus B_p(R)} \frac{d_{\tilde{M}}(g\tilde{x},\tilde{x})}{d(x,p)}$$

$CovSpec^{\infty}_{rs}(M) \in (0,1]$ is scale and basepoint invariant. $1 \in CovSpec^{\infty}_{rs}(M)$ if M has a handle. Other elements detect linearly opening holes.

Preprints soon to appear (S-Wei):

The Cut-off Covering Spectrum

Various Covering Spectra of Complete Spaces

Already published (S-Wei):

The covering spectrum of a compact length space
JDG 67 2004 (erratum JDG 74 2006, Ex 10.3)

All available on my webpage:

http://comet.lehman.cuny.edu/sormani

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