The Cut-off Covering Spectrum

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Abstract

We introduce the R cut-off covering spectrum and the cut-off covering spectrum of a complete length space or Riemannian manifold. The spectra measure the sizes of localized holes in the space and are defined using covering spaces called δ covers and R cut-off δ covers. They are investigated using δ homotopies which are homotopies via grids whose squares are mapped into balls of radius δ.

On locally compact spaces, we prove that these new spectra are subsets of the closure of the length spectrum. We prove the R cut-off covering spectrum is almost continuous with respect to the pointed Gromov-Hausdorff convergence of spaces and that the cut-off covering spectrum is also relatively well behaved. This is not true of the covering spectrum defined in our earlier work which was shown to be well behaved on compact spaces. We close by analyzing these spectra on Riemannian manifolds with lower bounds on their sectional and Ricci curvature and their limit spaces.

1 Introduction

THIS IS A PRELIMINARY PREPRINT, MORE SECTIONS WILL APPEAR AT THE END, INCLUDED STATEMENTS AND PROOFS HAVE BEEN CHECKED.

Complete length spaces and Riemannian manifolds are often studied using Gromov-Hausdorff convergence and Gromov's compactness theorem. However, this convergence, reviewed in Section 5, does not preserve the topology of the space. Thinner and thinner flat tori converge to circles, thus losing a generator of the fundamental group. Sequences of surfaces of higher and higher genus can converge to the Hawaii Ring, a space with an infinitely generated fundamental group and no universal cover [Example 2.1]. These surfaces have no lower bound on the sectional curvature. Menguy constructed examples of four dimensional manifolds with positive Ricci curvature which converge to length spaces of locally infinite topological type [Me].

In [SoWei3], the authors defined the covering spectrum of a compact length space, K. This spectrum measures the size of the one dimensional holes in the space and is closely related to the length spectrum: every element in the covering spectrum is half the length of a closed geodesic,

$$CovSpec(K) \subset (1/2)Length(K). \tag{1.1}$$

The covering spectrum is empty when the space is simply connected or is its own universal cover. It is determined using a sequence of covering spaces called δ *covers* which unravel curves that don't fit in balls of radius δ. We proved that when compact length spaces K_i converge in the Gromov-Hausdorff sense to a compact length space K, then their covering spectra converge in the Hausdorff sense:

$$d_H(CovSpec(K_i) \cup \{0\}, CovSpec(K) \cup \{0\}) \to 0. \tag{1.2}$$

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It is possible for elements to converge to 0 as they do on the sequence of thinner and thinner tori, at which point they disappear and are no longer in the covering spectrum. However, elements which converge to a positive value do not disappear in the limit. Furthermore, an element in $CovSpec(K)$ is the limit of elements in $CovSpec(K_i)$. In particular, the covering spectrum of the limit space, K, of a sequence of simply connected spaces, K_i, is empty [SoWei3].

When studying complete noncompact spaces, it is natural to employ *pointed Gromov-Hausdorff convergence*. The covering spectrum is not continuous with respect to this convergence. Sequences of manifolds, X_i with handles sliding out to infinity converge to a space X with no handles, so that we can have $\delta \in CovSpec(X_i) \forall i \in \mathbb{N}$ yet $CovSpec(X) = \emptyset$ [Example 5.1]. It is even possible for there to be an element in the covering spectrum of the limit space when $CovSpec(X_i) = \emptyset \forall i \in \mathbb{N}$ [Example 5.2]. These difficulties arise because the pointed Gromov-Hausdorff convergence is defined as the Gromov-Hausdorff limit of balls of radius R where the convergence can be slower as we take larger values of R.

Further difficulties are caused by the lack of compactness on a single space. Even on a locally compact space the covering spectrum is no longer closely related to the length spectrum on a noncompact space: there can be holes which extend to infinity and decrease in size [Examples 2.4 and 2.5]. Those that decrease to 0 are not detected by the covering spectrum and those that decrease to a constant cause an element in the covering spectrum to exist which is not 1/2 the length of a closed geodesic. This is explored in [SoWei4]. Here we define a new spectrum which resolves many of these difficulties.

In this paper we introduce the *R cut-off covering spectrum* and the *cut-off covering spectrum* to overcome these difficulties. The R cut-off covering spectrum of a pointed space (X, x) detects holes which do not extend outside the closed ball $\bar{B}_x(R)$. The cut-off covering spectrum detects holes which do not extend to infinity. A cylinder only has a hole extending to infinity, so its cut-off covering spectrum is empty. We prove that on locally compact length space X both of these spectra are contained in the closure of the length spectrum because the holes they detect are localized [Theorem 4.16 and Corollary 4.18]. Local compactness is seen to be necessary in Example 4.3.

We prove that the R cut-off covering spectrum is continuous with respect to the pointed Gromov-Hausdorff convergence of the locally compact spaces [Theorem 5.2]. This result is not an immediate extension of our compact results because the R cut-off spectrum is not uniformly localized: it detects any hole which passes into $\bar{B}_x(R)$ no matter how far out part of the hole extends. While the elements of the covering spectrum of a compact space are bounded above by the diameter of the space, there is no upper bound on an element in the R cut-off covering spectrum [Example 5.5]. In Example 5.5, one sees a sequence of X_i with increasingly large holes such that the hole snaps open to give a simply connected limit X. One aspect of our theorem says that if a sequence of spaces (X_i, x_i) have elements

$$\delta_i \in CovSpec^R_{cut}(X_i, x_i) \tag{1.3}$$

which diverge to infinity, then the holes they detect always snap open in the limit and are no longer holes at all.

Another difficulty in our noncompact setting arises from the fact that the R cut-off covering spectrum is defined using covering spaces and, as such, homotopies which extend far outside $\bar{B}_x(R)$ could influence the value of $CovSpec^R_{cut}(X, x)$. To handle this issue we develop the concept of the δ *homotopy* first introduced [SoWei1]. A closed curve is δ homotopic to a point if it lifts as a closed curve to the δ cover of the space. We introduce δ homotopies: maps from rectangular grids to the space which map squares into balls of radius δ [Lemma 3.4]. This allows us to control the location of the maps and, in particular, we prove that if a curve is δ homotopic to a point, then it is δ homotopic in a bounded region to a collection of possibly trivial loops lying near the boundary of that region [Lemma 3.7]. Later we apply this to localize subsets of the cut-off covering spectrum [Proposition 4.29]. We also bound the lengths of curves in a region A with certain δ homotopic properties in terms of the number of disjoint balls of radius $\delta/5$ that fit within A [Lemma 3.9]. This is useful later for uniformly bounding the size of holes which are detected by the R cut-off covering spectrum in a Gromov-Hausdorff converging sequence of balls.

Since the δ homotopy concepts and lemmas are of interest beyond their applications to the cut-off covering spectra, they are developed in Section 3 immediately following Section 2 which reviews the definition of the covering spectrum and provides a new simplified definition for spaces with universal covers [Theorem 2.11]. While the lemmas in Section 3 are very intuitive and have explanatory diagrams, the proofs are necessarily technical and may be skipped by the reader.

In Section 4 we introduce the cut-off covering spectra of pointed length spaces (X, x). We begin by defining the R cut-off δ covers, $\tilde{X}^{\delta,R}_{cut}$, which unravel curves that are not δ homotopic to loops outside $\bar{B}_x(R)$. We prove they have unique limits as R diverges to infinity and call these limits the cut-off δ covers[Prop 4.7]. These covers unravel holes which do not extend to infinity. The R cut-off covering spectrum, $CovSpec^R_{cut}(X, x)$ is defined using the R cut-off δ covers while the cut-off covering spectrum $CovSpec_{cut}(X)$ is defined using the cut-off δ covers and is basepoint invariant [Definitions 4.4 and 4.9].

In Section 4.3 we relate these spectra to the covering spectra and to each other, showing in particular that for any $R_1 < R_2$ and any basepoint $x \in X$ we have

$$CovSpec^{R_1}_{cut}(X, x) \subset CovSpec^{R_2}_{cut}(X, x) \subset CovSpec_{cut}(X) \subset CovSpec(X). \tag{1.4}$$

In Section 4.4, we prove Theorem 4.16 that for locally compact X, if $\delta \in CovSpec^R_{cut}(X)$ then $2\delta \in Length(X)$ which we write as $CovSpec^R_{cut}(X) \subset (1/2)Length(X)$. As a corollary we then show

$$CovSpec_{cut}(X) \subset Cl_{lower}((1/2)Length(X)) \tag{1.5}$$

where $Cl_{lower}(A)$ is the lower semiclosure of the set $A \subset \mathbb{R}$. The lower semiclosure is defined and explored in the appendix, where we prove any spectrum defined in a manner similar to these spectra are lower semiclosed sets [Theorem 6.5]. Example 4.2 demonstrates the necessity of the lower semiclosure in (1.5) .

In Section 4.5 we study various topological conditions on a complete length space. We first recall the loops to infinity property defined in [So] and relate this concept to the emptiness of the cut-off covering spectrum [Theorem 4.20 and Theorem 4.21]. Then we describe the cut-off covering spectrum on various topological spaces [Theorem 4.22].

In Section 4.6 we localize the R cut-off covering spectrum using the δ homotopies as mentioned above. Proposition 4.29 shows subsets of the R cut-off covering spectra agree on spaces with isometric balls of sufficient size.

In Section 4.7 we explore

$$CovSpec^{R_2}_{cut}(X) \setminus CovSpec^{R_1}_{cut}(X) \quad \text{when } R_2 > R_1. \tag{1.6}$$

In particular Propositions 4.30 and 4.31 together imply that these two spectra are equivalent for R_2 sufficiently close to R_1 on locally compact spaces .

In Section 5 we introduce Gromov-Hausdorff convergence, first reviewing the definitions. In Section 5.1 we provide examples demonstrating why the covering spectrum is not continuous with respect to pointed Gromov-Hausdorff convergence: elements can shrink to 0, disappear in the limit, suddenly appear in the limit, or diverge to infinity.

In Section 5.2 we prove the continuity of the R cut-off covering spectrum [Theorem 5.2] and provide examples clarifying why it is necessary to slightly change R to obtain this continuity. The proof requires two propositions: one controlling the fundamental groups of the R cut-off δ covers and the other proving the R cut-off δ covers converge. It also strongly relies on the results on δ homotopies and localization proven in the earlier sections.

In Section 5.3 we prove Theorem 5.5 which states that

$$\text{for any } \delta \in CovSpec_{cut}(X), \text{ there is } \delta_i \in CovSpec_{cut}(X_i) \tag{1.7}$$

such that $\delta_i \to \delta$. In particular if X_i are simply connected locally compact spaces that converge to a locally compact space X in the pointed Gromov-Hausdorff sense then $CovSpec_{cut}(X) = \emptyset$ [Corollary 5.6]. This limit space need not be simply connected as can be seen in Example 5.2.

In Section 5.4 we prove the pointed Gromov-Hausdorff limits of simply connected spaces either have the loops to infinity property or two ends [Theorem 5.7]. In Section 5.5 we investigate the cut-off covering spectra of tangent cones at infinity. WE WILL ADD SECTION 5.6 WITH FURTHER EXAMPLES.

We close the paper with Section **??** on applications to spaces with curvature bounds. THIS SECTION IS NOT YET COMPLETED.

2 Background

First we recall some basic definitions.

Definition 2.1 A *complete length space* is a complete metric space such that every pair of points in the space is joined by a length minimizing rectifiable curve. The distance between the points is the length of that curve. A *compact length space* is a compact complete length space (c.f. [BBI]).

Note that complete Riemannian manifolds are complete length spaces. Like geodesics in Riemannian manifolds, geodesics in length spaces are locally minimizing curves. Closed geodesics are geodesics from S^1 to the space.

Example 2.1 *A simple example of a complete length space that we will use repeatedly in this section is a collection of circles of various radii joined at a point, p. The distance between points on single circle is just the shorter arclength between them. Distances between points, q_1, q_2 on distinct circles is the sum of the shorter arclength from q_1 to p and the shorter arclength from q_2 to p. This space is called the Hawaii ring when the collection of radii is $\{1/j : j \in \mathbb{N}\}$.*

2.1 Spanier covers and δ-covers

Definition 2.2 We say $\bar{X}$ is a *covering space* of X if there is a continuous map $\pi : \bar{X} \to X$ such that $\forall x \in X$ there is an open neighborhood U such that $\pi^{-1}(U)$ is a disjoint union of open subsets of $\bar{X}$ each of which is mapped homeomorphically onto U by π (we say U is evenly covered by π).

Definition 2.3 [Sp, pp 62,83] We say $\tilde{X}$ is a *universal cover* of X if $\tilde{X}$ is a cover of X such that for any other cover $\bar{X}$ of X, there is a commutative triangle formed by a continuous map $f : \tilde{X} \to \bar{X}$ and the two covering projections.

Note that the Hawaii Ring does not have a universal cover. In fact the universal cover of a space need not be simply connected as can be seen by taking the double spherical suspension of the Hawaii Ring [Sp].

We now introduce a special collection of covers we will call Spanier covers as they are described in [Sp, Page 81].

Definition 2.4 *Let $\mathcal{U}$ be any collection of open sets covering Y. For any $p \in Y$, by [Sp, Page 81], there is a covering space, $\tilde{Y}_{\mathcal{U}}$, of Y with covering group $\pi_1(Y, \mathcal{U}, p)$, where $\pi_1(Y, \mathcal{U}, p)$ is a normal subgroup of $\pi_1(Y, p)$, generated by homotopy classes of closed paths having a representative of the form $\alpha^{-1} \circ \beta \circ \alpha$, where β is a closed path lying in some element of $\mathcal{U}$ and α is a path from p to $\beta(0)$.*

It is easy to see that a Spanier cover is a regular or Galois cover. That is, the lift of any closed loop in Y is either always closed or always open in a Spanier cover. In particular Spanier covers of a collection of circles of various radii will leave some or none of the circles as circles and unravel the other circles completely into a tree.

The following lemma is in Spanier [Sp, Ch.2, Sec.5, 8]:

Lemma 2.5 *Let $\mathcal{U}$ and $\mathcal{W}$ both be collections of open sets that cover Y. Suppose $\mathcal{U}$ refines $\mathcal{W}$ in the sense that for any open set W in $\mathcal{W}$ there is an open set $U \in \mathcal{U}$, such that $U \subset W$. Then the Spanier cover $\tilde{Y}_{\mathcal{U}}$ covers $\tilde{Y}_{\mathcal{W}}$.*

Spanier covers will be used to define various covering spaces in this paper as well as the δ covers first introduced by the authors in [SoWei1].

Definition 2.6 Given $\delta > 0$, the δ-cover, denoted $\tilde{Y}^\delta$, of a length space Y, is defined to be the Spanier cover, $\tilde{Y}_{\mathcal{U}_\delta}$, where $\mathcal{U}_\delta$ is the open covering of Y consisting of all open balls of radius δ.

The covering group will be denoted $\pi_1(Y, \delta, p) \subset \pi_1(Y, p)$.

In Example 2.1, the δ cover of the space consisting of circles of various sizes glued at a common point, is a covering space which unravels all the circles of circumference $2\pi r \geq 2\delta$ and keeps the smaller circles wrapped as circles. In particular, when X is the figure eight created by joining one circle of circumference 2π and one circle of circumference 4π at a common point: then $\tilde{X}^\delta$ is X itself when $\delta > 2\pi$, it is a real line with circles of circumference 2π glued at the points $\{2j\pi : j \in \mathbb{Z}\}$ when $\delta \in (\pi, 2\pi]$ and it is the universal cover $\tilde{X}$ when $\delta \leq \pi$.

The δ-covers of compact spaces are surveyed quickly in the background section of [SoWei3]. There we proved that δ covers of complete length spaces are monotone in the sense that if $\delta_1 < \delta_2$ then $\tilde{Y}^{\delta_1}$ covers $\tilde{Y}^{\delta_1}$ which just follows from Lemma 2.5. See [SoWei3, Lemma 2.6].

If one has a space where balls of radius δ_1 and δ_2 have the same topology, the covering spaces are the same. In fact, for compact spaces, we proved the δ covers are lower semicontinuous in the sense that for any $\delta_1 > 0$ there is a $\delta_2 < \delta_1$ sufficiently close to δ_1 such that the two delta covers agree [SoWei3, Lemma 2.7]. This is not true for complete noncompact spaces. In fact, the space of circles of circumference $2\pi r_i$ joined at a point have distinct delta covers for each $\delta_i = \pi r_i$ so that lower semicontinuity fails when there is a sequence r_i increasing to r_0 [SoWei3, Example 2.8].

