

WEAK CONVERGENCE AND CANCELLATION

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Abstract. In this article, we study the relationship between the weak limit of a sequence of integral currents in a metric space and the possible Hausdorff limit of the sequence of supports. Due to cancellation, the weak limit is in general supported in a strict subset of the Hausdorff limit. We exhibit sufficient conditions in terms of topology of the supports which ensure that no cancellation occurs and that the support of the weak limit agrees with the Hausdorff limit of the supports. We use our results to prove countable \mathcal{H}^m -rectifiability of the Gromov-Hausdorff limit of sequences of Lipschitz manifolds M_n all of which are λ -linearly locally contractible up to some scale r_0 . In the appendix, we show that the Gromov-Hausdorff limit need not be countably \mathcal{H}^m -rectifiable if the M_n have a common local geometric contractibility function which is only concave (and not linear). We also relate our results to work of Cheeger-Colding on the limits of noncollapsing sequences of manifolds with nonnegative Ricci curvature.

1. I

This article is concerned with the relationship between two notions of convergence, Hausdorff convergence for sets on the one hand, and weak convergence for (integral) currents on the other hand. These play an important role in problems in metric and Riemannian geometry and geometric measure theory, respectively. We give sufficient conditions in terms of topology on the sets (supports of the currents) that guarantee that these two notions ‘agree’.

We work in the generality of metric spaces (for reasons that will become clear later). In this setting, Ambrosio-Kirchheim have recently developed in [1] a powerful theory of metric integral currents which extends the classical Federer-Fleming theory [5] from Euclidean to (general) metric spaces Z . For $m \geq 0$, the space of metric integral m -currents in Z is denoted by $\mathbf{I}_m(Z)$. Similarly to the classical theory, elements $T \in \mathbf{I}_m(Z)$ are ‘functionals’ (on suitably generalized m -forms on Z) satisfying certain properties. As in the classical theory, there is a notion of mass, $\mathbf{M}(T)$, an associated measure, $\|T\|$, and, for dimension $m \geq 1$, a notion of boundary, ∂T , which is an element of $\mathbf{I}_{m-1}(Z)$. The support of T is the closed set

$$\text{spt } T := \{z \in Z : \|T\|(B(z, \varepsilon)) > 0 \text{ for all } \varepsilon > 0\},$$

where $B(z, \varepsilon)$ denotes the open ball of radius ε centered at z . Weak convergence is defined as pointwise convergence, as in the classical theory: T_n converges weakly to T if $T_n(f d\pi) \rightarrow T(f d\pi)$ for every Lipschitz m -form $f d\pi$ on Z . We refer to Section 2 for definitions and details concerning integral currents.

We now turn to the relationship between weak convergence of the T_n and Hausdorff convergence of the sets $\text{spt } T_n$. For the definition of Hausdorff convergence of sets we refer

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to Section 2.1. We say that $(T_n) \subset \mathbf{I}_m(Z)$ is a bounded sequence if

$$\sup_n [\mathbf{M}(T_n) + \mathbf{M}(\partial T_n)] < \infty.$$

It is well-known, see e.g. Lemma 2.7, that if T_n converges weakly to some $T \in \mathbf{I}_m(Z)$ and the sequence $(\text{spt } T_n)$ converges in the Hausdorff sense to a closed subset $X \subset Z$ then

$$\text{spt } T \subseteq X.$$

The inclusion may be strict as the simple Examples 6.1 – 6.3 illustrate. Strict inclusion is sometimes referred to as ‘cancellation’, see Section 6. In our first two theorems below we show that under suitable restrictions on the local topology of $\text{spt } T_n$ cancellation cannot occur, thus $\text{spt } T = X$. This is then used to prove metric structure results for the Gromov-Hausdorff limit of sequences of Riemannian manifolds which are uniformly linearly locally contractible up to some scale and sequences of noncollapsing Riemannian manifolds of nonnegative Ricci curvature.

We turn to the precise statements of our results and first recall the following definition.

Definition 1.1. *Given a metric space Y and $\lambda \geq 1$, a subset $A \subset Y$ is called λ -linearly locally contractible in Y if, for all $a \in A$ and $r > 0$, $A \cap \bar{B}(a, r)$ is contractible in $\bar{B}(a, \lambda r)$. Given $m \geq 0$, a subset $A \subset Y$ is called λ -linearly locally m -connected in Y if, for every $k \in \{0, \dots, m\}$, all $a \in A$ and $r > 0$, every continuous map $f : S^k \rightarrow Y$ with image in $A \cap \bar{B}(a, r)$ has a continuous extension $\bar{f} : \bar{B}^{k+1} \rightarrow Y$ with image in $\bar{B}(a, \lambda r)$. If for some $r_0 > 0$, every r_0 -ball in Y is λ -linearly locally contractible (m -connected) in Y then Y is called λ -linearly locally contractible (m -connected) up to scale r_0 .*

Here, S^k and \bar{B}^{k+1} denote the unit k -sphere and the closed unit ball in \mathbb{R}^{k+1} , respectively. We will also need the following definition, taken from [15].

Definition 1.2. *A metric space Y is Lipschitz m -connected in the small if there exist constants $\gamma > 0$ and $\delta > 0$ such that for every $k \in \{0, \dots, m\}$, every ν -Lipschitz map $f : S^k \rightarrow Y$ with $\nu \leq \delta$ has a $\gamma\nu$ -Lipschitz extension $\bar{f} : \bar{B}^{k+1} \rightarrow Y$.*

Clearly, every compact Riemannian n -manifold is Lipschitz m -connected in the small for every m ; the same is true for every compact metric space locally biLipschitz homeomorphic to an open set in \mathbb{R}^n .

In its most general form, the main result of our paper can be stated as follows. (For the convenience of the geometrically minded reader, we will include versions for Riemannian manifolds of all our theorems in the corollaries below.)

Theorem 1.3. *Let Z be a complete metric space, $m \geq 1$, and $(T_n) \subset \mathbf{I}_m(Z)$ a bounded sequence of integral currents weakly converging to some $T \in \mathbf{I}_m(Z)$. Suppose that the support $\text{spt } T_n$ of each T_n is compact, Lipschitz m -connected in the small, and has Nagata dimension $\leq m$. Let furthermore $z \in Z$ and $\lambda \geq 1$, $r > 0$. If there exists a sequence $z_n \rightarrow z$ with $z_n \in \text{spt } T_n$ and such that $B(z_n, r) \cap \text{spt } \partial T_n = \emptyset$ and $B(z_n, r) \cap \text{spt } T_n$ is λ -linearly locally m -connected in $\text{spt } T_n$ for all n large enough, then*

$$(1) \quad \|T\|(B(z, s)) \geq C \lambda^{-m(m+1)} s^m \quad \text{for all } 0 \leq s \leq 2^{-(m+6)} \lambda^{-(m+1)} r,$$

for some $C > 0$ depending only on m . In particular, $z \in \text{spt } T$ and $\|T\|$ has strictly positive lower m -dimensional density at z .

For the definition and properties of the Nagata dimension (also called Assouad-Nagata dimension in the literature) we refer to Section 2.1 and to [15]. We mention here that the

Nagata dimension of a metric space is always at least its topological dimension. Furthermore, the Nagata dimension of any compact m -dimensional Riemannian manifold is m . This is more generally true of any compact metric space locally homeomorphic to an open subset of \mathbb{R}^m . Thus, Theorem 1.3 implies:

Corollary 1.4. *Let (M_n) be a sequence of compact oriented m -dimensional Riemannian manifolds with uniformly bounded volume and volume of boundary. Let $p_n \in M_n$ and suppose that for some $\lambda \geq 1$ and $r > 0$ and each n , the ball $B_{M_n}(p_n, r)$ is λ -linearly locally contractible in M_n and does not intersect ∂M_n . Suppose further that Z is a metric space and $\varphi_n : M_n \hookrightarrow Z$ are isometric embeddings. If $\varphi_n(p_n)$ converges to some z and if the sequence of currents $T_n := \varphi_{n\#} \llbracket M_n \rrbracket$ weakly converges to some $T \in \mathbf{I}_m(Z)$ then*

$$\|T\|(B(z, s)) \geq C \lambda^{-m(m+1)} s^m \quad \text{for all } 0 \leq s \leq 2^{-(m+6)} \lambda^{-(m+1)} r,$$

for some $C > 0$ depending only on m . In particular, $z \in \text{spt } T$ and $\|T\|$ has strictly positive lower m -dimensional density at z .

For the definition of $\llbracket M_n \rrbracket$ see (4). The term *isometric* in the corollary means distance preserving (when M_n is viewed as a metric space). Throughout this paper we assume manifolds to be connected. In Example 5.3 we show that the hypothesis in Theorem 1.3 that $\text{spt } T_n$ has Nagata dimension $\leq m$ cannot be replaced by $\leq m+1$, even when $Z = \mathbb{R}^{m+1}$. Note also that in Theorem 1.3 the constants appearing in Definition 1.2 for $\text{spt } T_n$ are allowed to ‘degenerate’ as $n \rightarrow \infty$.

Theorem 1.3 can be used to prove the following result:

Theorem 1.5. *Let Z be a complete metric space, $m \geq 1$, and $(T_n) \subset \mathbf{I}_m(Z)$ a bounded sequence of integral currents weakly converging to some $T \in \mathbf{I}_m(Z)$, with $\partial T_n = 0$ for all n . Suppose that the support $\text{spt } T_n$ of each T_n is compact, Lipschitz m -connected in the small, and has Nagata dimension $\leq m$. If for some $\lambda \geq 1$ and $r > 0$, $\text{spt } T_n$ is λ -linearly locally m -connected up to scale r for all n large enough, then $\text{spt } T_n$ converges in the Hausdorff sense to $\text{spt } T$ and*

$$(2) \quad \|T\|(B(z, s)) \geq C \lambda^{-m(m+1)} s^m$$

for all $z \in \text{spt } T$ and $0 \leq s \leq 2^{-(m+6)} \lambda^{-(m+1)} r$, where $C > 0$ depends only on m . In particular, $\text{spt } T$ is countably \mathcal{H}^m -rectifiable.

We refer to Section 2.2 for the definition of countable \mathcal{H}^m -rectifiability. Note that the support of an arbitrary integral current T in Z need not be countably \mathcal{H}^m -rectifiable; indeed if Z is for example Z is a separable Banach space then an integral current T in Z can have $\text{spt } T = Z$. Clearly, (2) implies that $\|T\|$ has strictly positive lower m -dimensional density at every point $z \in \text{spt } T$. The last claim in the theorem follows from this together with Theorem 4.6 in [1]. Theorem 1.5 clearly implies the following (also compare with Corollary 1.8):

Corollary 1.6. *Let (M_n) be a sequence of closed oriented m -dimensional Riemannian manifolds with uniformly bounded volume and such that for some $\lambda \geq 1$ and $r > 0$, each M_n is λ -linearly locally contractible up to scale r . Suppose further that Z is a metric space and $\varphi_n : M_n \hookrightarrow Z$ are isometric embeddings. If the sequence of currents $T_n := \varphi_{n\#} \llbracket M_n \rrbracket$ weakly converges to some $T \in \mathbf{I}_m(Z)$ then $\varphi_n(M_n)$ Hausdorff converges to $\text{spt } T$. Furthermore $\|T\|$ has strictly positive lower m -dimensional density at every $z \in \text{spt } T$ and, in particular, $\text{spt } T$ is countably \mathcal{H}^m -rectifiable.*

A slightly more general form of Theorem 1.5 will be discussed in Remark 5.5. In Section 7 we will moreover prove an analog of Theorem 1.5 for sequences of oriented closed Riemannian manifolds with nonnegative Ricci curvature, with a uniform upper bound on diameter and strictly positive lower bound on volume. As is well-known, such sequences admit an isometric embedding into some compact metric space and therein give rise to a sequence of integral currents. However, they do not satisfy the hypotheses of Theorem 1.5 in general, see Section 7 for details and references. Nevertheless, Theorem 7.1 shows that the support of the weak limit is countably \mathcal{H}^m -rectifiable and coincides with the (Gromov-)Hausdorff limit of the sequence. Countable \mathcal{H}^m -rectifiability was proved before by Cheeger-Colding in [3]; Theorem 7.1 gives a new perspective of this fact.

Theorem 1.3 can be used to prove the following theorem whose statement does not involve any currents.

Theorem 1.7. *Let $m \geq 1$ and let (X_n) be a sequence of compact metric spaces such that each X_n is locally biLipschitz homeomorphic to the open unit ball $B \subset \mathbb{R}^m$. Suppose further that each X_n is orientable and that the X_n have uniformly bounded diameter and Hausdorff m -measure. If there exist $\lambda \geq 1$ and $r > 0$ such that X_n is λ -linearly locally contractible up to scale r for all n large enough, then there exists a subsequence X_{n_j} which converges in the Gromov-Hausdorff sense to a compact and countably \mathcal{H}^m -rectifiable space X with $0 < \mathcal{H}^m(X) < \infty$.*

Here, X_n orientable means that there exist finitely many biLipschitz maps $\varphi_i : B \hookrightarrow X_n$ such that the sets $\varphi_i(B)$ cover X_n and such that $\det(\nabla(\varphi_i^{-1} \circ \varphi_j)) > 0$ almost everywhere on $\varphi_j^{-1}(\varphi_i(B))$. As a simple consequence we obtain the following result.

Corollary 1.8. *Let (M_n) be a sequence of closed, orientable m -dimensional Riemannian manifolds of uniformly bounded volume. If there exist $\lambda \geq 1$ and $r > 0$ such that M_n is λ -linearly locally contractible up to scale r for all n large enough, then there exists a subsequence (M_{n_j}) which converges in the Gromov-Hausdorff sense to a compact and countably \mathcal{H}^m -rectifiable metric space X with $0 < \mathcal{H}^m(X) < \infty$.*

Note that no diameter bound is assumed in the corollary. The existence of a Gromov-Hausdorff limit in Corollary 1.8 was already proved by Greene-Petersen in [7] under the following weaker conditions: Let $r > 0$ and let $\varrho : [0, r) \rightarrow [0, \infty)$ be a continuous function with $\varrho(0) = 0$ and $\varrho(s) \geq s$ for all $s \in [0, r)$. Then ϱ is called a local geometric contractibility function for a given metric space X if for every $z \in X$ and every $s \in (0, r)$ the ball $B(z, s)$ is contractible in $B(z, \varrho(s))$. It is shown in [7] that a sequence of closed m -dimensional Riemannian manifolds with a uniform upper bound on volume and such that ϱ is a local geometric contractibility function for every M_n then a subsequence M_{n_j} converges in the Gromov-Hausdorff sense to a compact metric space X . Their proof relies on a lower bound – proved in [7] using an argument similar to one given by Gromov in [9] – for the filling radius of (smooth approximations to) distance spheres in such Riemannian manifolds. In Theorem 4.1 we prove a generalization of their filling radius estimate using a different approach and use it in the proof of our main theorem. Note that our assumption in Corollary 1.8 is that all M_n have local geometric contractibility function $\varrho(s) = \lambda s$. In the appendix of the present paper, Raanan Schul and the second author show that if in Corollary 1.8 the condition on λ -linear local m -connectedness is replaced by uniform geometric contractibility, then the Gromov-Hausdorff limit X need not be countably \mathcal{H}^m -rectifiable. If, in addition, no uniform bound on volume is assumed, then M_n can converge to an infinite dimensional space, as was shown by Ferry-Okun in [6].

Our initial motivation for studying the relationship between Hausdorff convergence and weak convergence originates from the article [21] in which we first introduced and studied a new distance, called flat intrinsic distance, between compact oriented Riemannian manifolds. Roughly, the flat intrinsic distance between M and M' is the infimal flat distance that can be achieved by isometrically embedding M and M' in a metric space Z . Flat intrinsic convergence of a sequence of Riemannian manifolds then amounts to flat convergence in a suitable metric space into which the sequence isometrically embeds. It is natural to ask what the relationship between the flat intrinsic and Gromov-Hausdorff convergence is. Since flat convergence implies weak convergence, our results above can be interpreted as giving sufficient conditions which ensure that a Gromov-Hausdorff limit coincides with the flat intrinsic limit, see [21] and also Remark 5.5 of the present paper.

The paper is organized as follows. Section 2 provides mostly well-known definitions and facts used throughout the paper. Proposition 2.3 is new, however, and due to U. Lang. It provides a relationship between Hausdorff and Nagata dimension for subsets in \mathbb{R}^n and is used in the proof of Theorem 4.1. In Section 3 we prove a Lipschitz extension theorem, Theorem 3.4, which is a variant of a result of Lang-Schlichenmaier [15]. In Section 4 we use this theorem to exhibit lower bounds for the absolute filling radius of slices with spheres of integral currents, generalizing a theorem in [7]. Section 5 is devoted to the proofs of the results stated in the introduction. In Section 7 we show that the (Gromov-) Hausdorff limit of sequences of Riemannian manifolds with nonnegative Ricci curvature agrees with the support of the weak limit and is countably \mathcal{H}^m -rectifiable. The content of the appendix was discussed above.

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2. P

2.1. Hausdorff and Gromov-Hausdorff distance, Nagata dimension. Let (Z, d) be a metric space and $A, B \subset Z$ closed. The Hausdorff distance between A and B in Z is

$$d_H(A, B) := \inf \{ \varepsilon \geq 0 : A \subset U_\varepsilon(B) \text{ and } B \subset U_\varepsilon(A) \},$$

where $U_\varepsilon(B)$ denotes the open ε -neighborhood of B . It is clear that d_H is symmetric and satisfies the triangle inequality. If $d_H(A, B) = 0$ then $A = B$.

Definition 2.1. A sequence (A_n) of closed subsets of Z is said to converge in the Hausdorff sense to a closed subset $A \subset Z$ if $d_H(A_n, A) \rightarrow 0$ as $n \rightarrow \infty$.