2.2 Review of the Covering Spectrum

In [SoWei3] we introduced the covering spectrum on compact metric spaces which is well defined on complete noncompact spaces as well.

Definition 2.7 *Given a complete length space X, the covering spectrum of X, denoted CovSpec(X) is the set of all $\delta > 0$ such that*

$$\tilde{X}^\delta \neq \tilde{X}^{\delta'} \tag{2.1}$$

for all $\delta' > \delta$.

The covering spectrum of a finite collection of circles of circumference $2\pi r_i$ joined (glued) at a common point is $\{\pi r_i\}$.

For a compact length space the covering spectrum is discrete and the only accumulation point of the covering spectrum that can occur outside of the covering spectrum is 0 [SoWei3, Prop. 3.2]. This happens for example with the Hawaii Ring where the circles have circumference $2\pi r_j = 2\pi/j$.

The covering spectra of complete noncompact spaces need not be discrete:

Example 2.2 *The covering spectrum of a complete noncompact length space can be $(0, \infty)$ as can be seen by joining the uncountable collection of circles of circumference $2\pi r$ for every $r \in (0, \infty)$ at a common point. This same covering spectrum can be achieved by taking a joined countable collection of circles of circumference $2\pi r$ for every $r \in \mathbb{Q}$.*

The following lemma is a simple exercise on the definition:

Lemma 2.8 *If $\delta_j \in CovSpec(X)$ and δ_j decrease to a positive limit $\delta_0 > 0$, then $\delta_0 \in CovSpec(X)$.*

Example 2.3 *Thus the covering spectrum of the joined collection of circles of circumference $2\pi r_j = 2\pi + 2\pi/j$, is $\{\pi(1+1/j) : j \in \mathbb{N}\} \cup \{\pi\}$. In contrast the covering spectrum of the joined collection of circles of circumference $2\pi r_j = 2\pi - 2\pi/j$ is just $\{\pi(1-1/j) : j \in \mathbb{N}\}$.*

This is just an indication of the complexity one encounters when studying the covering spectra of complete noncompact spaces. In the next section we explore this situation, and in subsequence sections we introduce alternative spectra which detect properties that the covering spectrum cannot detect on a complete noncompact space. Further review of the covering spectra of compact spaces will appear below.

2.3 The Covering Spectrum and Deck Transforms

In our prior papers, we did not like to assume the space had a universal cover in part because we were applying δ-covers and the covering spectrum to prove the existence of universal covers. However, if one does assume the existence of a universal cover there is a fairly beautiful new perspective on the meaning of the covering spectrum using its relationship with the group of deck transforms $\pi_1(Y)$ on the universal cover, $\tilde{Y}$.

Recall that a δ cover is defined using a covering group $\pi_1(Y,\delta)$, so with a universal cover we have:

$$\tilde{Y}^\delta = \tilde{Y}/\pi_1(Y,\delta). \tag{2.2}$$

This provides us with an equivalent definition for the covering spectrum:

Definition 2.9 *Given a complete length space X, with a universal cover, $\tilde{X}$, the covering spectrum of X is the set of all $\delta > 0$ such that*

$$\pi_1(X,\delta) \neq \pi_1(X,\delta') \qquad \forall \delta' > \delta \tag{2.3}$$

when viewed as subsets of $\pi_1(X)$.

The δ-covering group as described in the definition of the δ-cover is complicated without the assumption of the existence of a universal cover. With the universal cover, one knows it is a subgroup of the deck transforms and in fact we shall see it can be described using the length of these transforms.

Definition 2.10 *Given a complete length space X with universal cover $\tilde{X}$, for each element $g \in \pi_1(X)$, its length $L(g)$ is*

$$L(g) = \inf_{\tilde{x} \in \tilde{M}} d(\tilde{x}, g\tilde{x}). \tag{2.4}$$

Theorem 2.11 *The δ covering group $\pi_1(Y,\delta)$ is the subgroup of $\pi_1(Y)$ generated by elements g with $L(g) < 2\delta$. Thus $\delta \in CovSpec(Y)$ iff there exists δ_i decreasing to δ and $g_i \in \pi_1(Y)$ of length $L(g_i) < 2\delta_i$ such that g_i is not generated by elements of length $< 2\delta$.*

To prove this we need to relate the loops β in balls of radius δ that were used to define $\pi_1(Y,\delta)$ in Defn 2.6, to deck transforms of length $< 2\delta$. The difficulty is that the loops β might well be quite long and correspond to deck transforms of large length. So we apply the following lemma first proven in [SoWei3, Lemma 5.8].

Lemma 2.12 *Given a complete length space X, suppose $C : [0, L] \to B_q(\delta) \subset X$ where X is a complete length space, then C is freely homotopic in $B_q(\delta)$ to a product of curves of length $< 2\delta$ based at q.*

Note that this lemma does not require the existence of a universal cover. We include the proof since the idea is of some importance.

Proof: Since $B_q(\delta)$ is open and the image of C is closed there exists $\epsilon > 0$ such that $Im(C) \subset B_q(\delta - \epsilon)$. Take a partition, $0 = t_0 < t_1 < ... < t_k = L$, such that $t_{j+1} - t_j < \epsilon$, and let γ_j run minimally from q to $C(t_j)$ making sure to choose $\gamma_0 = \gamma_k$. Then C is clearly freely homotopic in $B_q(\delta)$ to the combination $\gamma_j C([t_j, t_{j+1}])\gamma_j^{-1}$, and each of these curves has length $< 2(\delta - \epsilon) + \epsilon < 2\delta$. ■

Applying this lemma we can prove Theorem 2.11 concerning spaces with universal covers.

Proof of Theorem 2.11: Let g be a generator of $\pi_1(Y, \delta)$ so it has a representative loop β in a ball od radius δ. By the Lemma 2.12, β is generated by loops of length $< 2\delta$, so g is generated by deck transforms of length $< 2\delta$

Now let $L(h) < 2\delta$ so h has a representative path β whose lift runs between some x and hx of length $< 2\delta$. But then β is in a ball of radius δ around its midpoint, so $g \in \pi_1(Y, \delta)$. ■

On compact spaces Theorem 2.11 can be combined with Arzela-Ascoli and Lemma 3.2 to prove $CovSpec(M) \subset (1/2)Length(M)$ where $Length(M)$ is the collection of lengths of closed geodesics $\gamma : S^1 \to M$ [SoWei3]. This is not true on complete spaces as the infimum in (2.4) need not be achieved:

Example 2.4 *Let M^2 be the warped product manifold $\mathbb{R} \times_{f(r)} S^1$ where*

$$f(r) = 2Arctan(-r) + 2\pi. \tag{2.5}$$

Here $\pi_1(M)$ is generated by a single element g whose length

$$L(g) = \inf_{r \in (-\infty, \infty)} f(r) = \pi \tag{2.6}$$

but there is no closed curve homotopic to a representative of g whose length is π.

On a compact Riemannian manifold, $CovSpec(M) = \emptyset$ implies M is simply connected [SoWei3]. Yet this is not true for complete manifolds:

Example 2.5 *Let M^2 be the warped product manifold $\mathbb{R} \times_{f(r)} S^1$ where*

$$f(r) = 2Arctan(-r) + \pi. \tag{2.7}$$

Given any $\delta > 0$, eventually $f(r) < 2\delta$, so any $g \in \pi_1(M)$ is represented by a loop of length $< 2\delta$. Thus by Theorem 2.11 the covering spectrum is empty.

Further implications of this perspective on the covering spectrum will be investigated in [SoWei4]. In that paper we will also investigate the slipping group:

Definition 2.13 *The slipping group of X denoted $\pi_{slip}(X)$ is generated by the elements $g \in \pi_1(X)$ such that $L(g) = 0$.*

3 Delta homotopies

In this section we develop the concept of the delta homotopy which we first defined in [SoWei1]:

Definition 3.1 *Two loops γ_1, γ_2 in X are called δ-homotopic if $\pi_\delta([\gamma_1]) = \pi_\delta([\gamma_2])$, where $\pi_\delta : \pi(X) \to \pi(X)/\pi(X,\delta)$. In particular γ_1 is δ homotopic to a point if*

$$[\gamma_1] \in \pi(X, \delta) \tag{3.1}$$

which means γ_1 lifts as a closed loop to $\tilde{X}^\delta$.

This concept can be used to produce closed geodesics.

Lemma 3.2 *If γ is not δ homotopic to a point and $L(\gamma) \le 2\delta$ then γ is a closed geodesic which is minimizing over any interval of half its length and has length 2δ.*

Proof: Since γ lifts as a closed loop to $\tilde{X}^\delta$ it does not fit in a ball of radius δ. In particular, for any $t \in S^1_{\delta/\pi}$ we have

$$Im(\gamma) \cap (X \setminus B_{\gamma(t)}(\delta)) \neq \emptyset. \tag{3.2}$$

However $L(\gamma) = 2\delta$ so the only point in (3.2) must be $\gamma(t+\delta)$ and $d(\gamma(t+\delta), \gamma(t))$ must be δ. Thus γ is minimizing on any subinterval of length δ including an interval centered at $t = 0$. ■

This lemma will be applied later when we prove our new spectra are in the length spectrum.

The remiander of this section will be dedicated to providing a more geometric understanding of δ homotopies. We will first relate δ homotopies to grids [Section 3.1], then describe how to localize δ homotopies [Section 3.2] and finally prove a few properties of δ homotopies that are localized in compact sets [Section 3.3].

While we apply the results in this section to study the cut-off covering spectrum, we prove them first because they apply in a much more general setting and should prove useful for those interested in other concepts. Those who are more interested in the cut-off covering spectrum may jump to Section 4 and only return to this section before continuing to Section 5 on Gromov-Hausdorff convergence. Alternatively one might skim through this section reading only the statements and viewing the acompanying diagrams.

3.1 Using grids to understand δ homotopies

Before we can transform our original somewhat algebraic definition of δ homotopy [Definition 3.1] into a geometric statemant about grids, we need to examine the definition closely. Clearly it is base point independent. So if a curve C is δ homotopic to a point the $\alpha C \alpha^{-1}$ is also δ homotopic to a point. So it is often easier to think of γ_1 as δ homotopic to γ_2 if we joint them to a common point via curves α_1 and α_2 and then say $\alpha_1\gamma_1\alpha_1^{-1}$ is δ homotopic to $\alpha_2\gamma_2\alpha_2^{-1}$ which is the same as saying

$$\alpha_1\gamma_1\alpha_1^{-1}(\alpha_2\gamma_2\alpha_2^{-1})^{-1} = \alpha_1\gamma_1\alpha_1^{-1}\alpha_2^{-1}\gamma_2^{-1}\alpha_2 \tag{3.3}$$

is δ homotopic to a point. In this sense we make the following definition:

Definition 3.3 *A collection of loops $\gamma_1, \gamma_2, ...\gamma_k$ is δ homotopic to a point if there exist curves α_i mapping a base point p to $\gamma_i(0)$ and such that*

$$\alpha_1\gamma_1\alpha_1^{-1}\alpha_2\gamma_2\alpha_2^{-1}...\alpha_k\gamma_k\alpha_k^{-1} \tag{3.4}$$

is δ homotopic to a point.

The ordering of the loops is important in this definition. If γ_1, γ_2 is δ homotopic to a point then γ_1 is δ homotopic to γ_2^{-1}.

Lemma 3.4 *A loop C of length L is δ homotopic to a point iff there is a δ homotopy $H : G \to X$ where G is an $N \times M$ grid of unit squares such that $H(0, y) = C(yL/M)$, $H(x, 0) = H(x, M) = H(N, y) = C(0)$ and such that the image under H of each square in the grid is contained in a ball of radius δ.*

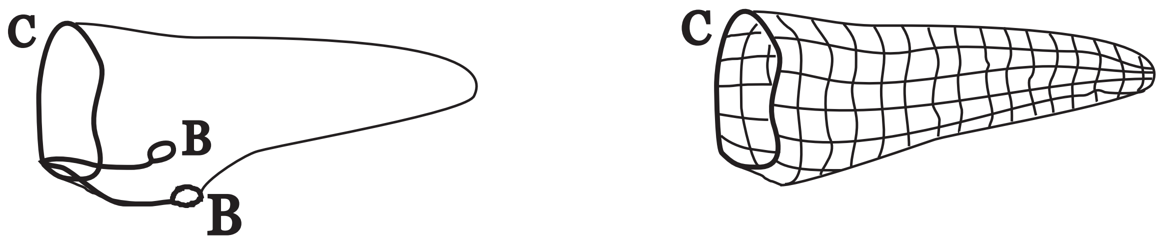

Figure 1:

In some sense this lemma is intuitively obvious. See Figure 1. Special cases of this lemma were used within some of the proofs in [SoWei1]. Writing out the proof is a bit technical and so first we set some notation. Let $\beta_{j,k}$ be image of the clockwise loop around the square $(j, k), (j, k + 1), (j + 1, k + 1), (j + 1, k)$. Let $\alpha_{j,k}$ be the image of the line segment from $(j, 0)$ to (j, k). Let $\bar{\alpha}_j$ is the image of the line segment from $(j, 0)$ to $(j - 1, 0)$.

Proof: If such a homotopy exists, then define $C_j(t)$ to be the loop $H(j, t)$ from t to M so $C_0(t) = C(tL/M)$ and $C_N(t)$ is a point. Note that C_0 is just C. Furthermore each

$$\alpha_{j,0}\beta_{j,0}\alpha_{j,0}^{-1}\alpha_{j,1}\beta_{j,1}\alpha_{j,1}^{-1}...\alpha_{j,M}\beta_{j,M}\alpha_{j,M}^{-1} \text{ is homotopic to } C_j(t)(\bar{\alpha}_j C_{j-1}(t)\bar{\alpha}_j^{-1})^{-1} \tag{3.5}$$

within the image of the grid. Thus by the definition of the δ cover,

$$C_j(t)(\bar{\alpha}_j C_{j-1}(t)\bar{\alpha}_j^{-1})^{-1} \tag{3.6}$$

lifts as a closed loop to the $\tilde{X}^\delta$ and so C_j and C_{j-1} are δ homotopic to each other. Thus C is δ homotopic to C_N which is a point.

Conversely, if C is δ homotopic to a point, then by the definition of the δ cover, C is homotopic to a collection of curves $\alpha_i\beta_i\alpha_i^{-1}$ where β_i are in balls of radius δ. So we take the homotopy $\bar{H} : [0, N] \times [0, M] \to X$ so that $\bar{H}(0, t) = C(tM/L)$, $\bar{H}(s, 0) = \bar{H}(s, 1) = C(0)$ and

$$\bar{H}(N, t) = \alpha_1\beta_1\alpha_1^{-1}\alpha_2\beta_2\alpha_2^{-1}...\alpha_k\beta_k\alpha_k^{-1}(t) \tag{3.7}$$

Using the uniform continuity of the homotopy $\bar{H}$ we can choose N and M large enough that each square in the grid is within a ball of radius δ. We can also insure, possibly by adding a few more columns to allow for a slow homotopy between reparametrizations, that each β_j starts at a t_j and ends at a $t_j + 1$ where t_j are integers.

We now add a gridded column of unit squares on the right side of the homotopy. The horizontal bars will have constant images. The verticals will agree with $\bar{H}(N, t)$ whenever this is part of an α curve but will take the value $\bar{H}(N, t_j)$ for $t \in [t_j, t_j + 1]$. In this way most of the new squares will be in subsegments of the α curves, and the selected new squares at the t_j points will have images equal to β_j and thus lie in balls of radius δ.

Finally we add a number more columns to allow for a homotopy from the curve

$$\alpha_1\alpha_1^{-1}\alpha_2\alpha_2^{-1}...\alpha_k\alpha_k^{-1} \tag{3.8}$$

to a point. This can be done just by contracting along each α_j. In this way we complete the homotopy. Then we restrict the homotopy to the grid points and we are finished. ■

Lemma 3.5 *If H is a δ homotopy, then there exists $\epsilon \in (0, \delta)$ sufficiently close to δ that H is an ϵ homotopy.*

In fact on compact spaces, one then has $\tilde{X}^\epsilon = \tilde{X}^\delta$ as proven in Lemma 2.7 of [SoWei3].