If X and Y are metric spaces then the Gromov-Hausdorff distance between X and Y is $d_{GH}(X, Y) := \inf \{ d_H(\iota_X(X), \iota_Y(Y)) : Z \text{ metric space, } \iota_X : X \hookrightarrow Z, \iota_Y : Y \hookrightarrow Z \text{ isometric} \}$.

A sequence of complete metric spaces X_n is said to converge in the Gromov-Hausdorff sense to a complete metric space X if $d_{GH}(X_n, X) \rightarrow 0$ as $n \rightarrow \infty$. By Gromov's famous compactness theorem for metric spaces, see [8], every uniformly compact sequence (X_n) of metric spaces admits an isometric embedding into a common compact metric space Z . It follows that a subsequence X_{n_j} converges in the Gromov-Hausdorff sense to a compact metric space, in fact to a closed subset of Z . Here, a sequence (X_n) of compact metric

spaces X_n is called uniformly compact if $\sup \text{diam } X_n < \infty$ and if for every $\varepsilon > 0$ there exists $N(\varepsilon)$ such that every X_n can be covered by at most $N(\varepsilon)$ balls of radius ε .

We turn to the definition and properties of the Assouad-Nagata dimension for a metric space Y . For a detailed account we refer to [15]. A family $(B_i)_{i \in I}$ of subsets of Y is called D -bounded if $\text{diam } B_i \leq D$ for all $i \in I$. For $s \geq 0$, the s -multiplicity of the family is the infimum of all $k \geq 0$ such that every subset of X with diameter $\leq s$ intersects at most k members of the family.

Definition 2.2. *The (Assouad-)Nagata dimension of a metric space Y is the infimum of all integers n with the following properties: there exists a constant $c > 0$ such that for all $s > 0$, Y has a cs -bounded covering with s -multiplicity at most $n + 1$.*

It can be shown that the Nagata dimension of Y is at least its topological dimension. If $Y = Y_1 \cup Y_2$ then the Nagata dimension of Y is the maximum of the Nagata dimensions of Y_1 and Y_2 . Every subset of \mathbb{R}^n with nonempty interior has Nagata dimension n . The Nagata dimension is invariant under biLipschitz homeomorphisms. It thus follows that every compact metric space locally biLipschitz homeomorphic to an open subset of \mathbb{R}^n has Nagata dimension n . In particular, every compact Riemannian n -manifold has Nagata dimension n . For proofs of all these statements and many more properties see [15].

In the proof of Theorem 4.1 we will need the following proposition whose statement and proof below are due to U. Lang [14].

Proposition 2.3. *Suppose $X \subset \mathbb{R}^n$ has Nagata dimension $m < n$ ($m \geq 0$, $n \geq 1$). Then X is uniformly porous; in particular, the Hausdorff dimension of X is strictly less than n .*

For the definition of Hausdorff dimension see Section 2.2; for the notion of porous sets and the last assertion in the above proposition, see e.g. the introduction of [19].

Proof. There exists a constant $c > 0$ such that for every $s > 0$, X admits a cs -bounded covering of the form $\mathcal{B} = \bigcup_{k=0}^m \mathcal{B}_k$, where each family \mathcal{B}_k has s -multiplicity 1 (cf. [15, Proposition 2.5(4)]). Let $s > 0$. Put $b := s/(2\sqrt{n})$ and let N be the smallest integer such that $l := Nb > cs + 2b$. Let $D \subset \mathbb{R}^n$ be a closed cube of edge length l , and consider the subdivision C into N^n closed cubes of edge length b . We claim that at least one $C \in C$ is disjoint from X . Otherwise we can associate to each $C \in C$ a color $k(C) \in \{0, \dots, m\}$, meaning that C intersects a member of the family $\mathcal{B}_{k(C)}$. For fixed k , consider the connected components of the union of all $C \in C$ with $k(C) = k$. Suppose that one such component of color k , L , intersects two opposite faces of D . Then L contains two cubes $C, C' \in C$ with $d(C, C') > cs$, so that $C \cap B \neq \emptyset \neq C' \cap B'$ for two distinct members B, B' of \mathcal{B}_k . Since L is connected, we can also choose C, C' with this last property such that $C \cap C' \neq \emptyset$. Then $\text{diam}(C \cup C') \leq s$, which contradicts the fact that \mathcal{B}_k has s -multiplicity 1. This shows that no connected component connects two opposite faces of D . By Lebesgue's Covering Theorem (cf. [11, p. 42]), among the connected components of all colors we find $n + 1$ with a common point. Since there are only $m + 1 < n + 1$ different colors, this is a contradiction. Hence, at least one $C \in C$ is disjoint from X . This holds for every $s > 0$, and N is independent of s . So X is uniformly porous. \square

2.2. Hausdorff measure and countable \mathcal{H}^m -rectifiability. Let Z be a metric space and $A \subset Z$. The Hausdorff m -dimensional measure of A is

$$\mathcal{H}^m(A) := \liminf_{\delta \searrow 0} \left\{ \sum_{i=1}^{\infty} \omega_m \left(\frac{\text{diam}(B_i)}{2} \right)^m : B \subset \bigcup_{i=1}^{\infty} B_i, \text{diam}(B_i) < \delta \right\},$$

where ω_m denotes the Lebesgue measure of the unit ball in \mathbb{R}^m . For $Z = \mathbb{R}^m$, \mathcal{H}^m agrees with the Lebesgue measure. The Hausdorff measure \mathcal{H}^s can also be defined for non-integer numbers s by replacing the power m in the above definition by s and ω_m by a suitable positive number ω_s , see for example [12]. The Hausdorff dimension of A is the infimum over all $s \geq 0$ such that $\mathcal{H}^s(A) = 0$.

Definition 2.4. An \mathcal{H}^m -measurable set $A \subset Z$ is said to be countably \mathcal{H}^m -rectifiable if there exist countably many Lipschitz maps $f_i : B_i \rightarrow Z$ from subsets $B_i \subset \mathbb{R}^m$ such that

$$\mathcal{H}^m\left(A \setminus \bigcup f_i(B_i)\right) = 0.$$

Finally, the m -dimensional lower density $\Theta_{*m}(\mu, x)$ of a finite Borel measure μ on Z at a point z is given by the formula

$$\Theta_{*m}(\mu, z) := \liminf_{r \searrow 0} \frac{\mu(B(z, r))}{\omega_m r^m}.$$

2.3. Currents in metric spaces. We recall the basic definitions from the theory of currents which we need in this paper. The general reference is [1].

Let (Z, d) be a complete metric space and $m \geq 0$ and let $\mathcal{D}^m(Z)$ be the set of $(m+1)$ -tuples (f, π_1, \dots, π_m) of Lipschitz functions on Z with f bounded. Sometimes, we write the short-hand $f d\pi$ for (f, π_1, \dots, π_m) . The Lipschitz constant of a Lipschitz function f on Z will be denoted by $\text{Lip}(f)$.

Definition 2.5. An m -dimensional metric current T on Z is a multi-linear functional on $\mathcal{D}^m(Z)$ satisfying the following properties:

(i) If π_i^j converges pointwise to π_i as $j \rightarrow \infty$ and if $\sup_{i,j} \text{Lip}(\pi_i^j) < \infty$ then

$$T(f, \pi_1^j, \dots, \pi_m^j) \rightarrow T(f, \pi_1, \dots, \pi_m).$$

(ii) If $\{z \in Z : f(z) \neq 0\}$ is contained in the union $\bigcup_{i=1}^m B_i$ of Borel sets B_i and if π_i is constant on B_i then

$$T(f, \pi_1, \dots, \pi_m) = 0.$$

(iii) There exists a finite Borel measure μ on Z such that

$$(3) \quad |T(f, \pi_1, \dots, \pi_m)| \leq \prod_{i=1}^m \text{Lip}(\pi_i) \int_Z |f| d\mu$$

for all $(f, \pi_1, \dots, \pi_m) \in \mathcal{D}^m(Z)$.

The space of m -dimensional metric currents on Z is denoted by $\mathbf{M}_m(Z)$ and the minimal Borel measure μ satisfying (3) is called mass of T and written as $\|T\|$. We also call mass of T the number $\|T\|(Z)$ which we denote by $\mathbf{M}(T)$. The support of T is, by definition, the closed set $\text{spt } T$ of points $z \in Z$ such that $\|T\|(B(z, r)) > 0$ for all $r > 0$.

Every function $\theta \in L^1(K, \mathbb{R})$ with $K \subset \mathbb{R}^m$ Borel measurable induces an element of $\mathbf{M}_m(\mathbb{R}^m)$ by

$$\llbracket \theta \rrbracket(f, \pi_1, \dots, \pi_m) := \int_K \theta f \det\left(\frac{\partial \pi_i}{\partial x_j}\right) d\mathcal{L}^m$$

for all $(f, \pi_1, \dots, \pi_m) \in \mathcal{D}^m(\mathbb{R}^m)$.