Proof: By Definition 3.6, every square $S_{i,j}$ in the domain of H is mapped into a ball $B_{q_{i,j}}(\delta)$. Since $H(S_{i,j})$ is a closed set, lying in an open ball, it fits in a smaller open ball $B_{q_{i,j}}(\delta_{i,j})$ with $\delta_{i,j} < \delta$. Let

$$\epsilon = \max\{\delta_{i,j} : i = 1..N,\ j = 1..M\}. \tag{3.9}$$

■

3.2 δ homotopies in subsets

The following extension of the definition of δ homotopy takes full advantage of Lemma 3.4. Note that this extension only requires X to be a length space so that distances between curves are not necessarily achieved by curves, just approached by a sequence of curves.

Definition 3.6 *A loop C of length L is δ homotopic to a point in a set A of a length space X iff there is a δ homotopy $H : G \to A$ where G is an $N \times M$ grid of unit squares such that $H(0, y) = C(yL/M)$, $H(x, 0) = H(x, M) = H(N, y) = C(0)$ and such that the image under H of each square in the grid is contained in a ball of radius δ.*

We say a curve C_0 is δ homotopic in A to a collection of curves $C_1, C_2, ...C_k$, if there exists paths α_j from $C_0(0)$ to $C_j(0)$ lying in A such that

$$C_0^{-1}\alpha_1 C_1 \alpha_1^{-1}\alpha_2 C_2 \alpha_2^{-1}...\alpha_k C_k \alpha_k^{-1} \tag{3.10}$$

is δ homotopic in A to a point. Similarly one can define δ homotopies in A between two collections of curves.

Suppose we have a curve which is δ homotopic in a set A to a point and we would like to restrict the δ homotopy to a set $B \subset A$. Parts of the δ homotopy may well leave B and so they need to be chopped off. This provides new curves where the homotopy is chopped. See Figure 4 for a glimpse at an application.

Lemma 3.7 *Given a δ homotopy in A, $H : G \to A$ from a curve γ to a point, and given a set B contained in A such that $\gamma \subset B$, then γ is δ homotopic in B to a collection of curves $\gamma_1, \gamma_2, ...\gamma_j$ such that each γ_j lies in B and the tubular neighborhood $T_{2\delta}(A \setminus B)$.*

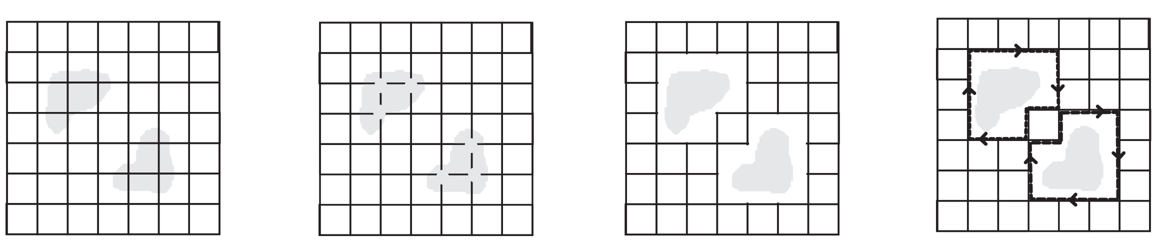

Figure 2:

Figure 2 depicts this lemma and the idea of the proof. The grey regions are the pullback of the $A \setminus B$ to the grid of the initial homotopy. In the figure the collection is just a pair of curves. Once we have the final picture in the figure, we can apply Lemma 3.8 It is possible that γ itself just lies in $T_{2\delta}(A \setminus B)$. Before we can prove of Lemma 3.7 rigorously, we need another lemma which justifies that images of the pair of curves produced in the last step of the picture are indeed δ homotopic to the initial curve.

Lemma 3.8 *Given a δ homotopy in A, $H : G \to A$ from a curve γ to a point, and a subset of squares $G' \subset G$ such that the image of $Cl(G \setminus G')$ is contained in a set $B \subset A$. Here by closure, we are including the boundary of G'. Suppose G' has connected components $G_1, ...G_k$. Let γ_j be the boundary of G_j running around clockwise so that the image of γ_j lies in B.*

Then γ is δ homotopic in B to the collection of curves $\alpha_j \gamma_j \alpha_j^{-1}$ where α_j are paths lying in B or, equivalently, is freely δ homotopic in B to the collection of curves γ_j.

Intuitively this can be seen because there are only squares that fit in balls of radius δ running between them. You might wish to skip the proof if you intuitively believe the process. For the intuitive idea see Figure 3.

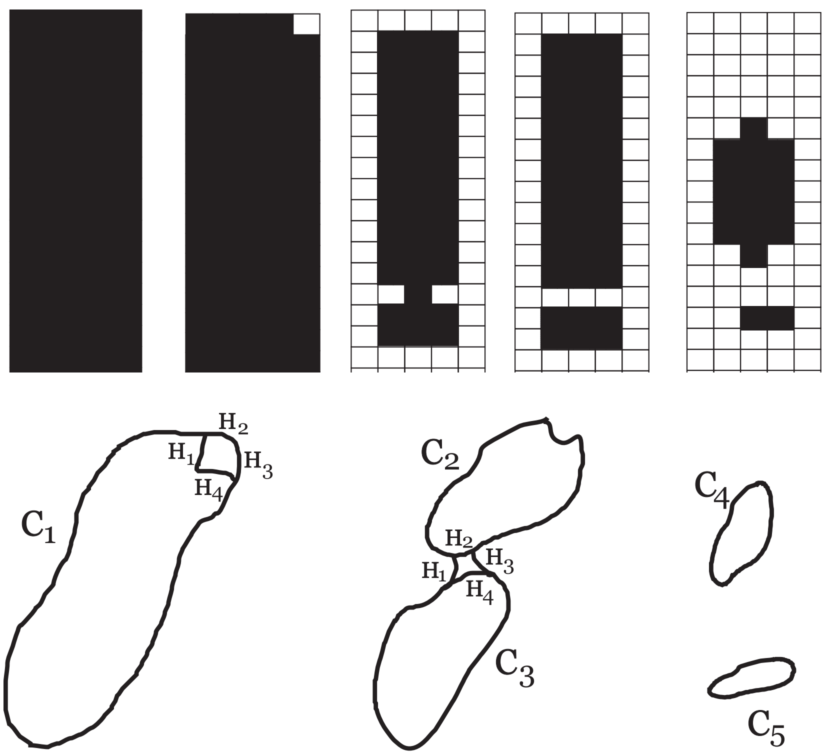

Figure 3: In this figure $\gamma_0 = C_1 H_2 H_3$ and we are proving it is δ homotopic in B to the pair of curves $\gamma_1 = C_4$ and $\gamma_2 = C_5$. The grids are drawn above. The first rectangle is filled in completely so we can view our γ_0 as the boundary of the full dark region. The last rectangle has G' darkened and it's two connected components G_1 above G_2. Intuitively we are saying that the two inner curves should be δ homotopic to the outer curve because of all the squares between them. The rectangles in between show how we can run through a sequence of subsets of the grid creating a δ homotopy from γ_0 to the pair γ_1 and γ_2.

Proof: We now rigorously construct a sequence of collections of curves so that each collection is δ homotopic to the next. We begin with γ_0 which is the image of the boundary of the entire grid $G_0 = G$. Each G_i will be a subset of G_{i-1} created by removing one square, and at each step our collection of curves will be the boundary of G_i. We know that we can create a sequence of G_i so that eventually we arrive at $G_I = G'$. We just need to verify that we have a δ homotopy running from each boundary to the next. There are three cases.

The first case we encounter occurs when removing a square does not change the number of connected components of the subgrid. This is seen in the first part of Figure 3. A square is removed from the side on one region. We need to show that a curve of the form $C_1H_2H_3$ is δ homotopic to $C_1H_1^{-1}H_4^{-1}$ when H_1, H_2, H_3H_4 is a loop in a ball of radius δ because it is the image of a single square. To construct the δ homotopy, we set $H(0,t)$ to be the required

$$C_1H_2H_3(C_1H_1^{-1}H_4^{-1})^-1 = C_1H_2H_3H_4H_1C_1^{-1}. \tag{3.11}$$

This time we put all of $H_2H_3H_4H_1$ into one integer segment and stretch the C_1 enough that each segment lies in a δ ball. We add the second column to the grid keeping everything as in the first column except for the $H_2H_3H_4H_1$ segment which is now just set to $H_2(0) = C_1(L)$. Thus the image of the grid thus far is contained in the images of the old curves which is in B and all the squares are in δ balls trivially. The rest of the homototy is a classical homotopy contracting $C_1C_1^{-1}$ to a point and we take as many columns as necessary so that everything is done slowly enough to fit in balls of radius δ. This portion is contained in $Im(C_1) \subset B$ so we are done. It is also possible that the square would be attached on only one side, but this is equally easy.

The second possible case, depicted in the center of Figure 3 is when the square which is removed creates divides a region into two connected components. So we must show that $C_3H_1C_2H_3$ is δ homotopic in B to the pair of curves $C_3H_4^{-1}$ and $C_2H_2^{-1}$ given that $H_1H_2H_3H_4$ is the image of a square and so lies in a ball of radius δ. Using H_4H_1 to run C_2H_2 to a common base point, we will construct a δ homotopy in B from

$$(C_3H_1C_2H_3)((H_4H_1)C_2H_2^{-1}(H_4H_1)^{-1})^{-1}(C_3H_4^{-1})^{-1} \tag{3.12}$$

to a point.This is already homotopic within its image to

$$C_3H_1C_2H_3(H_4H_1)H_2C_2^{-1}(H_4H_1)^{-1}H_4C_3^{-1} \tag{3.13}$$

which is homotopic within its range to

$$C_3H_1C_2H_3(H_4H_1)H_2C_2^{-1}H_1^{-1}C_3^{-1} \tag{3.14}$$

Once again we set this up as the first column so that each collection of curves H_j fit in a single unit segment and the C_j are spread out so that they divided into pieces of length less than δ. Our second column will be set up so that all the horizontal bars are constant and the new vertical line is the same as before except that the segment with $H_3(H_4H_1)H_2$ is not just the fixed point $H_3(0) = C_2(L_2)$. So our new column is

$$C_3H_1C_2C_2^{-1}H_1^{-1}C_3^{-1} \tag{3.15}$$

but this can be contracted via a homotopy lying on its image to a point, so we just provide that homotopy enough columns so that the images of all the squares lie in δ balls. So we are done. Note that the order of the new collection was important so that this last step would untangle.

In fact there are cases where the square that is removed might separate into three or even four regions. This follows exactly as above the the regions need to be selected in clockwise order around the square to get the last step to untangle.

The third case is the situation where removing a square removes a segment from the collection. That situation is trivial. Anytime a collection of curves includes a loop within a δ ball it is δ homotopic to the collection with the ball removed.

Thus we have shown that no matter how we remove the square, we can show that each collection of curves is δ homotopic to the next collection carefully replacing one curve by a new curve or a new curve by a collection of new curves in the right order until finally one has the boundary of the given region G'. ■

We can now return to the proof of Lemma 3.7. See Figure 2.

Proof of Lemma 3.7: Let H be the given δ homotopy. Remove all vertices in G which are mapped by H into $A \setminus B$. Remove all the squares touching these vertices. This gives our collection of squares G' which satisfies the condition of Lemma 3.8. So we obtain a collection $\gamma_1, ...\gamma_k$ which are δ homotopic in B to γ where each γ_j lies in the boundary of G'. Thus every point q which lies on a γ_j, is on the image of a square which includes one of the original removed points z. So q and z lie in a common ball of radius δ and $z \in A \setminus B$. Thus $q \in T_{2\delta}(A \setminus B)$. ■

3.3 Compactness and δ homotopies

One very nice attribute of δ homotopy classes of curves is that they interact well with compactness so that one can control the lengths of curves in a given class.

Lemma 3.9 *Let $\mathcal{C}$ be a set of loops in a length space Z which includes a trivial loop that is just a point. Suppose there is a curve C in Z which is not $\delta = 5\rho$ homotopic to any collection of curves in $\mathcal{C}$ has length. Suppose the number of disjoint balls of radius ρ lying in Z is bounded above by a finite number N. Then there exists a curve γ in Z which is not $\delta = 5\rho$ homotopic to any collection of curves in $\mathcal{C}$ and has length $\leq 5N\rho$.*

Note here that we cannot just take C to be a trivial loop (a point), because then it would be homotopic to the trivial loop in $\mathcal{C}$. In our application $\mathcal{C}$ will be all loops located outside a given set but this more general statement is equally valid and possibly useful to others. Our space Z will be a subset of a larger space using the induced length metric and thus might not be complete.

Proof: Take a maximal disjoint collection of balls of radius ρ centered at points

$$Y = \{y_j : j = 1..N\} \in Z. \tag{3.16}$$

So the tubular neighborhood of radius 2ρ of this finite collection of points contains all of Z.

Take $C : [0, L] \to Z \subset T_{2\rho}(Y)$ parametrized by arclength which is not 5ρ homotopic to any collection of curves in $\mathcal{C}$. We will use C to construct a shorter such curve. Define $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $t_j - t_{j-1} = \rho$ for $j < k$ and $t_k - t_{k-1} < \rho$. Define $\sigma : [0, L] \to Z$ so that $\sigma(t_j)$ is a point in Y closest to $C(t_j)$. Then

$$d_A(\sigma(t_j), \sigma(t_{j+1}) \leq d_A(\sigma(t_j), C(t_j) + \rho d_A(C(t_{j+1}), \sigma(t_{j+1}) < 5\rho \tag{3.17}$$

and we can join the points in σ by curves in A of length $< 5\rho$. We can also join $C(t_j)$ to $\sigma(t_j)$ by a curve h_j in A of length $< 2\rho$. Thus we have a collection of squares

$$h_j \sigma([t_j, t_j + 1]) h_{j+1}^{-1} C([t_j, t_{j+1}])^{-1} \subset B_{\sigma(t_j)}(5\rho). \tag{3.18}$$

So C is 5ρ homotopic to σ. Thus σ is not 5ρ homotopic to any collection of curves in $\mathcal{C}$. If $k \leq N$ then $L(\sigma) \leq 5k\rho \leq 5N\rho$ and we are done.

If $k > N$ then by the pigeon hole principle and the fact that $\sigma(t_j) \in Y$ for $j = 0..k$. with $\sigma(t_0) = \sigma(t_k)$. We see that there must exist a pair $m, n \in \{0, ...k-1\}$ with $|m - n| \leq N$ such that $\sigma(t_n) = \sigma(t_m)$. This allows us to break our loops σ into two loops one of which is of length $\leq 5\rho N$. If the other loop is longer, apply the pigeon hole principle to that loop, and break off another loop of length $\leq 5\rho N$. Repeating this at most finitely many times, we see that our original curve σ is really a concatenation of loops all of which have length $\leq 5\rho N$.

I claim one of these short loops must not be 5ρ homotopic to any collection of curves in $\mathcal{C}$. Otherwise, all the of them are 5ρ homotopic to some collection of curves in $\mathcal{C}$ and so their concatenation must be 5ρ homotopic to a concatenation of that collection. ■

4 The Cut-off Covering Spectrum

It is natural when studying complete noncompact spaces to remove the ends of the manifolds before beginning the analysis. In fact, it is standard to refer pointed spaces (X, x) with a special base point $x \in X$. In this vein of thought, we define the cut-off covering spectra. We begin by defining the R cut-off δ covers and R cut-off covering spectra, $CovSpec^R_{cut}(X)$, which are blind to everything outside a fixed ball of radius R as trivial. Next we define the cut-off δ covers by taking $R \to \infty$ and define the cut-off covering spectrum, $CovSpec_{cut}(X)$, based on them.

While the covering spectrum is not well related to the Length spectrum on complete noncompact spaces as was seen in Example 2.4, we do prove the $CovSpec^R_{cut}(X) \subset (1/2)L(X)$ and $CovSpec_{cut}(X) \subset Cl_{lower}((1/2)L(X))$. We then review the loops to infinity property, and prove such loops are not detected by the cut-off covering spectra. We close the section with two technical subsections: one establishing that the R cut-off covering spectrum is truly localized and the other describing how $CovSpec^R_{cut}(X)$ changes as one varies R. These results will be applied to establish the continuity properties of these cut-off spectra in Section 5.

4.1 The R cut-off δ covers and $CovSpec^R_{cut}(X)$

The R cut-off covering spectrum is a basepoint dependant concept. It is defined on pointed length spaces (X, x) which are length spaces with given basepoints. We begin with the corresponding covering spaces. Recall Defn 2.4 of a Spanier Cover.

Definition 4.1 *Given a pointed length space (Z, x), the R cut-off δ cover based at x, denoted $\tilde{X}^{\delta,R}_{cut}$ or $\tilde{X}^{\delta,R}_{cut\,x}$, is the Spanier cover corresponding to the open sets*

$$\{B_p(\delta) : p \in X\} \cup \{X \setminus \bar{B}_x(R)\}. \tag{4.1}$$

When the basepoint is obvious we will omit it.

Lemma 4.2 *The R cut-off δ cover based at x is covered by the δ cover. In fact*

$$\tilde{X}^{\delta,R}_{cut} = \tilde{X}^{\delta}/G(R) \tag{4.2}$$

where $G(R)$ is the subgroup of π_1 generated by elements with representative loops of the form $\alpha \circ \beta \circ \alpha$ where $\beta \in M \setminus \bar{B}_x(R)$.

Proof: By definition $\pi_1(\tilde{X}^{\delta,R}_{cut})$ is generated by loops of the form $\alpha \circ \beta \circ \alpha^{-1}$ where β is either in a ball of radius δ or in $M \setminus \bar{B}_x(R)$. So it is generated by elements in $\pi_1(\tilde{X}^{\delta})$ and elements in $G(R)$. Thus

$$\tilde{X}^{\delta,R}_{cut} = \tilde{X}/\pi_1(\tilde{X}^{\delta,R}_{cut}) = (\tilde{X}/\pi_1(\tilde{X}^{\delta}))/G(R) = \tilde{X}^{\delta}/G(R). \tag{4.3}$$

Lemma 4.3 *If $B_{x_1}(R_1) \subset B_{x_2}(R_2)$ and $\delta_1 \le \delta_2$, then $\tilde{X}^{\delta_1,R_1}_{cut}$ based at x_1 covers $\tilde{X}^{\delta_2,R_2}_{cut}$ based at x_2.*

Proof: Just apply Lemma 2.5 which is proven in Spanier. ■

Example 4.1 *A cylinder is its own R cut-off δ cover for all $R > 0$ and all $\delta > 0$.*

Definition 4.4 *Given a pointed length space (X, x), the R cut-off δ spectrum, denoted $CovSpec^R_{cut}(X)$ or $CovSpec^R_{cut}(X, x)$, is the collection of $\delta > 0$ such that*

$$\tilde{X}^{\delta_1,R}_{cut} \neq \tilde{X}^{\delta,R}_{cut} \tag{4.4}$$

for all $\delta_1 > \delta$.

Note that by Theorem 6.5 applied with fixed R, $CovSpec^R_{cut}(X)$ is a lower semiclosed set.
The following lemma was known for compact spaces in [SoWei3]:

Lemma 4.5 *Given a complete length space X, if $[\delta_1, \delta_0) \cap CovSpec^R_{cut}(X) = \emptyset$, then $\tilde{X}^{\delta_1,R}_{cut} = \tilde{X}^{\delta_0,R}_{cut}$.*

Proof: Let

$$A = \{\delta \in [\delta_1, \delta_0) : \tilde{X}^{\delta,R}_{cut} = \tilde{X}^{\delta_1,R}_{cut}\} \subset [\delta_1, \delta_0). \tag{4.5}$$

Claim, $\sup\{A\} = \delta_0$. If $\sup\{A\} = \delta' < \delta_0$. By assumption, $\delta' \notin CovSpec^R_{cut}(X, x)$. Therefore there is $\delta'' > \delta'$ such that $\tilde{X}^{\delta',R}_{cut} = \tilde{X}^{\delta'',R}_{cut}$, contradicting that δ' is the supremum. So there exist δ_i increasing to δ_0 such that

$$\tilde{X}^{\delta_1,R}_{cut} = \tilde{X}^{\delta_i,R}_{cut}. \tag{4.6}$$

To prove the lemma, we proceed by contradition, assuming

$$\tilde{X}^{\delta_0,R}_{cut} \neq \tilde{X}^{\delta_1,R}_{cut}. \tag{4.7}$$

Then there is a curve C which lifts closed to $\tilde{X}^{\delta_0,R}_{cut}$ but open to $\tilde{X}^{\delta_1,R}_{cut}$. Then C is δ_0 homotopic to a collection of curves outside $\bar{B}(x, R)$. Applying Lemma 3.5 we know that for δ_i sufficiently close to δ_0, H is δ_i homotopy. So C lifts closed to $\tilde{X}^{\delta_i,R}_{cut}$. By (4.6), C lifts closed to $\tilde{X}^{\delta_1,R}_{cut}$ which is a contradiction. ∎

4.2 The cut-off δ covers and $CovSpec_{cut}(X)$

The following definition will be shown to be well defined in Proposition 4.7 below.

Definition 4.6 *The cut-off δ cover of X, denoted $\tilde{X}^{\delta}_{cut}$ is the Gromov-Hausdorff limit of the R cut-off δ covers as $R \to \infty$.*

Note that as in the case with the cylinder, whose R cut-off δ covers are all just they cylinder itself, the cutoff δ cover is also just the cylinder. This is in contrast with the δ cover which is Euclidean space for small enough values of δ.

Proposition 4.7 *For any complete length space, the Gromov-Hausdorff limit of the R cut-off δ covers as $R \to \infty$ exists and does not depend on the base point x. Furthermore we have the following covering maps:*

$$\tilde{X}^{\delta} \mapsto \tilde{X}^{\delta}_{cut} \mapsto \tilde{X}^{\delta,R}_{cut}. \tag{4.8}$$

Proof: First we fix a base point $x \in X$. By Lemma 4.2 we have a sequence of covering maps

$$f_R : \tilde{X}^{\delta} \to \tilde{X}^{\delta,R}_{cut} \tag{4.9}$$

and a sequence of covering maps

$$h_R : \tilde{X}^{\delta,R}_{cut} \to X \tag{4.10}$$

both of which are isometries on balls of radius δ. Let number of disjoint balls of radius ϵ in a ball of radius r in a space Y be denoted $N(\epsilon, r, Y)$. By the covering maps we have

$$N(\epsilon, r, \tilde{X}^{\delta,R}_{cut}) \leq N(\epsilon, r, \tilde{X}^{\delta}) \tag{4.11}$$

so by Gromov's Compactness Theorem, a subsequence $\tilde{X}^{\delta,R_j}_{cut}$ converges. We call the limit $\tilde{X}^{\delta}_{cut}$. Furthermore by the Grove-Petersen Arzela-Ascoli Theorem subsequences of f_{R_j} and h_{R_j} converge to functions f and h such that

$$f : \tilde{X}^{\delta} \to \tilde{X}^{\delta}_{cut} \tag{4.12}$$

$$h : \tilde{X}^{\delta}_{cut} \to X \tag{4.13}$$

which are still isometries on balls of radius $\delta/2 > 0$ and are thus covering maps. This implies that any limit space satisfies (4.8).

To show we have a unique limit that doesn't depend on the base point, take an alternate base point x' and an alternate sequence $R'_j \to \infty$ and assume it converges to some other limit space Z. Taking a subsequence so that

$$B_x(R_j) \subset B_{x'}(R'_j) \subset B_x(R_{j+1}) \tag{4.14}$$

and applying Lemma 4.3 we have covering maps

$$f_j : \tilde{X}^{\delta,R_{j+1}}_{cut} \to \tilde{X}^{\delta,R'_j}_{cut} \tag{4.15}$$

$$h_j : \tilde{X}^{\delta,R'_j}_{cut} \to \tilde{X}^{\delta,R_j}_{cut} \tag{4.16}$$

which are isometries on δ balls. Subsequences converge by Grove-Petersen Arzela-Ascoli to covering maps:

$$f_\infty : \tilde{X}^{\delta}_{cut} \to Z \text{ and } h_\infty : Z \to \tilde{X}^{\delta}_{cut}. \tag{4.17}$$

So the covering maps are isometries and the limit is unique. ■

We leave the following proposition as an exercise as it can be proven using similar limits of covering maps:

Proposition 4.8 *For all $\delta_1 < \delta_2$ we have*

$$\tilde{X}^{\delta_1}_{cut} \mapsto \tilde{X}^{\delta_2}_{cut}. \tag{4.18}$$

Definition 4.9 *The cut-off covering spectrum, denoted $CovSpec_{cut}(X)$, is the collection of $\delta > 0$ such that*

$$\tilde{X}^{\delta_1}_{cut} \neq \tilde{X}^{\delta}_{cut} \tag{4.19}$$

for all $\delta_1 > \delta$.

Note that by Proposition 4.8, Theorem 6.5 and this definition, we have:

Lemma 4.10 *The cut-off covering spectrum is a lower semiclosed set.*

The following proposition is easy to prove from the definitions.

Proposition 4.11 *If X is a bounded metric space with $D = diam(X)$, then*

$$\tilde{X}^{\delta,R}_{cut} = \tilde{X}^{\delta} \qquad \forall R \geq D, \text{ and } \tilde{X}^{\delta}_{cut} = \tilde{X}^{\delta}. \tag{4.20}$$

So $CovSpec_{cut}(X) = CovSpec(X)$.

Thus the cut-off covering spectrum is really only useful to study complete length spaces which are not bounded.

In the next subsection we explore the distinction between these two spectra in general.

4.3 Relating the various spectra

The intuitive idea behind the next theorem is that the covering spectrum can detect any holes that the cut-off covering spectrum sees.

Theorem 4.12 *The cut-off covering spectrum of a complete length space is a subset of its covering spectrum.*

This follows from Proposition 4.13 and Proposition 4.14 which we now state and prove.

Proposition 4.13 *For any basepoint $x \in X$, $CovSpec^R_{cut}(X,x) \subset CovSpec(X)$, and*

$$CovSpec^{R_1}_{cut}(X,x) \subset CovSpec^{R_2}_{cut}(X,x) \text{ for } R_1 < R_2. \tag{4.21}$$

Proof: If $\delta \in CovSpec^R_{cut}(X)$, then $\tilde{X}^{\delta_1,R}_{cut} \neq \tilde{X}^{\delta,R}_{cut}$ for all $\delta_1 > \delta$. So there is a nontrivial loop γ which lifts to $\tilde{X}^{\delta,R}_{cut}$ nontrivially and lifts to $\tilde{X}^{\delta_1,R}_{cut}$ trivially. In particular we can choose γ which lies in a ball of radius δ_1. Otherwise if all such loops lift trivially to $\tilde{X}^{\delta,R}_{cut}$ then the covering groups are the same.

If $\delta \notin CovSpec(X)$, then $\tilde{X}^{\delta} = \tilde{X}^{\delta_1}$ for some $\delta_1 > \delta$. Then γ which lifts trivially to the δ_1 cover, also lifts trivially to the δ cover, and must then project trivially back down to $\tilde{X}^{\delta,R}_{cut}$ nontrivially. causing a contradiction.

Similarly if $\delta \notin CovSpec^{R_2}_{cut}(X)$, then $\tilde{X}^{\delta,R_2}_{cut} = \tilde{X}^{\delta_1,R_2}_{cut}$ for some $\delta_1 > \delta$ and we can lift γ trivially to both of these covers which contradicts that it lifts to $\tilde{X}^{\delta,R}_{cut}$ nontrivially. ■

Proposition 4.14 *If X is a complete length space then for any basepoint $x \in X$,*

$$\bigcup_{R>0} CovSpec^R_{cut}(X,x) \subset CovSpec_{cut}(X). \tag{4.22}$$

In particular, $CovSpec_{cut}(X) \subset CovSpec(X)$.

Proof: If $\delta \in CovSpec^{R_0}_{cut}(X)$, by Proposition 4.13, then $\delta \in CovSpec^R_{cut}(X)$ for all $R \geq R_0$. So the covering map

$$\pi_R : \tilde{X}^{\delta,R}_{cut} \to \tilde{X}^{\delta_1,R}_{cut} \tag{4.23}$$

is nontrivial for all $\delta_1 > \delta$. Then as $R \to \infty$, the limit map

$$\pi : \tilde{X}^{\delta}_{cut} \to \tilde{X}^{\delta_1}_{cut} \tag{4.24}$$

is nontrivial. So $\delta \in CovSpec_{cut}(X)$. Hence $\bigcup_{R>0} CovSpec^R_{cut}(X) \subset CovSpec_{cut}(X)$. ■

Thus Theorem 4.12 is proven.

At first one might think that the union in Proposition 4.14 is equal to the covering spectrum. This is not true.

Example 4.2 *Let X be a line with circles attached at the integers $j \neq 0$ of circumference $2\pi r_j$ where $r_j = 1 + 1/|j|$. Using 0 as the base point we have*

$$CovSpec^R_{cut}(X) = \{\pi + \pi/j : j \in \mathbb{N}, j + 1 \leq R\} \tag{4.25}$$

because R cut-off δ covers unravel all loops such that $j+1 \leq R$ and $\pi + \pi/j \geq \delta$. Taking the Gromov-Hausdorff limit of these covers we see that the cut off δ covers of X unravel all loops $\pi + \pi/j \geq \delta$. Thus $CovSpec_{cut}(X)$ is the lower semiclosure of $\{\pi + \pi/j : j \in \mathbb{N}, j + 1 \leq R\}$ which includes the number π because for all $\delta' > \pi$ we have $\tilde{X}^{\delta'}_{cut} \neq \tilde{X}^{\pi}_{cut}$. However the union of $CovSpec^R_{cut}(X)$ over all $R > 0$ does not include the number π.

Proposition 4.15 *If X is a complete length space then the lower semiclosure of the union of all R cut-off spectra is the cut-off covering spectrum:*

$$Cl_{lower}\left(\bigcup_{R>0} CovSpec^R_{cut}(X)\right) \cup \{0\} \ = \ CovSpec_{cut}(X) \cup \{0\}. \tag{4.26}$$

Proof: Suppose $\delta > 0$ is not in the lower semiclosure of $\bigcup_{R>0} CovSpec^R_{cut}(X)$. So by Lemma 6.4 there exists $\epsilon > 0$ such that

$$[\delta, \delta+\epsilon) \cap \bigcup_{R>0} CovSpec^R_{cut}(X) = \emptyset. \tag{4.27}$$

So for all $R > 0$,

$$[\delta, \delta+\epsilon) \cap CovSpec^R_{cut}(X) = \emptyset \tag{4.28}$$

which implies (by Lemma 4.5) that

$$\tilde{X}^{\delta+\epsilon,R}_{cut} = \tilde{X}^{\delta,R}_{cut}. \tag{4.29}$$

Taking the $R \to \infty$ and the Gromov-Hausdorff limits of these spaces, we get

$$\tilde{X}^{\delta+\epsilon}_{cut} = \tilde{X}^{\delta}_{cut} \tag{4.30}$$

which implies that $\delta \notin CovSpec_{cut}(X)$.

To show the $\delta > 0$ which is in the lower semiclosure is in $CovSpec_{cut}(X)$, we just use Proposition 4.14, apply the lower semiclosure to both sides of the equation and note the fact that $CovSpec_{cut}(X)$ is already lower semiclosed by Theorem 6.5. ■

4.4 The length spectrum and the cut-off spectrum

Recall that a locally compact metric space is a metric space whose closed bounded sets are compact, i.e. complete locally compact length spaces. Riemannian manifolds, for example, are locally compact metric spaces. The length space created by connecting a countable collection of circles of equal size are not locally compact since they are themselves closed and bounded but not compact. Example 4.2 is locally compact.

Theorem 4.16 *If X is locally compact then*

$$CovSpec^R_{cut}(X) \subset (1/2)L(X). \tag{4.31}$$

That is, if $\delta \in CovSpec^R_{cut}(X)$ then $2\delta \in L(X)$.

The assumption that the space be locally compact is necessary:

Example 4.3 *Let X be the collection of circles of circumeference $2\pi + 2\pi/k$, then*

$$CovSpec_{cut}(X) = CovSpec(X) = \{\pi + \pi/k : \ k \in \mathbb{N}\} \cup \{\pi\} \tag{4.32}$$

while the (1/2) length spectrum of the collection of circles is all finite sums:

$$(1/2)Length(X) = \{\sum_{k=1}^{\infty} a_k\pi(1+1/k) : a_k \in \mathbb{N}\} \tag{4.33}$$

which does not include π.