The restriction of $T \in \mathbf{M}_m(Z)$ to a Borel set $A \subset Z$ is given by

$$(T \llcorner A)(f, \pi_1, \dots, \pi_m) := T(f\chi_A, \pi_1, \dots, \pi_m).$$

This expression is well-defined since T can be extended to a functional on tuples for which the first argument lies in $L^\infty(Z, \|T\|)$. It can be shown that $T(f, \pi_1, \dots, \pi_m)$ only depends on the values of f, π_1, \dots, π_m on $\text{spt } T$.

If $m \geq 1$ and $T \in \mathbf{M}_m(Z)$ then the boundary of T is the functional

$$\partial T(f, \pi_1, \dots, \pi_{m-1}) := T(1, f, \pi_1, \dots, \pi_{m-1}).$$

It is clear that ∂T satisfies conditions (i) and (ii) in the above definition. If ∂T also satisfies (iii) then T is called a normal current. By convention, elements of $\mathbf{M}_0(Z)$ are also called normal currents.

The push-forward of $T \in \mathbf{M}_m(Z)$ under a Lipschitz map φ from Z to another complete metric space Y is given by

$$\varphi_\# T(g, \tau_1, \dots, \tau_m) := T(g \circ \varphi, \tau_1 \circ \varphi, \dots, \tau_m \circ \varphi)$$

for $(g, \tau_1, \dots, \tau_k) \in \mathcal{D}^m(Y)$. This defines an m -dimensional metric current on Y . It follows directly from the definitions that $\partial(\varphi_\# T) = \varphi_\#(\partial T)$.

Definition 2.6. A sequence (T_n) in $\mathbf{M}_m(Z)$ is said to converge weakly to $T \in \mathbf{M}_m(Z)$ if

$$T_n(f, \pi_1, \dots, \pi_m) \rightarrow T(f, \pi_1, \dots, \pi_m)$$

for all $(f, \pi_1, \dots, \pi_m) \in \mathcal{D}^m(Z)$.

In this article we will exclusively work with integral currents, defined below: An element $T \in \mathbf{M}_0(Z)$ is called integer rectifiable if there exist finitely many points $x_1, \dots, x_n \in X$ and $\theta_1, \dots, \theta_n \in \mathbb{Z} \setminus \{0\}$ such that

$$T(f) = \sum_{i=1}^n \theta_i f(x_i)$$

for all bounded Lipschitz functions f .

A current $T \in \mathbf{M}_m(Z)$ with $m \geq 1$ is said to be integer rectifiable if the following properties hold:

- (i) $\|T\|$ is concentrated on a countably \mathcal{H}^m -rectifiable set and vanishes on \mathcal{H}^m -negligible Borel sets.
- (ii) For any Lipschitz map $\varphi : Z \rightarrow \mathbb{R}^m$ and any open set $U \subset Z$ there exists $\theta \in L^1(\mathbb{R}^m, \mathbb{Z})$ such that $\varphi_\#(T \llcorner U) = \llbracket \theta \rrbracket$.

Integer rectifiable normal currents are called integral currents. The corresponding space is denoted by $\mathbf{I}_m(Z)$. If $A \subset \mathbb{R}^m$ is a Borel set of finite measure and finite perimeter then $\llbracket \chi_A \rrbracket \in \mathbf{I}_m(\mathbb{R}^m)$. Here, χ_A denotes the characteristic function. If $T \in \mathbf{I}_m(Z)$ and if $\varphi : Z \rightarrow Y$ is a Lipschitz map into another complete metric space then $\varphi_\# T \in \mathbf{I}_k(Y)$. If $Z = \mathbb{R}^n$ then there is a natural isometric isomorphism between the space of compactly supported Ambrosio-Kirchheim integral (integer rectifiable) m -currents and the space of Federer-Fleming integral (integer rectifiable) m -currents in \mathbb{R}^n , see [1].

Given a Riemannian manifold N and an oriented m -dimensional submanifold M of N , possibly with boundary. Let $\tau = \tau_1 \wedge \dots \wedge \tau_m$ be the orientation of M , where $\{\tau_1, \dots, \tau_m\}$ is an m -tuple of orthonormal vector fields. If M has finite volume and finite volume of the boundary then it gives rise to an integral current $\llbracket M \rrbracket \in \mathbf{I}_m(N)$ by

$$(4) \quad \llbracket M \rrbracket(f, \pi_1, \dots, \pi_m) := \int_M f \langle d\pi_1 \wedge \dots \wedge d\pi_m, \tau \rangle d\mathcal{H}^m$$

for all $(f, \pi_1, \dots, \pi_m) \in \mathcal{D}^m(N)$. By definition,

$$\langle d\pi_1 \wedge \dots \wedge d\pi_m, \tau \rangle(z) := \det(d_z \pi_i(\tau_j(z)))$$

for all $z \in M$ at which each π_i is differentiable (which is for \mathcal{H}^m -almost every z by Rademacher's theorem). Note that $\mathcal{H}^m \llcorner M$ agrees with the Riemannian volume on M .

We conclude this section with the following easy result.

Lemma 2.7. *Let $(T_m) \subset \mathbf{I}_m(Z)$ be a bounded sequence converging weakly to some $T \in \mathbf{I}_m(Z)$. Then $\text{spt } T$ is contained in the set of points $z \in Z$ for which there exists a sequence z_n converging to z with $z_n \in \text{spt } T_n$ for all n large enough. In particular, if $(\text{spt } T_n)$ converges in the Hausdorff sense to a closed subset $X \subset Z$ then $\text{spt } T \subset X$.*

Proof. Let $z \in \text{spt } T$ and $\varepsilon > 0$. Since $\|T\|(B(z, \varepsilon/2)) > 0$, there exist $f, \pi_1, \dots, \pi_m \in \text{Lip}(Z)$ such that $\text{spt } f \subset B(z, \varepsilon)$ and $T(f, \pi_1, \dots, \pi_m) \neq 0$. It follows that $T_n(f, \pi_1, \dots, \pi_m) \neq 0$ for n large enough and we thus conclude from Proposition 2.7 in [1] that $\|T_n\|(B(z, \varepsilon)) > 0$ and $\text{spt } T_n \cap B(z, \varepsilon) \neq \emptyset$. \square

3. A L

The main purpose of this section is to prove Theorem 3.4, a Lipschitz extension theorem which will be needed in the proof of Theorem 1.3. The results in this section are variants of results of Lang-Schlichenmaier in [15] and our proofs here follow closely those in [15].

Definition 3.1. *Let Y be a metric space, $n \geq 0$, and $\varrho \geq 1$. A subset $A \subset Y$ will be called ϱ -linearly locally weakly Lipschitz n -connected in Y if for every $\nu' > 0$ there exists $\gamma > 0$ such that for all $k \in \{0, 1, \dots, n\}$, all $a \in A$ and $r > 0$, every ν -Lipschitz map $f : S^k \rightarrow Y$ with $\nu \geq \nu'$ and image in $A \cap \bar{B}(a, r)$ has a $\gamma\nu$ -Lipschitz extension $\bar{f} : \bar{B}^{k+1} \rightarrow Y$ with image in $\bar{B}(a, \varrho r)$.*

The following proposition is a variation of Theorem 5.1 in [15].

Proposition 3.2. *Let Y be a compact metric space, $\varrho \geq 1$, $r_0 > 0$, and $n \geq 0$. Suppose that Y is Lipschitz n -connected in the small. If a ball $\bar{B}(y_0, r_0)$ in Y is ϱ -linearly locally n -connected in Y then $\bar{B}(y_0, r_0)$ is (2ϱ) -linearly locally weakly Lipschitz n -connected.*

Up to a minor adjustment, the proof is the same as that in [15]. It consists of three parts, only the first of which needs modifying.

Proof. Since Y is Lipschitz n -connected in the small, there exist $\delta', \gamma' > 0$ such that for all $l \in \{0, \dots, n\}$ every δ -Lipschitz map $\varphi : \partial[0, 1]^{l+1} \rightarrow Y$ with $\delta \leq \delta'$ has a $\gamma'\delta$ -Lipschitz extension to $[0, 1]^{l+1}$. Fix $l \in \{0, \dots, n\}$ and put $D := [0, 1]^{l+1}$. Let $f : \partial D \rightarrow Y$ be a λ -Lipschitz map with image in $\bar{B}(y, r) \cap \bar{B}(y_0, r_0)$ for some $y \in \bar{B}(y_0, r_0)$ and some $r \leq 2r_0$. We show that f has a Lipschitz extension $\bar{f} : D \rightarrow Y$ with image in $\bar{B}(y, \frac{3}{2}\varrho r)$. For this, pick a continuous extension $g : D \rightarrow Y$ of f with image in $\bar{B}(y, \varrho r)$. This is possible because $\bar{B}(y_0, r_0)$ is ϱ -linearly locally n -connected. Let $\varepsilon > 0$ be small enough, to be determined later. Equip D with the structure of a piecewise Euclidean polyhedral complex whose 0-skeleton $D^{(0)}$ is $D \cap ((1/N)\mathbb{Z})^{l+1}$ and whose $(l+1)$ -cells are cubes of edge length $1/N$, where $N \geq 2$ is chosen large enough so that $\lambda \leq \varepsilon N$ and $\text{diam } g(C) \leq \varepsilon$ for every $(l+1)$ -cell C of D . For $k = 0, \dots, n$, we successively find extensions $f^{(k)} : \partial D \cup D^{(k)} \rightarrow Y$ of f such that $f^{(k)}|_{\partial C}$ is $(2\gamma')^k \varepsilon N$ -Lipschitz on every $(k+1)$ -cell C not contained in ∂D and such that