Before we prove Theorem 4.16, we prove the corresponding proposition which does not require local compactness:

Proposition 4.17 *If X is a complete length space and $\delta \in CovSpec^R_{cut}(X,x)$ then there exist δ_j decreasing to δ and loops, σ_j, which are not $R\delta$ homotopic to a point such that $L(\sigma_j) < 2\delta_j$.*

Proof of Proposition 4.17: Given $\delta \in CovSpec^R_{cut}(X,x)$ we know there exists δ_j decreasing to δ such that

$$\tilde{X}^{\delta,R}_{cut} \neq \tilde{X}^{\delta_j,R}_{cut}. \tag{4.34}$$

So there exist loops C_j in X which are δ_j homotopic to curves outside $\bar{B}_x(R)$ but are not δ homotopic to such a curve. Note that C_j is homotopic to a combination of curves $\alpha\beta\alpha^{-1}$ where β lie outside $\bar{B}_x(R)$ or inside $B_p(\delta_j)$. If all the β curves lie outside $\bar{B}_x(R)$ then C_j is δ homotopic to such curves, so this is impossible. In fact there must be a β_j which lies in a ball $B_{p_j}(\delta_j)$ which is not δ homotopic to a collection of curves outside $\bar{B}_x(R)$.

By Lemma 2.12 β_j is freely homotopic to a collection of curves of length $< 2\delta_j$. At least one of these curves is not δ homotopic to a collection of curves outside $\bar{B}_x(R)$ because β_j is not. This is the curve σ_j. ■

We can now add the condition that the space is locally compact:

Proof of Theorem 4.16: By Proposition 4.17 we have a sequence of curves σ_j in X. Note that $Im(\sigma_j) \cap \bar{B}_x(R)$ is nonempty for all j. Since $L(\sigma_j) < 2\delta_j < 4\delta$ for j sufficiently large

$$\sigma_j : [0, L(\sigma_j)] \to \bar{B}_x(R + 2\delta). \tag{4.35}$$

By the local compactness this closed ball is compact for j , so we can apply the Arzela-Ascoli theorem to produce a converging subsequence and a limit curve σ_∞.

It is easy to construct a δ homotopy from σ_∞ to σ_j for j sufficiently large so σ_∞ is also not δ homotopic to a curve outside $\bar{B}_x(R)$ and, in particular, not δ homotopic to a point. Since

$$L(\sigma_\infty) \le \liminf_{i\to\infty} L(\sigma_i) \le \liminf_{i\to\infty} 2\delta_i = 2\delta \tag{4.36}$$

we can apply Lemma 3.2 to say that σ_∞ is a closed geodesic and has length 2δ so $2\delta \in Length(X)$. ■

Corollary 4.18 *For a complete locally compact length space X,*

$$CovSpec_{cut}(X) \subset (1/2)Cl_{lower}(L(X)). \tag{4.37}$$

That is, if $h/2 \in CovSpec^R_{cut}(X)$ then either $h \in L(X)$ or there exist $h_j \in L(X)$ such that h_j decrease to h.

Example 4.2 shows that the lower semiclosure is needed here.

4.5 Topology and the $CovSpec_{cut}(X)$

In this section we prove that the cut-off covering spectrum is empty given certain topological conditions on the space X: particlarly Theorem 4.20 and its converse and Theorem 4.22. Recall that the covering spectrum of a simply connected compact metric space is empty while the cut-off covering spectrum of a cylinder is empty. We begin with the loops to infinity property defined in [So]:

Definition 4.19 *Given a complete length space, X, a loop $\gamma : S^1 \to X$ is said to have the* loops to infinity *property, if for every compact set $K \subset X$, there is another loop $\sigma : S^1 \to X \setminus K$ freely homotopic to γ.*

The space X is said to have the loops to infinity property if all its noncontractible loops have this property.

Theorem 4.20 *A complete length space X with the loops to infinity property has an empty cut-off covering spectrum.*

Proof: If X is simply connected it has an empty cut-off covering spectrum. So we assume X is not simply connected.

Let us assume X has the loops to infinity property. Fix $x_0 \in X$ and $\delta > 0$. For every $R > 0$ let $K = B_{x_0}(R)$ and for any $g \in \pi_1(X, x_0)$ let γ be a representative of g based at x_0. So there exists β freely homotopic to γ outside K which means there is a curve $\alpha \circ \beta \circ \alpha^{-1}$ which represents g such that $\beta \subset X \setminus \bar{B}_p(R)$. So every $g \in \pi_1(X, x_0)$ is in the covering group of $\tilde{X}^{\delta,R}_{cut}$, which means $\tilde{X}^{\delta,R}_{cut} = X$. Taking the limit $R \to \infty$ we get $\tilde{X}^{\delta}_{cut} = X$ for all δ so the cut-off covering spectrum is trivial. ■

This theorem is applied to manifolds with nonnegative Ricci curvature in Theorem **??**. Such manifolds have only one end.

Recall that a length space X is said to have k ends if for all sufficiently large compact sets K, $X \setminus K$ has k path connected components.

Theorem 4.21 *Let X be a complete simply connected Riemannian manifold with an empty cut-off covering spectrum, then any curve in X is homotopic to a product of curves which have the loops to infinity property. Furthermore, if X has an empty cut-off covering spectrum and only one end then it has the loops to infinity property.*

Proof: If the cut-off covering spectrum is empty then all the $\tilde{X}^{\delta}_{cut} = X$ and, by Proposition 4.7, $\tilde{X}^{\delta,R}_{cut}$ is between these two spaces, so it is isometric to X as well. Thus for all $\delta > 0$ and for all $R > 0$, the fundamental group of X is generated by elements of the form $\alpha \circ \beta \circ \alpha$ where β is either in a ball of radius δ or in $X \setminus \bar{B}_{x_0}(R)$.

Choose any nontrivial loop γ and any compact set $K \subset X$. Take $R > 0$ large enough that

$$K \cup Im(\gamma) \subset B_{x_0}(R/2). \tag{4.38}$$

Since X is a Riemannian manifold, we can take $\delta > 0$ small enough that balls of radius δ in $B_{x_0}(R)$ are semilocally simply connected so that any loop β in such a ball is contractible. Thus $[\gamma] \in \pi_1(X, x_0)$ must be generated by loops of the form $\alpha \circ \beta \circ \alpha^{-1}$ where

$$\beta \in X \setminus \bar{B}_{x_0}(R) \subset X \setminus K. \tag{4.39}$$

Since X has only one end, the set $X \setminus K$ is path connected, thus the various β used to generate X can be connected via new paths $\alpha \in X \setminus K$ to a point $x_1 \in X \setminus K$. Thus we have constructed $\sigma \in X \setminus K$ which is freely homotopic to γ. ■

Example 4.4 *One end is necessary as can be seen by taking the length space X formed by joining two closed half cylinders at a point. The loop γ running around a figure eight which goes once around each cylinder, does not have the loops to infinity property. It is generated by 2 different loops β_j each of which goes to infinity in a different direction. This can be made smooth by taking the connected sum of two manifolds that are not simply connected that have only one end each, like Nabonnand's example [Na].*

Theorem 4.22 *If a complete length space X is homeomorphic to the product of complete length spaces, $M \times N$, then X has the loops to infinity property and $CovSpec_{cut}(X) = \emptyset$ if either of the following holds:*

i) both M and N are noncompact

ii) M is noncompact and $CovSpec(M) = \emptyset$.

Proof: Let C be a loop in X, so $C = (a, b)$ where a and b are closed loops in M and N respectively. C is freely homotopic to (a, b_0) followed by (a_0, b) where $b_0 = b(0)$ and $a_0 = a(0)$.

In both cases M is complete and noncompact, so there exists $p_j \in M$ which diverge to infinity and there exist minimal paths σ_j from any fixed point p_0 to p_j. If b is a loop in N, then (p_0, b) is freely homotopic to (p_i, b) via (σ_j, b). Any compact $K \subset X$, is a subset of the image of $K_M \times K_N$ where K_M is compact in M, taking $p_j \in M \setminus K_M$ we have (p_j, b) outside K. Thus (p_0, b) has the loops to infinity property.

In case i, N is also noncompact so both (a, b_0) and (a_0, b) have the loops to infinity property. So any loop C in X is a combination of curves with the loops to infinity property and we just apply Theorem 4.20.

Before we begin case ii we note that: *if a has the loops to infinity property then so does (a, b_0).* This is seen by taking the homotopies h_i from a to a_i that diverge to infinity. Mapping them to X, we get homotopies (h_i, b_0) from (a, b_0) to (a_i, b_0). So for any compact set $K \subset X$, we have $K \subset K_M \times K_N$ where K_M is compact. So we can choose a_i in $M \setminus K_M$ and have (a_i, b_0) outside K.

In case ii, we don't have the ray σ, but $CovSpec_{cut}(M) = \emptyset$. Applying Theorem 4.21, we see that the loop a in M is freely homotopic to a combination of loops which have the loops to infinity property. Thus (a, b) is freely homotopic to a combination of loops (a_i, b) each of which is homotopic to $(a_i(0), b)$ following (a_i, b_0). Each $(a_i, b(0))$ has the loops to infinity property via the loops to infinity property of each a_i. As above the cases, each $(a_i(0), b)$ has the loops to infinity property via rays in M based at $a_i(0)$. So $CovSpec_{cut}(X) = \emptyset$ here as well. ∎

Corollary 4.23 *If X is a complete noncompact length space homeomorphic to $M \times \mathbb{R}$ then*

$$CovSpec_{cut}(X) = \emptyset. \tag{4.40}$$

4.6 Localizing the R cut-off covering spectrum

In this section we show that one can compute $CovSpec^R_{cut}(X, x) \cap [0, D]$ using only the information contained in $B(x, r)$ when r is taken sufficiently large [Prop 4.29]. In fact we give a precise estimate on r independant of X which will allow us to stufy sequences of spaces.

Note that there is a complete hyperbolic manifold M of constant sectional curvature -1 such that for any r, there exists a contractible curve lying in $B(p, 1)$ which is not homotopically trivial in $B(p, r)$ [BoMe][Po]. In other words, the homotopies required to contract these loops to a point extend further and further out in M. A simpler example with this property is formed by taking the Hawaii Ring with circles of circumference $1/k$ and attaching a cylinders of length k to the k^{th} circle and then capping off the cylinder. This is a simply connected space none of whose balls about the basepoint are simply connected.

The covering spectrum of these spaces could not be computed using a localization process like the one we obtain here for the R cut-off covering spectrum. It is crucial that we can chop off homotopies as in Figure 4 when computing the R cut-off covering spectrum.

Recall the definition of δ homotopy in Definition 3.1 and Lemma 3.4 and Defn 3.6. Now we define:

Definition 4.24 *Two loops γ_1, γ_2 in X are R-cutoff δ homotopic in X if $\pi_{\delta,R}(\gamma_1) = \pi_{\delta,R}(\gamma_2)$, where $\pi_{\delta,R} : \pi(X) \to \pi(X)/\pi(X, \delta, R)$.*

It is not hard to see from the definition of the R cut-off δ cover that we have the following simpler description which will allow us to apply the lemmas from the section on δ homotopies to study this new kind of homotopy:

Lemma 4.25 *A loop γ is R-cutoff δ homotopic to a point in A iff it is δ homotopic in A to a collection of curves β_j lying outside $\bar{B}_p(R)$.*

Our next lemma will useful for localizing the δ homotopies so that we can use compactness to control them.

Lemma 4.26 *Given $\delta > 0, R > 0$, and a loop C in a complete length space X, if C is δ-homotopic in X to a collection of curves $\alpha\beta\alpha^{-1}$ where β are in δ-balls or outside $\bar{B}(x,R)$, then C is δ-homotopic in $B(x, R+2\delta)$ to a collection of curves $\alpha\beta\alpha^{-1}$ where β are outside $\bar{B}(x,R)$. So C is R cutoff δ homotopic to a point in $\bar{B}(x,R)$.*

See Figure 4 where the darker balls are $B(x,R)$ and the lighter balls are $B(x, R+2\delta)$.

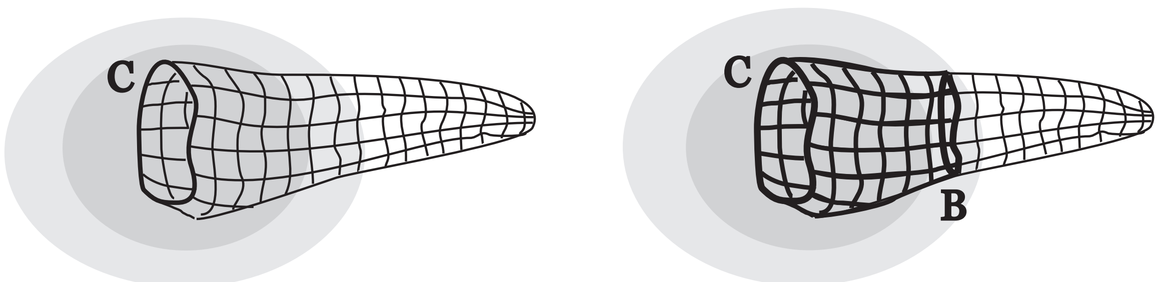

Figure 4: Here C is δ homotopic to a single $\beta = B$ outside $\bar{B}_x(R)$.

Proof: This proof follows from Lemma 3.7 where our set $A = X$ and $B = B(x, R+2\delta)$ so we see that C is δ homotopic in B to a collection of curves $\gamma_1, \gamma_2, ...\gamma_j$ such that each γ_j lies in B and the tubular neighborhood

$$T_{2\delta}(A \setminus B). \tag{4.41}$$

In particular the γ_i lie outside $\bar{B}(x,R)$. Thus by Lemma 4.25, C is a curve which is R-cutoff δ-homotopic in $B(x, R+2\delta)$ to a collection of curves $\alpha\beta\alpha^{-1}$ where β are in δ-balls or outside $\bar{B}(x,R)$. ■

Using Lemma 4.26 we have the following relation between the R-cutoff spectrums of balls and the total space which will be very useful later.

Lemma 4.27 *If*

$$\delta \in CovSpec^R_{cut}(B(x,r)) \tag{4.42}$$

for some $r \geq 3(R+2\delta)$ then

$$\delta \in CovSpec^R_{cut}(X). \tag{4.43}$$

Proof: We prove the contrapositive. Assume $\delta \notin CovSpec^R_{cut}(X)$. By the definition, there exists $\delta' > \delta$ such that

$$\tilde{X}^{\delta,R}_{cut} = \tilde{X}^{R,\delta'}_{cut}. \tag{4.44}$$

This means that any curve C whose image lies in a ball of radius δ' centered in $B_x(R+\delta)$ is δ-homotopic in X to a path created as a combination of $\alpha\dot{\beta}\dot{\alpha}^{-1}$ where β are either in a ball of radius δ or lie outside $\bar{B}_x(R)$.

By Lemma 4.26 C is a curve which is δ-homotopic in $B(x, R+2\delta)$ to a collection of curves $\alpha\beta\alpha^{-1}$ where β are in delta balls or outside $\bar{B}(x,R)$.

When $r > 3(R+2\delta)$ the metric on $B_x(R+2\delta)$ restricted from the induced length metric on $B_x(r)$ agrees with its metric restricted from X, we have that any such curve C lifts closed to $\tilde{B}(x,r)^{\delta,R}_{cut}$.