$$f^{(k)}(\partial D \cup D^{(k)}) \subset \bar{B}\left(y, \varrho r + 2^{-1} \varepsilon \gamma' \tau(k)\right)$$

where $\tau(k) = \frac{(2\gamma')^k - 1}{2\gamma' - 1}$. For $k = 0$, the extension $f^{(0)} := g|_{\partial D \cup D^{(0)}}$ satisfies these properties. Now, suppose that for a given k , $f^{(k)}$ is an extension with these properties. If $(2\gamma')^k \varepsilon \leq \delta'$

then there exists an extension $f^{(k+1)} : \partial D \cup D^{(k+1)} \rightarrow Y$ of f such that $f^{(k+1)}|_C$ is $\gamma'(2\gamma')^k \varepsilon N$ -Lipschitz for every $(k+1)$ -cell C . In particular, $f^{(k+1)}(C)$ is contained in the $2^{-1}\gamma'(2\gamma')^k \varepsilon$ -neighborhood of $f^{(k)}(\partial C)$. If $k+1 \leq l$, it follows that for every $(k+2)$ -cell C' that is not contained in ∂D , $f^{(k+1)}|_{\partial C'}$ is $(2\gamma')^{k+1} \varepsilon N$ -Lipschitz. If $\varepsilon > 0$ is chosen sufficiently small so that, in particular, $\varepsilon\gamma'\tau(l+1) \leq \varrho r$, this iterative procedure gives the desired extension $\bar{f} := f^{(l+1)}$ of f . This proves the first part. The second and third parts of the proof are the same as in [15]. \square

The next theorem is an adjustment of Theorem 5.2 in [15] to our setting. We will indicate below the changes one has to make in Lang-Schlichenmaier's proof to obtain our theorem.

Theorem 3.3. *Let X and Y be metric spaces, and let $\alpha, \beta > 0$ and $n \geq 1$. Suppose that $Z \subset X$ is a nonempty closed set and $(B_i)_{i \in I}$ is a covering of $X \setminus Z$ by subsets of $X \setminus Z$ such that*

- (i) $\text{diam } B_i \leq \alpha d(B_i, Z)$ for every $i \in I$,
- (ii) every set $D \subset X \setminus Z$ with $\text{diam } D \leq \beta d(D, Z)$ meets at most $n+1$ members of $(B_i)_{i \in I}$.

Suppose that Y is Lipschitz $(n-1)$ -connected in the small and that $\bar{B}(y_0, r_0)$ is a ball in Y which is ϱ -linearly locally weakly Lipschitz $(n-1)$ -connected in Y . Then for all $\lambda \geq 1$ and $0 < r < \frac{r_0}{16\lambda\varrho^n}$, every λ -Lipschitz map $f : Z \rightarrow Y$ with image in $\bar{B}(y_0, r_0/2)$ has a Lipschitz extension $\bar{f} : U_r(Z) \rightarrow Y$ with image in $U_{r'}(f(Z))$ where $r' = 8\lambda\varrho^n r$.

Here, $U_r(Z)$ denotes the open r -neighborhood of Z in X . If Z has Nagata dimension $\leq n-1$ then a covering $(B_i)_{i \in I}$ of $X \setminus Z$ satisfying the properties in the above theorem is constructed in the proof of Theorem 1.6 in [15]. Therefore, we obtain the following result.

Theorem 3.4. *Let X and Y be metric spaces, let $n \geq 1$, $\varrho \geq 1$, and $r_0 > 0$. Suppose that $Z \subset X$ is a nonempty closed set of Nagata dimension $\leq n-1$. Suppose further that Y is Lipschitz $(n-1)$ -connected in the small and that $\bar{B}(y_0, r_0)$ is a ball in Y which is ϱ -linearly locally weakly Lipschitz $(n-1)$ -connected in Y . Then for every $\lambda \geq 1$ and $0 < r < \frac{r_0}{16\lambda\varrho^n}$, every λ -Lipschitz map $f : Z \rightarrow Y$ with image in $\bar{B}(y_0, r_0/2)$ has a Lipschitz extension $\bar{f} : U_r(Z) \rightarrow Y$ with image in $U_{r'}(f(Z))$ where $r' = 8\lambda\varrho^n r$.*

Proof of Theorem 3.3. The proof follows that of Theorem 5.2 in [15]. We need to make several modifications, however. Let $f : Z \rightarrow Y$ be a λ -Lipschitz map with image in $\bar{B}(y_0, r_0/2)$. Set $\delta := \beta/(2(\beta+1))$ and, for each $i \in I$, let $\sigma_i : X \setminus Z \rightarrow \mathbb{R}$ be the 1-Lipschitz function defined by

$$\sigma_i(x) := \max\{0, \delta d(B_i, Z) - d(x, B_i)\}.$$

It can then be shown (see [15]) that for every $x \in X \setminus Z$, there are at most $n+1$ indices $i \in I$ such that $\sigma_i(x) > 0$. Now define $\bar{\sigma} := \sum_{i \in I} \sigma_i$ and note that $\bar{\sigma} > 0$ on $X \setminus Z$ because $(B_i)_{i \in I}$ covers $X \setminus Z$. Now define $g : X \setminus Z \rightarrow \ell^2(I)$ by

$$g(x) := \left(\frac{\sigma_i(x)}{\bar{\sigma}(x)} \right)_{i \in I}$$

and note that the image of g lies in the n -skeleton $\Sigma^{(n)}$ of the simplex $\Sigma := \{(v_i)_{i \in I} : v_i \geq 0, \sum_{i \in I} v_i = 1\} \subset \ell^2(I)$. For every $i \in I$, choose a point $z_i \in Z$ such that

$$d(z_i, B_i) \leq (2-\delta)d(B_i, Z).$$

Let $h^{(0)} : \Sigma^{(0)} \rightarrow Y$ be the map defined by $h^{(0)}(e_i) := f(z_i)$, where e_i is the i -th vertex of Σ . So far, the proof was exactly the same as the one in [15].

Now, let $\hat{\Sigma}$ be the subcomplex of Σ consisting of those simplices $[e_{i_0}, \dots, e_{i_k}] \subset \Sigma^{(k)}$ for which there exists $x \in U_r(Z)$ with $\sigma_{i_j}(x) > 0$ for every $j = 0, \dots, k$. Note that actually $g(U_r(Z) \setminus Z)$ lies in $\hat{\Sigma}$. Denote by $h^{(0)}$ again the restriction to $\hat{\Sigma}^{(0)}$ of the map $h^{(0)}$ defined above. Since Y is Lipschitz $(n-1)$ -connected in the small there exist constants $\delta' \in (0, 1/2)$ and $\gamma' \geq 1$ such that for all $k \in \{0, \dots, n-1\}$, every ν -Lipschitz map $\varphi : \partial\Delta^{k+1} \rightarrow Y$ with $\nu \leq \delta'$ has a $\gamma'\nu$ -Lipschitz extension $\bar{\varphi} : \Delta^{k+1} \rightarrow Y$. Set $\nu' := \delta'\gamma'^{-1} \min\{r, 1\}$ and choose $\gamma \geq 1$ such that for all $k \in \{0, \dots, n-1\}$, every ν -Lipschitz map $\psi : \partial\Delta^{k+1} \rightarrow Y$ with $\nu \geq \nu'$ and image in $\bar{B}(y, s) \cap \bar{B}(y_0, r_0)$ for some $y \in \bar{B}(y_0, r_0)$ and $s \leq 2r_0$ has a $\gamma\nu$ -Lipschitz extension $\bar{\psi} : \Delta^{k+1} \rightarrow Y$ with image in $\bar{B}(y, \varrho s)$. This is possible because $\bar{B}(y_0, r_0)$ is ϱ -linearly locally weakly Lipschitz $(n-1)$ -connected. Next, let $[e_i, e_j] \subset \hat{\Sigma}^{(1)}$ be a 1-simplex. There then exists $x \in U_r(Z)$ with $\sigma_i(x) > 0$ and $\sigma_j(x) > 0$ and thus

$$\begin{aligned} d(z_i, z_j) &\leq d(z_i, B_i) + d(B_i, x) + d(x, B_j) + d(B_j, z_j) \\ &\leq d(z_i, B_i) + \delta d(B_i, Z) + \delta d(B_j, Z) + d(B_j, z_j) \\ &\leq 2[d(B_i, Z) + d(B_j, Z)] \\ &\leq 8r. \end{aligned}$$