Now any curve σ which lifts closed to $\tilde{B}(x,r)^{R,\delta'}_{cut}$ is homotopic in $B(x,r)$ to a collection of $\alpha\dot{\beta}\dot{\alpha}^{-1}$ where β are now either in a ball of radius δ' or outside $\bar{B}_x(R)$. Note that any β which lies outside

$\bar{B}_x(R)$ lifts as a closed loop to $\tilde{B}(x,r)^{\delta,R}_{cut}$. Those β which pass within $\bar{B}_x(R)$ and fit in a ball of radius δ', satisfy the conditions for C and also lifts closed to $\tilde{B}(x,r)^{\delta,R}_{cut}$. Since σ is homotopic in $B(x,r)$ to a combination of curves which lift as closed loops to $\tilde{B}(x,r)^{\delta,R}_{cut}$, then σ must do the same. Thus by the Curve Lifting property (c.f. [Ma] page 123) and Lemma 4.8 we see that

$$\tilde{B}(x,r)^{R,\delta'}_{cut} = \tilde{B}(x,r)^{\delta,R}_{cut} \tag{4.45}$$

and so $\delta \notin CovSpec^R_{cut}(B(x,r))$. ∎

In the opposition direction we have

Lemma 4.28 *If*

$$\delta \in CovSpec^R_{cut}(X) \tag{4.46}$$

then for all $r \geq 3(R+2\delta)$,

$$\delta \in CovSpec^R_{cut}(B(x,r)). \tag{4.47}$$

Proof: The idea is if

$$\delta \in CovSpec^R_{cut}(X) \tag{4.48}$$

we produce a C_i of length $\leq 2\delta$ in $B(x,R+2\delta)$ which lifts open to $\tilde{X}^{\delta_i,R}_{cut}$ and closed to $\tilde{X}^{\delta,R}_{cut}$ with $\delta_i \to \delta$. Due to its length, C_i lifts as a closed loop to the $\tilde{B}(x,r)^{\delta,R}_{cut}$. We need only show C_i lifts as an open path to $\tilde{B}(x,r)^{\delta_i,R}_{cut}$. If not, it would have a homotopy to — and any homotopy in $B(x,r)$ is a homotopy in X and would cause it to lift closed to $\tilde{X}^{\delta_i,R}_{cut}$ which contradicts the choice of C_i. Taking $i \to \infty$, we are done. ∎

An immediate consequence of these two lemmas is the following:

Proposition 4.29 *Given two length spaces X and Y with isometric balls, $B(x,r) = B(y,r)$, then*

$$CovSpec^R_{cut}(X,x) \cap [0,D] = CovSpec^R_{cut}(Y,y) \cap [0,D] \tag{4.49}$$

whenever $3(R+2D) \leq r$.

Proof: If $\delta \in CovSpec^R_{cut}(X,x) \cap [0,D]$, then $\delta \leq D$ so apply Lemma 4.28 and have

$$\delta \in CovSpec^R_{cut}(B(x,r)) = CovSpec^R_{cut}(B(y,r)). \tag{4.50}$$

Then apply Lemma 4.27 gives the result. ∎

Remember that the R cut-off covering spectrum of a capped cylinder and a cylinder are both empty regardless of basepoint while the ordinary covering spectrum of the cylinder is nonempty..

The without restricting to a uniform $[0,D]$, the R cut-off covering spectrum will not match. This can be seen in the following example:

Example 4.5 *Let X_s be a unit interval with x_s on one end and a circle of circumference $2\pi s$ on the other end. Let Y be a unit interval with y at one end and two half lines at the far end. Taking $R = 2$ and $s > 1$ we have $CovSpec^R_{cut}(X_s,x_s) = \{\pi s\}$ and $CovSpec^R_{cut}(Y,y) = \emptyset$. Yet for any r we have $B(x_s,r)$ isometric to $B(y,r)$ for $s > r$.*

4.7 Varying R in the R cut-off covering spectra

In our section on the Gromov-Hausdorff convergence of metric spaces and the cut-off covering spectra we need to relate the R cut-off covering spectra for various values of R.

Proposition 4.30 *Given $R_0 < R_1$,*

$$\delta \in CovSpec_{cut}^{R_1}(X) \setminus CovSpec_{cut}^{R_0}(X) \tag{4.51}$$

implies

$$\tilde{X}_{cut}^{\delta,R_1} \to \tilde{X}_{cut}^{\delta,R_0}. \tag{4.52}$$

is nontrivial.

Proof: If $\delta \in CovSpec_{cut}^{R_1}(X)$, then $\tilde{X}_{cut}^{\delta_1,R_1} \neq \tilde{X}_{cut}^{\delta,R_1}$ for all $\delta_i > \delta$. So there is a nontrivial loop γ_i which lifts to $\tilde{X}_{cut}^{\delta,R_1}$ nontrivially and lifts to $\tilde{X}_{cut}^{\delta_i,R_1}$ trivially. Since R_1 is the same for both covering spaces, we can choose γ_i which lies in a balls of radius δ_i. Otherwise if all such loops lift trivially to $\tilde{X}_{cut}^{\delta,R}$ then the covering groups are the same.

Suppose $\delta \notin CovSpec_{cut}^{R_0}(X)$ where $R_0 < R_1$. Then for i sufficiently large,

$$\tilde{X}_{cut}^{\delta_i,R_0} = \tilde{X}_{cut}^{\delta,R_0}. \tag{4.53}$$

Since γ_i lies in a ball of radius δ_i it lifts trivially to the first cover, and thus also the second. So we have a nontrivial covering:

$$\tilde{X}_{cut}^{\delta,R_1} \to \tilde{X}_{cut}^{\delta,R_0}. \tag{4.54}$$

If $\delta \notin CovSpec(X)$, then $\tilde{X}^{\delta} = \tilde{X}^{\delta_1}$ for some $\delta_1 > \delta$. Then γ which lifts trivially to the δ_1 cover, also lifts trivially to the δ cover, and must then project trivially back down to $\tilde{X}_{cut}^{\delta,R}$ nontrivially. causing a contradiction.

Similarly if $\delta \notin CovSpec_{cut}^{R_2}(X)$, then $\tilde{X}_{cut}^{\delta,R_2} = \tilde{X}_{cut}^{\delta_1,R_2}$ for some $\delta_1 > \delta$ and we can lift γ trivially to both of these covers which contradicts that it lifts to $\tilde{X}_{cut}^{\delta,R}$ nontrivially. ■

In the next proposition we assume our space Y is compact. To apply this proposition to complete noncompact spaces X which are only locally compact we will use our localization results from the last section.

Proposition 4.31 *If Y is a compact space, and $CovSpec_{cut}^{R_i}(Y) \cap [\delta_1, \delta_2) = \emptyset$ for a sequence of R_i decreasing to R_1, then for R_i sufficiently close to R_1 we have*

$$\tilde{Y}_{cut}^{\delta_1,R_i} \to \tilde{Y}_{cut}^{\delta_2,R_1} \tag{4.55}$$

is trivial. In particular, without any assumption on the spectrum, we have

$$\tilde{Y}_{cut}^{\delta,R_i} \to \tilde{Y}_{cut}^{\delta,R_1} \tag{4.56}$$

is trivial whenever R_i is sufficiently close to R_1.

Combining this with Proposition 4.13, we only need to assume there exists $R_2 > R_1$ with $CovSpec_{cut}^{R_2}(Y) \cap [\delta_1, \delta_2) = \emptyset$ to conclude (4.55). In fact, by Proposition 4.14 we could assume $CovSpec_{cut}(Y) \cap [\delta_1, \delta_2) = \emptyset$ and draw the same conclusion. See Theorem 5.2 for an application of this proposition.

Proof: Assume on the contrary that $\tilde{Y}^{\delta_1,R_2} \to \tilde{Y}^{\delta_2,R_1}$ is not trivial. So there is a γ which lifts trivially to the latter cover, but not to the first. In particular we can either choose γ to lie inside a ball of radius δ_2, or outside $\bar{B}_p(R_1)$.

In the first case, γ lifts trivially to $\tilde{Y}^{\delta_2,R_2}$ which implies $CovSpec_{cut}^{R_2}(Y) \cap [\delta_1, \delta_2)$ is nonempty.

In the second case γ lies outside $\bar{B}_p(R_1)$ and is not δ_1 homotopic to a loop outside $\bar{B}_p(R_2)$. In particular $l(\gamma) \geq 2\delta_1$ and γ is not δ homotopic to a loop outside $\bar{B}_p(R_2)$ for any $\delta \leq \delta_1$.

Suppose we take $R_2 = R_i$ decreasing to R_1 and have nontrivial covers. So we get a sequence of γ_i, each γ_i lies outside $\bar{B}_p(R_1)$ and is not δ_1 homotopic to a loop outside $\bar{B}_p(R_i)$.

Note that $Y \setminus \bar{B}_p(R_1)$ is precompact. It is still a precompact length space if we give it the induced length structure (c.f. [BBI]). So there exists some finite number N such that it can be covered by at most N balls of radius $\delta_1/5$. Note that balls in the induced length metric are smaller than those in the metric on Z, so γ_i is also not δ_1 homotopic in the space $Z = Y \setminus \bar{B}_p(R_1)$ to a loop outside $\bar{B}_p(R_i)$. For the rest of the proof we will use the induced length metric on Z when referring to the δ_1 homotopies..

Applying Lemma 3.9, we see that we can always find a γ_i in Z with $L(\gamma_i) \leq N\delta_1$ in which is not δ_1 homotopic to a loop outside $\bar{B}_p(R_i)$.

Since the γ_i have length bounded above uniformly then since Y is compact, by Arzela Ascoli we have a subsequence which converges to some γ_∞. Note that γ_∞ need not be located outside $\bar{B}_p(R_1)$, so instead of relating γ_i to γ_∞, we will use the fact that γ_i must be a Cauchy sequence in Z. That is , there exists N' sufficiently large such that γ_i are $\delta_1/2$ homotopic to γ_j for all $i, j \geq N'$. Fix this N, note that γ_N lies outside the closed ball $\bar{B}_p(R_1)$ and R_j are decreasing to R_1, so γ_N is outside $\bar{B}_p(R_j)$ for j sufficiently large. This contradicts γ_j is not δ_1 homotopic to a loop outside $\bar{B}_p(R_j)$. ■

Note that the compactness here is essential as the following example shows.

Example 4.6 *Let Y be the Hawaii ring with circles of circumference $2\pi \pm \frac{\pi}{j}$, γ_j, all attached at a point. Take $\delta = \pi/2$, $R_i = (1 + 1/i)\pi$, $R_1 = \pi$, then the cover $\tilde{Y}^{\delta,R_i}_{cut} \to \tilde{Y}^{\delta,R_1}_{cut}$ is nontrivial for all i. This Y is not a compact length space.*

5 Gromov-Hausdorff Convergence

In [SoWei3], we proved that when compact spaces M_j converge to a compact limit M in the Gromov-Hausdorff sense then $CovSpec(M_i) \cup \{0\}$ converges to $CovSpec(M) \cup \{0\}$ in the Hausdorff sense as subsets of the real line. In particular, if M_j are simply connected, then the limit space has an empty covering spectrum and is its own universal cover.

In the next subsection we provide examples demonstrating that we do not get such a strong result when the spaces are noncompact. In fact the limit space of simply connected M_i might be a cylinder [Example 5.2].

In the subsequent sections we prove the continuity of the cut-off covering spectra [Theorem 5.2]. In particular the limit of simply connected manifolds will be seen to have an empty cut-off covering spectrum [Corollary 5.6].

First recall the definition of Gromov Hausdorff distance:

Definition 5.1 *Given a pair of compact length spaces X_i and Y we say X_i converges to Y in the Gromov Hausdorff sense if there exists δ_i Hausdorff approximations $f : X_i \to Y$ such that*

$$d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| < \delta_i \tag{5.1}$$

and $Y \subset T_{\delta_i}(f(X_i))$ with $\delta_i \to 0$. Note that once this is true there are also δ_i' Hausdorff approximations from Y to X_i with $\delta_i' \to 0$.

When complete noncompact spaces are said to converge in the Gromov-Hausdorff sense, they are considered as pointed spaces. We write (X_i, x_i) converges in the pointed Gromov-Hausdorff sense to (X, x) when for every $R > 0$, the closed balls with the restricted metric $\bar{B}_{x_i}(R) \subset X_i$ converge to balls in the limit space $\bar{B}_x(R) \subset X$.

Gromov's Compactness Theorem says that whenever X_i have a uniform bound on the number of disjoint balls of radius r in any ball of radius R, then a subsequence of the X_i converge in this sense to a complete pointed length space X. Crucial here is that the balls do piece together to form a complete limit space. However, one must keep in mind that the balls can converge at different rates. The next section depicts a few examples where this aspect of the pointed Gromov-Hausdorff convergence is crucial.

5.1 Examples

First recall that even for the covering spectrum on compact spaces it is possible for a sequence of spaces to become simply connected in the limit:

Example 5.1 *Let M be a simply connected surface and let X_k be created by adding a small handle onto M, such that the handle fits inside a ball of radius $1/k$. These X_k converge to M as $k \to \infty$. Note that both $CovSpec(X_k) = \{\delta_k\}$ and $CovSpec_{cut}(X_k) = \{\delta_k\}$ where $\delta_k < 1/k$ while $CovSpec(M) = \emptyset$.*

Nevertheless in [SoWei3] we proved that the difficulty seen here was the only cause for a lack of continuity in the covering spectrum. We proved for compact M_i converging to compact limits Y, then if $\lambda_j \in CovSpec(M_i)$ converge to $\lambda > 0$ then $\lambda \in CovSpec(Y)$ and if $\lambda \in CovSpec(Y)$ there exists $\lambda_j \in CovSpec(M_i)$ such that $\lambda_j \to \lambda$. In particular, the M_i are simply connected, Y has an empty covering spectrum.

Without the assumption of compactness, however, we can have simply connected manifolds which have a limit with a nonempty covering spectrum:

Example 5.2 *Let M be a capped off cylinder and let $p_i \in M$ diverge to infinity. Then the sequence (M_i, p_i) converges in the pointed Gromov-Hausdorff sense to a cylinder because the cap has disappeared off to infinity.*

Thus we have a sequence $M_i \to Y$ such that $CovSpec(M_i) = \emptyset$ but $CovSpec(Y) = \{\pi\}$. Now by Propositions 4.13 and 4.14 $CovSpec_{cut}(M_i) = CovSpec^R_{cut}(M_i) = \emptyset$ as well. Since a cylinder has the loops to infinity property $CovSpec_{cut}(Y) = CovSpec^(_{cut}Y)$ are also empty.

While in the above example, the limit gained an element in its covering spectrum due to longer and longer homotopies, it is also possible to gain an element in the covering spectrum without changing the topology of the space:

Example 5.3 We construct an example where an element of the covering spectrum appears in the limit. *As in Example 2.5, let M^2 be the warped product manifold $\mathbb{R} \times_{f(r)} S^1$ where*

$$f(r) = 2Arctan(-r) + \pi. \tag{5.2}$$

Since $\lim_{r\to\infty} f(r) = 0$, $\pi_{slip}(M) = \pi_1(M)$ and the covering spectrum is empty.

Now let $(X_i, x_i) = (M, p_i)$ where $r(p_i) = r_i \to -\infty$. Note then $\bar{B}_{x_i}(R)$ is in a warped product with warping function

$$f_i(r) = f(r - r_i) = 2Arctan(-r + r_i) + \pi \tag{5.3}$$

which converges uniformly on $[-R, R]$ to

$$f_\infty(r) = 2\pi. \tag{5.4}$$

Thus the pointed Gromov Hausdorff limit is the standard cylinder whose covering spectrum is $\{\pi\}$. As above the cut-off covering spectra of these examples is empty both for the X_i and the limit space.

Next we construct an example where an element of the covering spectrum disappears in the limit without decreasing to 0. This issue is not immediately solved by using the cut-off covering spectrum.

Example 5.4 *Let M^2 be a cylinder with a small handle near a point p. Let $(X_i, x_i) = (M, p_i)$ where $d(p_i, p) \to \infty$. Then $CovSpec(X_i) = CovSpec(M)$ since the covering spectrum does not depend on the base point and the spectrum has its first element, $\lambda_1 < \pi$ corresponding to the small handle. Yet for all $R > 0$ just take N_R large enough that*

$$d(p_i, p) \geq 2R \forall i \geq N_R. \tag{5.5}$$

Then $\bar{B}_{p_i}(R)$ are all isometric to balls of radius R in a cylinder. So (X_i, x_i) converge to a cylinder with a point. So the covering spectrum of the limit space does not include λ_1 and its only element is π. So we have locally compact X_i converging to locally compact X with

$$\delta_i \in CovSpec(X_i) \text{ such that } \delta_i = \lambda_1 \to \delta \notin CovSpec(X). \tag{5.6}$$

In fact we have

$$\delta_i = \lambda_1 \in CovSpec_{cut}(X_i) \text{ such that } \delta_i \to \delta \notin CovSpec_{cut}(X). \tag{5.7}$$

On the other hand any $R > 0$, there exists N_R sufficiently large that $CovSpec^R_{cut}(X_i, x_i) = \{\pi\}$ because the handle is located outside $\bar{B}_{x_i}(R)$.

Finally we have the possibility that elements of the covering spectrum can grow to infinity. In this example we see that in essence a hole could expand until it snaps and is no longer a hole in the limit space:

Example 5.5 *Let (X_r, x_r) be formed where X_r is a unit interval with x_r on one end and a circle of circumference $2\pi r$ attached to the other end with a half line attached on the opposite side of the circle. Then $CovSpec(X) = \{\pi r\}$.*

Note that if one takes a sequence of r_i diverging to infinity, (X_{r_i}, r_i), converges in the pointed Gromov-Hausdorff sense to (X_∞, x_∞) where X_∞ is a unit interval attached to x_∞ at one end and two half lines at the other end. So X_∞ is simply connected and has an empty covering spectra.

This example is not simplified by using the cut-off covering spectra. In fact for any $R \geq 1$ $CovSpec^R_{cut}(X) = \{\pi r\}$ and so $CovSpec_{cut}(X) = \{\pi r\}$.

5.2 Convergence of the R cut-off covering spectrum

In light of the above examples, it is natural to try to to prove continuity of the R cut-off covering spectra and then perhaps to apply this continuity to prove some form of continuity for the cut-off covering spectrum. Surprisingly the statement of the continuity theorem for the cut-off spectrum is somewhat tricky:

Theorem 5.2 *Let (X_i, x_i) be locally compact metric spaces converging in the pointed Gromov-Hausdorff sense to a locally compact space (X, x). Bounded elements do not disappear: if we have a converging sequence,*

$$\delta_i \in CovSpec^{R_1}_{cut}(X_i, x_i) \text{ and } \delta_i \to \delta > 0, \text{ then } \delta \in CovSpec^{R_1}_{cut}(X, x). \tag{5.8}$$

Nor do elements suddenly appear: for any $R_2 > R_1$ and if we have an element,

$$\delta \in CovSpec^{R_1}_{cut}(X, x) \text{ there are } \delta_i \in CovSpec^{R_2}_{cut}(X_i, x_i) \text{ such that } \delta_i \to \delta. \tag{5.9}$$

Examples 5.1 and 5.5 demonstrate why one must assume δ_i converge in $(0, \infty)$. We now present examples demonstrating why we cannot take $R_1 = R_2$ in this theorem.

Example 5.6 Appearing element in the R cut-off covering spectrum. *Let (X_r, x_r) be formed by attaching a line segment of length r to a circle of circumference 2π, and then continuing with a half line on the opposite side of the circle. The point x_r will be the endpoint of the line segment not attached to the circle. If $r_i \to r_\infty$ it is easy to see that (X_{r_i}, x_{r_i}) converges to $(X_{r_\infty}, x_{r_\infty})$.*

Note that $CovSpec(X_{r_i}) = \{\pi\}$ and so does $CovSpec_{cut}(X_r)$. However $CovSpec_{cut}^R(X_r) = \emptyset$ when $r > R$ because then the circle is contained in $X_r \setminus \bar{B}_{x_r}(R)$. Otherwise $CovSpec_{cut}^R(X_r) = \{\pi\}$. Thus the sequence X_{r_j} with r_j decreasing to R_1 has

$$\delta = \pi \in CovSpec_{cut}^{R_1}(X_{r_\infty}) \tag{5.10}$$

but $CovSpec_{cut}^{R_1}(X_{r_i}) = \emptyset$. However, taking $R_2 > R_1$, eventually we have $r_i < R_2$ so we have

$$\delta_i = \pi \in CovSpec_{cut}^{R_2}(X_{r_i}). \tag{5.11}$$

The next example also illustrates the same phenomenon with a distinct cause:

Example 5.7 *Fix M be a warped product manifold of the form $\mathbb{R} \times_f S^1$ where $f(t) = e^{-t^2}$. Fix p in the level $t = 0$.*

Let $X_r = \bar{B}_p(r)$. So it is a closed ball and if we wish to make it noncompact, we just attach a half line to it. We give it the induced length metric from M.

Let r_i decrease to some $r_\infty > \pi$. Then $X_i = X_{r_i}$ converges to $X_\infty = X_{r_\infty}$.

Let $R_1 = r_\infty$. The R_1 cutoff covering spectrum of X_∞ includes δ equal to half the length of one of the components of the boundary of $B(p, R_1)$, because this curve is not homotopic to anything outside $\bar{B}_p(R_1)$. However the R_1 cut-off covering spectra of the X_i are all empty because the loop is homotopic to a loop in $\partial B_p(r_i)$ which is outside $B(p, R_1)$. So once again we need $R_2 > R_1$ and need to wait for $r_i < R_2$ to get the cut-off covering spectra to converge.

One might also construct manifolds M_i converging to X_∞ by taking smoothed tubular neighborhoods of the X_i in five dimensional Euclidean space.

Note that in the above examples, if one were to take r_j increasing to R_0 the covering spectrum are all $\{\pi\}$. The difficulty arises because r_j decreasing to R_∞ are leaving the open set (R_∞, ∞) in the limit.

At first we thought we needed to take $R_2 > R_1$ in (5.8) as well as (5.9) but due to the lack of examples proving this was necessary, we investigated further and discovered we could boost our proof of (5.8) using the local compactness of the limit space. See Section **??**.

In order to prove this theorem we need extend several results for covering spaces of compact spaces to R-cutoff spaces. The first is an adaption of Theorem 3.4 in [SoWei1].

Proposition 5.3 *Let $B(p_i, s_i) \subset B(p_i, S_i) \subset Y_i, i = 1, 2$ be balls each with intrinsic metrics. Let $G(p_1, s_1, S_1, \delta_1)$ be the group of deck transformations of $\tilde{B}(p_1, S_1)_{cut}^{\delta_1, s_1}$.*

If there is a pointed ϵ-Hausdorff approximation $f : B(p_1, S_1) \to B(p_2, S_2)$ then for any $\delta_1 > 10\epsilon$ and $\delta_2 > \delta_1 + 10\epsilon$ and $s_2 < s_1 - 5\epsilon$, there is a surjective homomorphism, $\Phi : G(p_1, s_1, S_1, \delta_1) \to G(p_2, s_2, S_2, \delta_2)$

Proof of Proposition 5.3: We begin by describing a map for closed curves. For a closed curve $\gamma : [0, 1] \to B(p_1, S_1)$ with $\gamma(0) = \gamma(1) = p_1$, construct a 5ϵ-partition of γ as follows. On $\Gamma := \gamma([0, 1])$ choose a partition $0 = t_0 \le t_1 \le \cdots \le t_m = 1$ such that for $x_i = \gamma(t_i)$, one has $d(x_i, x_{i+1}) < 5\epsilon$ for $i = 0, \cdots, m - 1$. $\{x_0, \cdots, x_m\}$ is called a 5ϵ-partition of γ.

Let $y_m = y_0 = p_2$ and for each x_i, we set $y_i = f(x_i), i = 1, \cdots, m-1$. Connect y_i and y_{i+1} by minimal geodesics in $B_{p_2}(S_2)$. This yields a closed curve $\bar{\gamma}$ in $B(p_2, S_2)$ based at p_2 consisting of m minimizing segments each having length $\leq 6\epsilon$.

Any $\alpha \in G(p_1, s_1, S_1, \delta_1)$ can be represented by some rectifiable closed curve γ in $B(p_1, S_1)$, so we can hope to define

$$\Phi(\alpha) = \Phi([\gamma]) := [\bar{\gamma}] \in G(p_2, s_2, S_2, \delta_2).$$

First we need to verify that Φ doesn't depend on the choice of γ such that $[\gamma] = \alpha$.

Using the facts that $18\epsilon < \delta_2$ and loops which fit in balls of radius δ_2 do not effect the representative of a class in $G(p_2, s_2, S_2, \delta_2)$, one easily see that $[\bar{\gamma}]$ doesn't depend on the choice of minimizing curves $\bar{\gamma}_i$, nor on the special partition $\{x_1, \cdots, x_m\}$ of $\gamma([0,1])$.

Moreover using additionally the uniform continuity of a homotopy one can similarly check that if γ and γ' are homotopic in $B(p_1, S_1)$, then $[\bar{\gamma}] = [\bar{\gamma}']$ in $G(p_2, s_1 - 5\epsilon, S_2, \delta_2)$. That is, we can take a homotopy $h : [0,1] \times [0,1] \to B(p_1, S_1)$, we can take a grid on $[0,1] \times [0,1]$ small enough that homotopy maps the grid points to points $x_{i,j}$ that are less than 5ϵ apart from the images of their grid neighbors. Then we take $y_{i,j} = f(x_{i,j})$ and connect neighbors according to the rules in the first paragraph. Finally we use the argument in the paragraph above this to see that the net created using the $y_{i,j}$ is a δ_2 homotopy so $[\bar{\gamma}] = [\bar{\gamma}']$ in $G(p_2, s_2 - 5\epsilon, S_2, \delta_2)$. Thus we see that Φ is a homomorphism from $\pi_1(B(p_1, S_1), p_1)$ to $G(p_2, s_2, S_2, \delta_2)$. However $\alpha \in G(p_1, s_1, S_1, \delta_1)$ not $\pi_1(B(p_1, s_1), p_1)$.

Suppose γ_1 and γ_2 are both representatives of $\alpha \in G(p_1, s_1, S_1, \delta_1)$. Then $\gamma_1 * \gamma_2^{-1}$ is, in $B(p_1, S_1)$, homotopic to a loop γ_3 generated by loops of the form $\alpha * \beta * \alpha^{-1}$, where β is a closed path lying in a ball of radius δ_1 or in $B(p_1, S_1) \setminus \bar{B}(p_1, s_1)$. So $[\bar{\gamma_1}] = [\bar{\gamma_3}] * [\bar{\gamma_2}]$ in $\pi_1(B(p_1, s_1), p_1)$. So we need only show that $[\bar{\gamma_3}]$ is trivial in $G(p_2, s_2, S_2, \delta_2)$.

In fact $\bar{\gamma_3}$ can be chosen as follows. The y_i's corresponding to the x_i's from the β segments of γ_3 are all within $\delta_1 + \epsilon$ of a common point and the minimal geodesics between them are within $\delta_1 + (1 + 6/2)\epsilon < \delta_2$. Furthermore, the y_i's corresponding to the x_i's from the α and α^{-1} segments of the curve can be chosen to correspond. Thus $\bar{\gamma_3}$ is generated by loops of the form $\alpha * \beta * \alpha^{-1}$ lying in $B(p_2, S_2)$, where β is a closed path lying in a ball of radius δ_2 or $B(p_1, S_2) \setminus B(p_1, s_1 - 5\epsilon)$ and α is a path from p_2 to $\beta(0)$. So it is trivial.

Last, we need to show that Φ is onto. If $\bar{\alpha} \in G(p_2, s_2, S_2, \delta_2)$, it can be represented by some rectifiable closed curve σ in $B(p_2, S_2)$ based at p_2. Choose an ϵ-partition $\{y_0, \cdots, y_m\}$ of σ. Since $f : B(p_1, S_1) \to B(p_2, S_2)$ is an ϵ-Hausdorff approximation, there are $x_i \in B(p_1, s_1)$, $y_i' = f(x_i) \in B(p_2, S_2)$ where $y_0' = y_m' = p_2$, $x_0 = x_m = p_1$ and $d_{B(p_2, S_2)}(y_i, y_i') \leq \epsilon$. Connect y_i', y_{i+1}' with a length minimizing curve in $B(p_2, S_2)$; this yields a piecewise length minimizing closed curve σ' in $B(p_2, S_2)$ based at p_2, each segment has length $\leq 3\epsilon$. So $[\sigma'] = [\sigma]$ in $G(p_2, s_1 - 5\epsilon, S_2, \delta_2)$. Now connect x_i, x_{i+1} by length minimizing curves in $B(p_1, S_1)$ this yields a piecewise length minimizing $\gamma : [0,1] \to B(p_1, S_1)$ with base point p_1, each segment has length $\leq 4\epsilon$. So the curve γ allows a 5ϵ-partition and $[\gamma] \in G(p_1, s_1, S_1, \delta_1)$. By the construction, $\Phi([\gamma]) = \bar{\alpha}$.

Therefore Φ is surjective. ■

Proposition 5.4 *If a sequence of locally compact complete length spaces X_i converges to a length space X in the Gromov-Hausdorff topology, then for any $\delta > 0, R > 0, r > 3R$ there is a subsequence of X_i and a sequence $r_i \to r$ such that $\tilde{B}(x_i, r_i)_{cut}^{\delta,R}$ also converges in the pointed Gromov-Hausdorff topology. Moreover, the limit space $B(x, r)_{cut}^{\delta,R}$ is a covering space of $B(x, r)$ satisfying*

$$\tilde{B}(x, r)_{cut}^{\delta,R} \to B(x, r)_{cut}^{\delta,R} \to \tilde{B}(x, r)_{cut}^{\delta',R'} \tag{5.12}$$

for all $0 < R' < R$ and $\delta' > \delta$.

Proof: By the Appendix of [SoWei2] we know that for a sequence r_i converging to r, $B(x_i, r_i)$ converge with the induced length metric to $B(x, r)$.

By [SoWei3][Proposition 7.3] and the fact that the closed balls $B(x_i, r_i)$ are compact sets, we know that $\tilde{B}(x_i, r_i)^{\delta}$ have a converging subsequence. So by Gromov's compactness theorem, they have a uniform bound $N(a, b)$, the number of disjoint balls of radius a in a ball of radius b. By Proposition 4.7, $\tilde{B}(x_i, r_i)^{\delta}_{cut}$ covers $\tilde{B}(x_i, r_i)^{\delta,R}_{cut}$, so $N(a, b)$ can be used to count balls in $\tilde{B}(x_i, r_i)^{\delta,R}_{cut}$ as well. So by Gromov's compactness theorem, a subsequence of these spaces converges and we will denote the limit space: $B(x, r)^{\delta,R}_{cut}$.

To complete the proof we adapt Theorem 3.6 of [SoWei1]. The fact $\tilde{B}(x_i, r_i)^{\delta,R}_{cut}$ were isometries on balls of radius δ and outside $\bar{B}_p(R)$ guarantees that the limit is as well, so $B(x, r)^{\delta,R}_{cut}$ is a covering space for $B(x, r)$.

The isometries also guarantee it is covered by $\tilde{B}(x, r)^{\delta,R}_{cut}$. This can be seen using the Unique Lifting Theorem (c.f. [Ma] Lemma 3.1, p123) and noting that if C is a closed curve in $B(x, r)$ whose lift to $\tilde{B}(x, r)^{\delta,R}_{cut}$ then it is homotopic to a curve which is created from curves of the form $\alpha \cdot \beta \cdot \alpha^{-1}$ where the β are either in a ball of radius δ or outside $\bar{B}_x(R)$. So its lift to $B(x, r)^{\delta,R}_{cut}$ is also closed since π^{δ} is an isometry on δ-balls and an isometry outside $\bar{B}_x(R)$. Therefore $\tilde{B}(x, r)^{\delta,R}_{cut}$ covers $B(x, r)^{\delta,R}$

To complete the proof we apply the Unique Lifting Theorem by contradiction. We assume there is $\delta' > \delta$ and $R' < R$ and C is a curve which lifts closed to $B(x, r)^{\delta,R}_{cut}$ but lifts open to $\tilde{B}(x, r)^{\delta',R'}_{cut}$. Since this lift of C is not closed, $[C] \in G(x, r, R', \delta')$ is nontrivial.

Let $\epsilon > 0$ be chosen sufficiently small that

$$\epsilon < \min\{\delta/10, (\delta - \delta')/10, (R - R')/5\}. \tag{5.13}$$

Take i sufficiently large that we have an ϵ-Hausdorff approximation $f_i : B(x_i, r_i) \to B(x, r)$. Applying Proposition 5.3, we know there are a surjective homomorphisms, $\Phi : G(x_i, r_i, R, \delta) \to G(x, r, R', \delta')$, so there are closed loops $C_i \in B(x_i, r_i)$ such that $\Phi([C_i]) = [C]$.

By the construction of Φ, C_i can be chosen so these lifted curves $\tilde{C}_i$ converge to the lift of the limit of the curves, $\tilde{C}$ in $B(x, r)^{\delta,R}_{cut}$ and

$$d_{B(x,r)^{\delta,R}_{cut}}(\tilde{C}(0), \tilde{C}(1)) = \lim_{i\to\infty} d(\tilde{C}_i(0), \tilde{C}_i(1)). \tag{5.14}$$

However the $[C_i]$ are nontrivial, so their lifts to $\tilde{B}(x_i, r_i)^{\delta,R}$ run between points $\tilde{C}_i(0) \neq \tilde{C}_i(1)$ satisfying

$$d(\tilde{C}_i(0), \tilde{C}_i(1)) \geq \delta. \tag{5.15}$$

Combining this with (5.14), we see that $\tilde{C}$ is not closed and we have a contradiction. ■

At this point we could imitate the proof of Theorem 8.4 in [SoWei3] to prove Theorem 5.2 for X_i which are compact balls. However, this would not help us prove Theorem 5.2 for noncompact spaces as the cut-off covering spectrum of a ball does not match the cut-off covering spectrum of the space. Recall Examples 5.5 and 5.2 demonstrate that not only can holes become increasingly large, but homotopies may as well. One needs to control such phenomenon to complete the proof.