The last inequality follows from the fact that

$$d(Z, B_i) \leq d(Z, x) + d(x, B_i) < r + \delta d(B_i, Z)$$

and that $(1 - \delta)^{-1} < 2$. It follows that $d(h^{(0)}(e_i), h^{(0)}(e_j)) \leq \lambda d(z_i, z_j) \leq 8r\lambda$. For $m = 0, 1, \dots, n-1$, successively extend $h^{(m)}$ to $h^{(m+1)} : \hat{\Sigma}^{m+1} \rightarrow Y$ as follows. Let $S \subset \hat{\Sigma}^{(m+1)}$ be an $(m+1)$ -simplex and suppose that $h^{(m)}(\partial S)$ is contained in the $8\lambda\varrho^m r$ -ball around $h^{(m)}(e_i) = f(z_i)$ for some $e_i \in S^{(0)}$. (Note that this is the case for $m = 0$.) We distinguish two cases: If $\text{Lip}(h^{(m)}|_{\partial S}) < \nu'$ then $h^{(m)}|_{\partial S}$ has a γ' $\text{Lip}(h^{(m)}|_{\partial S})$ -Lipschitz extension $h^{(m+1)}|_S$ to S . In particular, $h^{(m+1)}(S)$ is contained in the r -ball around $h^{(m)}(e_i)$. If, on the other hand, $\text{Lip}(h^{(m)}|_{\partial S}) \geq \nu'$ then $h^{(m)}|_{\partial S}$ has a $\gamma \text{Lip}(h^{(m)}|_{\partial S})$ -Lipschitz extension $h^{(m+1)}|_S$ to S with image in the $8\lambda\varrho^{m+1} r$ -ball around $h^{(m)}(e_i) = f(z_i)$. We set $h := h^{(n)} : \hat{\Sigma}^{(n)} \rightarrow Y$ and note that h is Lipschitz on every simplex S of $\hat{\Sigma}^{(n)}$ with Lipschitz constant

$$\text{Lip}(h|_S) \leq C_1 \text{Lip}(h|_{S^{(0)}}) = \frac{C_1 \text{diam } h(S^{(0)})}{\sqrt{2}}$$

for some constant C_1 depending on $r, \lambda, \gamma, \gamma', \delta', \varrho$, and n . Furthermore, $h(\hat{\Sigma})$ is contained in $U_{r'}(f(Z))$ with $r' = 8\lambda\varrho^n r$. Finally, we define the extension $\bar{f} : U_r(Z) \rightarrow Y$ of f such that $\bar{f} = h \circ g$ on $U_r(Z)$. The Lipschitz property of \bar{f} follows now exactly as in [15] after (5.6). This completes the proof of our theorem. \square

4. F

m -

Before stating the main result of this section we recall the notion of filling radius (first introduced and studied by Gromov in [9]). Let Z be a complete metric space and let $l^\infty(Z)$ be the Banach space of bounded functions on Z with the supremum norm. Fix $z_0 \in Z$. Then the map $\iota : Z \hookrightarrow l^\infty(Z)$ given by $\iota(z) := d(z, \cdot) - d(z_0, \cdot)$ defines an isometric embedding, called Kuratowski embedding. Let now $m \geq 0$ and let $T \in \mathbf{I}_m(Z)$ be an integral current. The absolute filling radius of T is

$$\text{Fillrad}_\infty(T) = \inf \{ \varepsilon \geq 0 : \exists S \in \mathbf{I}_{m+1}(l^\infty(Z)) \text{ with } \partial S = \iota_\# T \text{ and } \text{spt } S \subset U_\varepsilon(\iota(\text{spt } T)) \},$$

where $U_\varepsilon(\iota(\text{spt } T))$ denotes the open ε -neighborhood of $\iota(\text{spt } T)$ in $l^\infty(Z)$. It is not difficult to see that $\text{Fillrad}_\infty(T)$ remains unchanged if $l^\infty(Z)$ is replaced by any other injective metric

space Z' into which $\text{spt } T$ isometrically embeds. This justifies the word ‘‘absolute’’ filling radius. The main theorem of the present section is:

Theorem 4.1. *Let Z be a complete metric space and $m \geq 1$. Let $T \in \mathbf{I}_m(Z)$ and suppose $\text{spt } T$ is compact, Lipschitz m -connected in the small, and has Nagata dimension $\leq m$. Let $z \in \text{spt } T$ and suppose that for some $\lambda \geq 1$ and $r_0 > 0$ the ball $\bar{B}(z, r_0) \cap \text{spt } T$ is λ -linearly locally m -connected in $\text{spt } T$. Then*

$$\text{Fillrad}_\infty(\partial(T \llcorner \bar{B}(z, r))) \geq \frac{r}{8(2\lambda)^{m+1}}$$

for almost every $r \in [0, \min\{2^{-(m+6)}\lambda^{-(m+1)}r_0, d(z, \text{spt } \partial T)\}]$.

In particular, it follows that

$$\|T\|(\bar{B}(z, r)) \geq \text{Fillvol}_\infty(\partial(T \llcorner \bar{B}(z, r))) \geq \frac{C'r^m}{[8(2\lambda)^{m+1}]^m}$$

for some $C' > 0$ depending only on m , see [9] or [1], [22].

In the setting of Riemannian manifolds, Theorem 4.1 was essentially proved by Greene-Petersen in [7]. Indeed, let M be an m -dimensional Riemannian manifold, $z \in M$, and let f_ε be smooth approximations of the distance function to z on M . Then for almost every r the subset $B_\varepsilon(r) := \{f_\varepsilon \leq r\}$ has smooth boundary. Greene-Petersen use an extension argument similar to the one in the proof of Gromov’s Lemma 1.2.B in [9] to show that if M has a local geometric contractibility function $\varrho : [0, \alpha) \rightarrow [0, \infty)$ then $\partial B_\varepsilon(r)$ has filling radius bounded below by a function $r' = r'(\varrho, r)$. If ϱ is linear as in our case then $r' \geq \delta r$. Their result can thus be used to obtain Theorem 4.1 in the special case that $\text{spt } T$ is isometric to a Riemannian manifold. Our proof for the general case uses slicing for integral currents instead of smoothing and the Lipschitz extension theorems from Section 3 instead of the extension argument used in [7].

Proof. Throughout the proof we view Z as a subset of $X := I^\infty(Z)$. Let

$$0 < r < \min\{2^{-(m+6)}\lambda^{-(m+1)}r_0, d(z, \text{spt } \partial T)\}$$

be such that $T' := \partial(T \llcorner \bar{B}(z, r))$ is an integral current with support in $S(z, r) \cap \text{spt } T$, where $S(z, r)$ denotes the metric sphere. By the slicing theorem [1, Theorems 5.6 and 5.7] this is the case for almost every r . Suppose, by contradiction, that $\text{Fillrad}_\infty(T') < Ar$ where $A := \frac{1}{8(2\lambda)^{m+1}}$. We will first construct an integral current S' with $\partial S' = T'$ as follows. If $T' = 0$ then set $S' := 0$. If $T' \neq 0$ then choose $S \in \mathbf{I}_m(X)$ with $\partial S = T'$ and $\text{spt } S \subset U_\varepsilon(S(z, r)) \subset X$ for some $\varepsilon < Ar$. Note that $\text{spt } T'$ is a nonempty closed subset of $\text{spt } T$ of Nagata dimension at most m . Furthermore, $\text{spt } T$ is Lipschitz m -connected in the small and $\bar{B}(z, r_0) \cap \text{spt } T$ is λ -linearly locally m -connected in $\text{spt } T$. Therefore, by Proposition 3.2, $\bar{B}(z, r_0) \cap \text{spt } T$ is 2λ -linearly locally weakly Lipschitz m -connected in $\text{spt } T$. Let $f : \text{spt } T' \rightarrow \text{spt } T$ be the inclusion map. By Theorem 3.4, there thus exists a Lipschitz extension $\tilde{f} : U_\varepsilon(S(z, r)) \rightarrow \text{spt } T$ of f with image in $U_{8(2\lambda)^{m+1}\varepsilon}(S(z, r)) \cap \text{spt } T$. Finally, set $S' := \tilde{f}_\# S$ and note that $\partial S' = T'$. This completes the construction of S' . It is now clear that in either case

$$(5) \quad \text{spt } S' \subset U_{8(2\lambda)^{m+1}\varepsilon}(S(z, r)) \cap \text{spt } T.$$

Since $8(2\lambda)^{m+1}\varepsilon < r$ and since $z \in \text{spt } T$, it thus follows with (5) that $T'' := T \llcorner \bar{B}(z, r) - S' \neq 0$; finally we have $\partial T'' = 0$ and $\text{spt } T'' \subset \bar{B}(z, 2r) \cap \text{spt } T$. Now, set $C := \bar{B}(z, 2r) \cap \text{spt } T$, endow $X' := [0, 2r] \times C$ with the product metric and define a subset of X' by $Z' := \{0, 2r\} \times C$. Let $f' : Z' \rightarrow \text{spt } T$ be the 1-Lipschitz map given by $f'(z', 0) := z'$ and $f'(z', 2r) := z$

for all z' . Since $2r < Ar_0$ there exists, as above, a Lipschitz extension $\tilde{f}' : X' \rightarrow \text{spt } T$ of f . Finally, define $g(t, z') := \tilde{f}'(2rt, z')$ for all $t \in [0, 1]$ and $z' \in C$ and set $W := g_{\#}([0, 1] \times T')$, where $[0, 1] \times T'$ is the product of currents defined in Section 2.3 of [22]. Note that $W \in \mathbf{I}_{m+1}(\text{spt } T)$ and that $\partial W = -T''$. Now, since $W \neq 0$, it follows from Theorem 4.5 in [1] that there exists $K \subset \mathbb{R}^{m+1}$ compact, with $\mathcal{L}^{m+1}(K) > 0$, and $\psi : K \rightarrow \text{spt } T$ biLipschitz. Since $\text{spt } T$ has Nagata dimension $\leq m$, also $\psi(K)$ and K have Nagata dimension $\leq m$. However, by Proposition 2.3, this implies that K has Hausdorff dimension $< m + 1$ and thus $\mathcal{L}^{m+1}(K) = 0$, a contradiction. This completes the proof. \square

Now, for a compact metric space Y and $\varepsilon > 0$, denote by $N(Y, \varepsilon)$ the smallest number of ε -balls needed to cover Y .

Corollary 4.2. *Let Z be a complete metric space and $m \geq 1$. Let $T \in \mathbf{I}_m(Z)$ with $\partial T = 0$ and suppose $\text{spt } T$ is compact, Lipschitz m -connected in the small, and has Nagata dimension $\leq m$. If $\text{spt } T$ is λ -linearly locally m -connected up to scale r_0 for some $\lambda \geq 1$ and $r_0 > 0$ then*

$$N(\text{spt } T, \varepsilon) \leq \frac{\lambda^{m(m+1)} \mathbf{M}(T)}{C \varepsilon^m} \quad \text{for all } \varepsilon \in (0, 2^{-(m+6)} \lambda^{-(m+1)} r_0),$$

where C only depends on m .

Proof. Fix $0 < \varepsilon < 2^{-(m+6)} \lambda^{-(m+1)} r_0$ and let $\{x_1, \dots, x_k\} \subset \text{spt } T$ be an ε -separated net, that is, $d(x_i, x_j) > \varepsilon$ for all $i \neq j$. Then the balls $\bar{B}(x_j, \varepsilon/2)$ are pairwise disjoint and, by Theorem 4.1, we have

$$\|T\|(\bar{B}(x_j, \varepsilon/2)) \geq \frac{C' \varepsilon^m}{[16(2\lambda)^{m+1}]^m}$$

for every j and consequently

$$(6) \quad k \leq \frac{[16(2\lambda)^{m+1}]^m \mathbf{M}(T)}{C' \varepsilon^m}.$$

The statement follows. \square

5. P

For the proof of our main result we will need the following lemma.

Lemma 5.1. *Let Z be a complete metric space and $(T_n) \subset \mathbf{I}_m(Z)$ a bounded sequence weakly converging to some $T \in \mathbf{I}_m(Z)$. Let $z \in Z$ and let $(z_n) \subset Z$ satisfy $z_n \rightarrow z$. Then for almost every $r > 0$ there exists a subsequence T_{n_j} such that $T_{n_j} \llcorner \bar{B}(z_{n_j}, r)$ is a bounded sequence of integral currents converging weakly to $T \llcorner \bar{B}(z, r)$, which is an integral current.*

The proof is a simple variation of the proof of Lemma 8.4 in [1], see also Proposition 6.6 in [13].

Proof of Theorem 1.3. Let T , T_n , and z , z_n be as in the statement of the theorem. In the following, we view Z as a subset of $l^\infty(Z)$. By Lemma 5.1 and Theorem 4.1, for almost every $0 < s < 2^{-(m+6)} \lambda^{-(m+1)} r$ there exists a subsequence T_{n_j} such that $T_{n_j} \llcorner \bar{B}(z_{n_j}, s)$ is a bounded sequence of integral currents converging weakly to the integral current $T \llcorner \bar{B}(z, s)$ and

$$\text{Fillvol}_\infty(\partial(T_{n_j} \llcorner \bar{B}(z_{n_j}, s))) \geq \frac{C' s^m}{[8(2\lambda)^{m+1}]^m}$$

for all j . Since $\partial(T_{n_j} \llcorner \bar{B}(z_{n_j}, s))$ converges weakly to $\partial(T \llcorner \bar{B}(z, s))$ and since $l^\infty(Z)$ admits local cone type inequalities, Theorem 1.4 in [23] implies that

$$\text{Fillvol}_\infty(\partial(T_{n_j} \llcorner \bar{B}(z_{n_j}, s)) - \partial(T \llcorner \bar{B}(z, s))) \rightarrow 0$$

and thus

$$\|T\|(\bar{B}(z, s)) \geq \text{Fillvol}_\infty(\partial(T \llcorner \bar{B}(z, s))) \geq \frac{C' s^m}{[8(2\lambda)^{m+1}]^m}.$$

Now, (1) easily follows and hence that

$$\text{spt } T = \{z \in Z : \|T\| \text{ has strictly positive } m\text{-dimensional lower density at } z\}.$$

The countable \mathcal{H}^m -rectifiability of the set on the right-hand side was proved in Theorem 4.6 in [1]. \square

Remark 5.2. The above proof in particular shows the following: Let $(T_n) \subset \mathbf{I}_m(Z)$ be a bounded sequence weakly converging to some $T \in \mathbf{I}_m(Z)$ and suppose $z \in Z$ and $z_n \in \text{spt } T_n$ are such that $z_n \rightarrow z$ for all n . If there exists $\delta > 0$ such that each T_n satisfies

$$(7) \quad \text{Fillvol}_\infty(\partial(T_n \llcorner \bar{B}(z_n, s))) \geq \delta s^m$$

for almost all $s \in (0, \varepsilon)$ then $z \in \text{spt } T$ and $\|T\|(\bar{B}(z, s)) \geq \delta s^m$ for all $s \in (0, \varepsilon)$. It would be desirable to find weaker conditions than the λ -linear local m -connectedness conditions we impose in this article which still guarantee that (7) holds.

The following example shows that if in Theorem 1.3 one only requires that each $\text{spt } T_n$ has Nagata dimension $\leq m + 1$ instead of $\leq m$, then the theorem becomes false.

Example 5.3. Let $(T_n) \subset \mathbf{I}_m(\mathbb{R}^{m+1})$ be a sequence with $\partial T_n = 0$ and $\text{spt } T_n = \bar{B}(0, 1)$, the closed unit ball in \mathbb{R}^{m+1} , for every n and such that $\mathbf{M}(T_n) \rightarrow 0$. Such a sequence can easily be obtained as a suitable infinite sum of m -spheres in $B(0, 1)$. It follows that T_n satisfies all the assumptions in the theorem except that $\text{spt } T_n$ has Nagata dimension $m + 1$. Furthermore, for any $z \in B(0, 1)$ the constant sequence $z_n = z$ and $\lambda = 1$ and $r = 1 - |z|$ satisfy the properties of the theorem. However, the weak limit T of T_n is clearly 0.

It is not clear whether Theorem 1.3 remains true if we only require that $\text{spt } T_n$ has topological dimension at most m .

Next, we turn to Theorem 1.5, the proof of which uses the following lemma.

Lemma 5.4. *Let Z be a complete metric space, $m \geq 1$, and $\varepsilon > 0$. Suppose $T \in \mathbf{I}_m(Z)$ satisfies $\partial T = 0$ and $\text{Fillvol}_Z(T) < \varepsilon$. Then for every $z \in \text{spt } T$ and all $r \geq 0$, $\delta > 0$ there exists a set $\Omega \subset (r, r + \delta)$ of strictly positive Lebesgue measure such that $\partial(T \llcorner \bar{B}(z, s)) \in \mathbf{I}_{m-1}(Z)$ and*

$$\text{Fillvol}_Z(\partial(T \llcorner \bar{B}(z, s))) \leq \frac{\varepsilon}{\delta}$$

for all $s \in \Omega$.

Proof. Choose $S \in \mathbf{I}_m(Z)$ with $\partial S = T$ and $\mathbf{M}(S) < \varepsilon$. Define $\varrho(y) := d(z, y)$ and note that, by the slicing theorem [1, Theorem 5.6],

$$\int_0^\infty \mathbf{M}(\langle S, \varrho, t \rangle) dt \leq \mathbf{M}(S).$$

There thus exists a subset $\Omega \subset (r, r + \delta)$ of strictly positive measure, such that $\langle S, \varrho, t \rangle \in \mathbf{I}_m(Z)$ and $\mathbf{M}(\langle S, \varrho, t \rangle) \leq \varepsilon/\delta$ for all $t \in \Omega$, since otherwise

$$\int_0^\infty \mathbf{M}(\langle S, \varrho, t \rangle) dt \geq \int_r^{r+\delta} \mathbf{M}(\langle S, \varrho, t \rangle) dt > \varepsilon,$$

a contradiction. Now, since $\partial(\langle S, \varrho, t \rangle) = -\langle \partial S, \varrho, t \rangle = -\partial(T \llcorner \bar{B}(z, t))$, the proof is complete. \square

We are ready for the proof of Theorem 1.5.