Proof of Theorem 5.2: In order to prove the first statement (5.8) we first prove that given any $R_2 > R_1$ if

$$\delta_i \in CovSpec^{R_1}_{cut}(X_i) \text{ and } \delta_i \to \delta > 0, \text{ then } \delta \in CovSpec^{R_2}_{cut}(X). \tag{5.16}$$

Later we will boost this result to (5.8).

Assume

$$\delta_i \in CovSpec^{R_1}_{cut}(X_i) \tag{5.17}$$

and $\delta_i \to \delta > 0$. By Lemma 4.28, $\delta_i \in CovSpec^{R_1}_{cut}(B(x_i, r))$ for $r \geq 3(R_1 + 2\delta_i)$. So $\tilde{B}(x_i, r)^{\delta_i,R_1}_{cut} \to \tilde{B}(x_i, r)^{\delta',R_1}_{cut}$ is nontrivial for all $\delta' > \delta_i$. So for all $\delta' > \delta > 0$ and $\epsilon \in (0, \delta)$ we have $\delta - \epsilon < \delta_i < \delta'$ for i sufficiently large and

$$\tilde{B}(x_i, r)^{\delta-\epsilon,R_1}_{cut} \to \tilde{B}(x_i, r)^{\delta',R_1}_{cut} \tag{5.18}$$

is nontrivial. Now take the limit as $i \to \infty$ and we get

$$B(x,r)_{cut}^{\delta-\epsilon,R_1} \to B(x,r)_{cut}^{\delta',R_1} \tag{5.19}$$

is nontrivial.

This is true for all $\epsilon \in (0,\delta)$ and $\delta' > \delta$. Now by the properties of limit covers in Proposition 5.4 we have for all $\epsilon \in (0,\delta)$ and $\delta'' > \delta'$, $R' \in (R_1, R_2)$, $\tilde{B}(x,r)_{cut}^{\delta-\epsilon,R_1} \to B(x,r)_{cut}^{\delta-\epsilon,R_1}$ and $B(x,r)_{cut}^{\delta',R_1} \to \tilde{B}(x,r)_{cut}^{\delta'',R'}$. Therefore $\tilde{B}(x,r)_{cut}^{\delta-\epsilon,R_1} \to \tilde{B}(x,r)_{cut}^{\delta'',R'}$ is nontrivial.

By Proposition 4.31, we then know that since $B(x,r)$ is compact and $R_2 > R'$ we have

$$CovSpec_{cut}^{R_2}(B(x,r)) \cap [\delta-\epsilon, \delta'') \neq \emptyset. \tag{5.20}$$

Taking ϵ to 0 and δ'' to δ, we get

$$\delta \in CovSpec_{cut}^{R_2}(B(x,r) \tag{5.21}$$

This is true for all sufficiently large r, so by Lemma 4.27,

$$\delta \in CovSpec_{cut}^{R_2}(X) \tag{5.22}$$

which completes proof of (5.16).

We now boost the statement (5.16) to prove (5.8). Again fix $R_1 > 0$. Suppose

$$\delta_i \in CovSpec_{cut}^{R_1}(X_i) \tag{5.23}$$

and $\delta_i \to \delta > 0$, Let X be the Gromov-Hausdorff limit of the X_i.

Proposition 4.31 says that that for $R_2 = R_i$ sufficiently close to R_1

$$\tilde{X}_{cut}^{\delta,R_2} = \tilde{X}_{cut}^{\delta,R_1} \tag{5.24}$$

putting this together with Proposition 4.30 says

$$\delta \notin CovSpec_{cut}^{R_2}(X) \setminus CovSpec_{cut}^{R_1}(X) \tag{5.25}$$

We apply (5.16 to say

$$\delta \in CovSpec_{cut}^{R_2}(X). \tag{5.26}$$

But then

$$\delta \in CovSpec_{cut}^{R_1}(X), \tag{5.27}$$

which gives us (5.8).

Now we prove the second statement (5.9): given

$$\delta \in CovSpec_{cut}^{R_1}(X) \tag{5.28}$$

and any $R_2 > R_1$, show there exists

$$\delta_i \in CovSpec_{cut}^{R_2}(X_i) \tag{5.29}$$

such that $\delta_i \to \delta$.

We assume on the contrary that there is a gap:

$$\exists \epsilon > 0 \text{ such that } CovSpec_{cut}^{R_2}(X_i) \cap (\delta - 2\epsilon, \delta + 2\epsilon) = \emptyset. \tag{5.30}$$

By Lemma 4.27, for $r \geq 3(R_2 + 2\delta + 4\epsilon)$,

$$CovSpec_{cut}^{R_2}(B(x_i,r)) \cap (\delta - 2\epsilon, \delta + 2\epsilon) = \emptyset. \tag{5.31}$$

By Lemma 4.13 we then have for any $R_2' \le R_2$:

$$CovSpec_{cut}^{R_2'}(B(x_i,r)) \cap (\delta-2\epsilon, \delta+2\epsilon) = \emptyset. \tag{5.32}$$

So the covering

$$\tilde{B}(x_i,r)_{cut}^{\delta-\epsilon,R_2'} \to \tilde{B}(x_i,r)_{cut}^{\delta+\epsilon,R_2'} \tag{5.33}$$

is trivial. By Proposition 5.4 we have a subsequence of the i such that:

$$\tilde{B}(x_i,r)_{cut}^{\delta-\epsilon,R_2'} \to B(x,r)_{cut}^{\delta-\epsilon,R_2'} \tag{5.34}$$

and

$$\tilde{B}(x_i,r)_{cut}^{\delta+\epsilon,R_2'} \to B(x,r)_{cut}^{\delta+\epsilon,R_2'}. \tag{5.35}$$

since the sequence of the covering map is trivial, the covering limit map

$$B(x,r)_{cut}^{\delta-\epsilon,R_2'} \to B(x,r)_{cut}^{\delta+\epsilon,R_2'} \tag{5.36}$$

is also trivial.

By Proposition 5.4 for any $R_2 \ge R_2' > R_1$

$$B(x,r)_{cut}^{\delta-\epsilon,R_2'} \to \tilde{B}(x,r)_{cut}^{\delta,R_1} \to \tilde{B}(x,r)_{cut}^{\delta+\epsilon,R_1} \to B(x,r)_{cut}^{\delta+\epsilon,R_1}. \tag{5.37}$$

By Proposition 4.31, for any $R_2' > R_1$ sufficiently close to R_1, the covering

$$\tilde{B}(x,r)_{cut}^{\delta-\epsilon,R_2'} \to \tilde{B}(x,r)_{cut}^{\delta+2\epsilon,R_1} \tag{5.38}$$

is trivial. Using Proposition 5.4, we have for $R_2'' < R_2'$, the covering

$$B(x,r)_{cut}^{\delta-\epsilon,R_2''} \to B(x,r)_{cut}^{\delta+\epsilon,R_1} \tag{5.39}$$

is trivial.

Apply this R_2'' to (5.37) we get trivial covers in 5.37). So $\delta \notin CovSpec_{cut}^{R_1}(B(x,r))$.

By Lemma 4.28, $\delta \notin CovSpec_{cut}^{R_1}(X)$. That is a contradiction. ∎

5.3 Convergence of the cut-off covering spectrum

Theorem 5.2 combined with Proposition 4.15 gives the following result that elements in the cut-off covering spectrum do not suddenly appear in limits. Example 5.4 demonstrates that elements of the cut-off covering spectrum can disappear in the limit by sliding out to infinity. Unlike the R cut-off covering spectrum, all handles are now visible.

Theorem 5.5 *Let (X_i, x_i) be locally compact metric spaces converging in the pointed Gromov-Hausdorff sense to a locally compact space (X,x), then*

$$\text{for any } \delta \in CovSpec_{cut}(X), \text{ there is } \delta_i \in CovSpec_{cut}(X_i) \tag{5.40}$$

such that $\delta_i \to \delta$.

This provides an immediate application:

Corollary 5.6 *If X_i are simply connected locally compact length spaces converging in the pointed Gromov-Hausdorff sense to a locally compact space (X,x) then $CovSpec_{cut}(X) = \emptyset$.*

Proof of Theorem 5.5: If $\delta \in CovSpec_{cut}(X)$, by Proposition 4.15,

$$\delta \in Cl_{lower} \cup_{R>0} CovSpec^{R}_{cut}(X). \tag{5.41}$$

So there are R_k increasing to infinity and

$$\delta_k \in CovSpec^{R_k}_{cut}(X) \tag{5.42}$$

such that $\delta_k \to \delta$. By Proposition 5.2 and $R_{k+1} > R_k$, for each δ_k, we have

$$\delta^i_k \in CovSpec^{R_{k+1}}_{cut}(X_i) \subset CovSpec_{cut}(X_i) \tag{5.43}$$

such that $\delta^i_k \to \delta_k$.

By a diagonal process, we have

$$\delta_i = \delta^i_{k_i} \in CovSpec_{cut}(X_i) \tag{5.44}$$

such that $\delta_i \to \delta$. ■

The condition that X_i and X be locally compact is shown to be necessary in Example **??**.

5.4 Applications of Convergence

In this section we observe the following topological consequence of our convergence results:

Theorem 5.7 *If X_i are locally compact metric spaces that satisfy the loops to infinity property and converge in the pointed Gromov-Hausdorff sense to a locally compact limit space X then either X has at least two ends or X has the loops to infinity property.*

Proof: By Theorem 4.20, $CovSpec_{cut}(X_i)$ are trivial. So by Theorem 5.5, $CovSpec_{cut}(X)$ must be trivial. To complete the proof we just apply Theorem 4.21. ■

One can think of this theorem as the complete version of the theorem in [SoWei1] which says that compact Gromov-Hausdorff limits of simply connected compact manifolds are simply connected.

Example 5.8 *The [SoWei1] theorem is not true for noncompact limits with pointed Gromov-Hausdorff convergence as can be seen by taking sequences of ellipsoids M^2_j which stretch out to a cylinder $S^1 \times \mathbb{R}$ or $M^2_j \times \mathbb{R}$ converging to $S^1 \times \mathbb{R}^2$. Thanks to our new theorem we see that while holes may form in a limit they cannot be handles.*

Example 5.9 *Notice if one takes a disk and stretches two points out to infinity then the limit is a disk with two cusps, which is no longer simply connected. Nor does it has the loops to infinity property. This is because a loop wrapping once around each cusp is not homotopic to loops approaching infinity. However the fundamental group of the space is generated by elements with the loops to infinity property.*

Example 5.10 *Note that one can have compact M_j with $\pi_2(M_j)$ converging to a space with nontrivial π_2. This can be seen by taking M_j diffeomorphic to the plane with warped product metrics*

$$dr^2 + f^2_j(r)d\theta^2 \tag{5.45}$$

where $f(r) = r((1-r)^2 + (1/k))$, so that the Gromov-Hausdorff limit as $k \to \infty$ is homoemorphic to a sphere attached to a plane. So we cannot hope to control higher homotopy, although an investigation of [ShSo1] reveals a close relationship between the loops to infinity property and the codimension one integer homology of the space.

5.5 Tangent Cones at Infinity

A complete noncompact space, X, is said to have a tangent cone at infinity if the Gromov-Hausdorff limit of a sequence of inward rescalings $(X/r_j, x)$ with $r_j \to \infty$, has a limit in the pointed Gromov-Hausdorff sense. While this limit space is called a cone, it is not a metric cone except in very special situations, like when X has nonnegative sectional curvature REF. In fact the tangent cone at infinity of a manifold need not even be simply connected:

Example 5.11 *Let M^2 be created by taking a cone, smoothing off the tip and adding handles, r_iH, at a distance r_i from the old tip. We write r_iH because we are rescaling the handle H by r_i, so that the handles are growing. Then M^2/r_i converges to a cone with a handle attached at a distance 1 from the tip. If $\lim r_{i+1}/r_i = \infty$, then the tangent cone has only one handle, but if $\lim r_{i+1}/r_i = d$, then the tangent cone has infinitely many handles located at $\{d^j : j \in \mathbb{Z}\}$, so the tangent cone at infinity, Y, has locally infinite topological type at its tip. Furthemore Y has no universal cover and $CovSpec(Y) = CovSpec_{cut}(Y)$ have infinitely many elements.*

Remark 5.8 *Menguy has created similar examples demonstrating that the tangent cone at infinity of a manifold with nonnegative Ricci curvature can have locally infinite topological type, although his examples are simply connected because his handles are higher dimensional (c.f. [ShSo2]). In [SoWei3] we proved the tangent cones at infinity of manifolds with Ricci ≥ 0 have universal covers.*

Using our results we can prove

Proposition 5.9 *If X is a locally compact metric space and $CovSpec_{cut}(X)$ is bounded then any tangent cone at infinity for X has a trivial cut-off covering spectrum.*

Proof: If we rescale a space dividing the metric by r then by Definition REF

$$CovSpec^{R/r}_{cut}(X/r) = CovSpec^{R}_{cut}(X)/r \tag{5.46}$$

So applying Proposition 4.15 we have

$$CovSpec_{cut}(X/r) = CovSpec_{cut}(X/r) \subset [0, Max(CovSpec(X))/r] \tag{5.47}$$

Any tangent cone at infinity, Y, is the Gromov-Hausdorff limit of X/r_i with $r_i \to \infty$, so by Theorem 5.5, $CovSpec_{cut}(Y) \subset \{0\}$ and is, thus, trivial. ■

This proposition implies that the tangent cones at infinity of manifolds with bounded covering spectra have trivial cutoff covering spectra. However, they need not have trivial covering spectra even when the manifold has a trivial covering spectrum:

Example 5.12 *We begin with a cone smoothed at the tip and at a distance R_i from the old tip we smoothly attach R_iX_i where X_i are the surfaces from Example 5.3 chopped off so they are warped products of $[r_i, \infty)$ so each has a loop to infinity which is in the slipping group of our space. So we have a smooth surface, M, with a trivial covering spectrum. Yet when we rescale M/R_i we converge to a cone with $\lim(X_i, x)$ attached at a distance 1. Which is a half cylinder, so the covering spectrum of the limit space includes the element $\{\pi\}$. The cut off covering spectrum is trivial.*

6 Appendix

As the concept of lower semiclosure does not seem to appear in the literature, we include a brief exposition here.

Definition 6.1 *A lower semiclosed subset of the real line is a set A such that $lim_{j\to\infty} a_j \in A$ whenever a_j is a decresing sequence of elements of A.*

Definition 6.2 *The lower semiclosure of a set A, denoted $Cl_{lower}(A)$, is the intersection of all lower semiclosed sets containing A.*

Lemma 6.3 *The lower semiclosure of A is the union of A and the limits of any decreasing sequence of $a_j \in A$.*

Lemma 6.4 *If $x \notin A$ and A is lower semiclosed, then there exists $\epsilon > 0$ such that*

$$[x, x+\epsilon) \cap A = \emptyset. \tag{6.1}$$

The following theorem implies that $CovSpec(X)$, $CovSpec_{cut}(X)$, and $CovSpec_{rescaled}(X)$ are all lower semiclosed subset of (R).

Theorem 6.5 *Let X_s be a collection of metric spaces parametrized by a real line, $s \in (R)$, such that whenever $s_1 < s_2$ we have X_{s_1} covers X_{s_2}. Any set A defined as follows:*

$$A := \{s : \forall s' > s X_{s'} \neq X_s\} \tag{6.2}$$

then A is lower semiclosed.

Proof: Let $s_j \in A$ a decreasing sequence converging to s_∞. We need to show $s_\infty \in A$. Let $s' > s_\infty$. Then for j sufficiently large, we have $s' > s_j$. Since $s_j \in A$ this means $X_{s'} \neq X_{s_j}$ so X_{s_j} is a nontrivial cover of $X_{s'}$. And since X_{s_∞} covers X_{s_j}, it must be a nontrivial cover of $X_{s'}$ as well. ■

Example 6.1 *If A_j are all lower semiclosed sets, the $\bigcup_{j\in\mathbb{N}} A_j$ need not be lower semiclosed. For example, let $A_j = \{1 + 1/k : k = 1, 2, ...j\}$.*

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