Proof of Theorem 1.5. Let $z \in \text{spt } T$. By Lemma 2.7, there exists a sequence (z_n) converging to z and such that $z_n \in \text{spt } T_n$ for all n . It follows from Theorem 1.3 that (2) holds and in particular that $\text{spt } T$ is compact.

Next, view Z as an isometric subset of $l^\infty(Z)$. We show that for every $\varepsilon > 0$ the $\text{spt } T_n$ eventually lie in the ε -neighborhood of $\text{spt } T$. For this, set $A := \text{spt } T$ and let $A(\varepsilon) := \{z \in Z : d(z, A) \leq \varepsilon\}$. By Proposition 8.3 in [1] there exists for almost every $\varepsilon > 0$ a subsequence T_{n_j} such that $T_{n_j} \llcorner A(\varepsilon)$ is a bounded sequence of integral currents weakly converging to T . In particular, it follows that $T'_j := T_{n_j} \llcorner [A(\varepsilon)]^c$ forms a bounded sequence converging weakly to 0. By Theorem 1.4 in [23], $\text{Fillvol}_\infty(\partial T'_j) \rightarrow 0$ as $j \rightarrow \infty$. In particular, there exists $S_j \in \mathbf{I}_m(l^\infty(Z))$ with $\partial S_j = \partial T'_j$ and $\mathbf{M}(S_j) \rightarrow 0$. By the remark following Lemma 3.4 in [22], we may assume that $\text{spt } S_j \subset A(2\varepsilon)$ for all j large enough. Note that $T''_j := T'_j - S_j$ satisfies $\partial T''_j = 0$, has uniformly bounded mass, and weakly converges to 0. Therefore, $v_j := \text{Fillvol}_{l^\infty(Z)}(T''_j) \rightarrow 0$, again by Theorem 1.4 in [23]. Suppose now that for some j large enough (to be determined later) there exists $z \in \text{spt } T_{n_j}$ with $d(z, A) \geq 3\varepsilon$. Set

$$r' := \frac{1}{2} \min \left\{ \varepsilon, 2^{-(m+6)} \lambda^{-(m+1)} r \right\}.$$

Since $T''_j \llcorner \bar{B}(z, s) = T_{n_j} \llcorner \bar{B}(z, s)$ for all $s < \varepsilon$, it follows from Theorem 4.1 that

$$\text{Fillvol}_\infty(\partial(T''_j \llcorner \bar{B}(z, s))) \geq \frac{C' s^m}{[8(2\lambda)^{m+1}]^m}$$

for almost every $s \in (r', 2r')$. On the other hand, Lemma 5.4 implies that

$$\text{Fillvol}_\infty(\partial(T''_j \llcorner \bar{B}(z, s))) \leq v_j / r'$$

for all s in a subset of $(r', 2r')$ of strictly positive measure. If j is large enough this is clearly a contradiction. This shows that $\text{spt } T_n$ eventually lies in the 3ε -neighborhood of $\text{spt } T$. Finally, let $\varepsilon > 0$ and let $\{z_1, \dots, z_k\}$ be a finite and ε -dense set in $\text{spt } T$. By Lemma 2.7, there exist sequences $z_n^j \rightarrow z_j$ with $z_n^j \in \text{spt } T_n$. This shows that $\text{spt } T$ lies in the 3ε -neighborhood of $\text{spt } T_n$ for all n large enough. This shows that $\text{spt } T_n$ converges in the Hausdorff distance to $\text{spt } T$. \square

Remark 5.5. We can also deduce the following result, related to Theorem 1.5. Let (X_n) be a sequence of complete metric spaces, $T_n \in \mathbf{I}_m(X_n)$ with $\partial T_n = 0$ and such that

$$\sup_n [\mathbf{M}(T_n) + \text{diam}(\text{spt } T_n)] < \infty.$$

Suppose further that $\text{spt } T_n$ is compact, Lipschitz m -connected in the small, and has Nagata dimension $\leq m$ for every n . If for some $\lambda \geq 1$ and $r_0 > 0$, every $\text{spt } T_n$ is λ -linearly locally m -connected up to scale r_0 then it follows from Corollary 4.2 that the sequence $(\text{spt } T_n)$ is uniformly compact. In particular, it follows from Gromov's compactness theorem that there exists a compact metric space Z and isometric embeddings $\varphi_n : \text{spt } T_n \hookrightarrow Z$. Moreover, there exists a subsequence T_{n_j} such that $\varphi_{n_j}(\text{spt } T_{n_j})$ converges in the Hausdorff sense to some closed subset $X \subset Z$ and $\varphi_{n_j\#} T_{n_j}$ converges weakly to some $T \in \mathbf{I}_m(Z)$. (The latter follows from the closure and compactness theorems in [1].) We conclude from Lemma 2.7 and Theorem 1.3 that $X = \text{spt } T$, that (1) holds at every $z \in \text{spt } T$, and thus, by [1], that $\text{spt } T$ is countably \mathcal{H}^m -rectifiable. Note that X is the Gromov-Hausdorff limit of the sequence $(\text{spt } T_{n_j})$.

Note that in [24] it is shown that whenever (X_n) is a sequence of complete metric spaces and $T_n \in \mathbf{I}_m(X_n)$ is such that

$$(8) \quad \sup_n [\mathbf{M}(T_n) + \mathbf{M}(\partial T_n) + \text{diam}(\text{spt } T_n)] < \infty$$

then there exists a complete metric space Z , isometric embeddings $\varphi_n : \text{spt } T_n \hookrightarrow Z$, and a subsequence T_{n_j} such that $\varphi_{n_j\#} T_{n_j}$ converges with respect to the flat distance to some $T \in \mathbf{I}_m(Z)$. In this greater generality, where only (8) is assumed, the sequence $(\text{spt } T_n)$ of supports need not be uniformly compact of course. Moreover, we cannot hope that a subsequence of $\text{spt}(\varphi_{n_j\#} T_{n_j})$ converges in the Hausdorff sense. If it does then in general $\text{spt } T$ is a strict subset.

Next, we give the proof of Theorem 1.7.

Proof of Theorem 1.7. For each n let T_n be the integral m -current on X_n obtained by integration over X_n (the same way as the current induced by an oriented Riemannian manifold). It follows that $\partial T_n = 0$ and, from Lemma 9.2 and Theorem 9.5 in [1], that $\text{spt } T_n = X_n$ and $\mathbf{M}(T_n) \leq 2^m \omega_m^{-1} \mathcal{H}^m(X_n)$ for all n , where ω_m denotes the volume of the unit ball in \mathbb{R}^m . It thus follows from Corollary 4.2 that (X_n) is uniformly compact. In particular, by Gromov's compactness theorem, there exists a compact metric space and isometric embeddings $\varphi_n : X_n \hookrightarrow Z$. By the compactness and closure theorems of Ambrosio-Kirchheim [1], possibly after passing to a subsequence, $\varphi_{n\#} T_n$ weakly converges to some $T \in \mathbf{I}_m(Z)$. After passing to a further subsequence we may assume that $\varphi_n(X_n) = \text{spt}(\varphi_{n\#} T_n)$ converges in the Hausdorff distance to a closed subset $X \subset Z$. By Lemma 2.7 we have that $\text{spt } T \subset X$. Furthermore, if $x \in X$ then there exists $x_n \in \text{spt}(\varphi_{n\#} T_n)$ with $x_n \rightarrow x$ and thus, by Theorem 1.3, we obtain that $x \in \text{spt } T$ and $\|T\|$ has strictly positive m -dimensional lower density at x . This shows that $\text{spt } T = X$ and that $\text{spt } T$ is countably \mathcal{H}^m -rectifiable and this concludes the proof. (Note that we could have used Theorem 1.5 that $\varphi_n(X_n) = \text{spt}(\varphi_{n\#} T_n)$ to achieve the same.) \square

In order to deduce Corollary 1.8 from Theorem 1.7 we must show that the M_n have uniformly bounded diameter. For this let $x, y \in M_n$ and let γ be a length-minimizing geodesic joining x and y , parametrized by arc-length. Choose points $x_1, \dots, x_k \in \text{Im}(\gamma)$ with mutual distance at least

$$\varepsilon := \frac{r}{32(2\lambda)^{m+1}}.$$

It follows from Corollary 4.2 that k is uniformly bounded and thus that the length of γ is uniformly bounded. This establishes the uniform upper bound on diameter.

6. E

Let (T_n) be a bounded sequence of integral currents in some metric space Z weakly converging to an integral current T . It was shown in Lemma 2.7 that if the sequence $(\text{spt } T_n)$ of supports converges in the Hausdorff sense to a closed subset $X \subset Z$ then $\text{spt } T \subset X$. In the following we illustrate with some simple examples that the inclusion may be strict in general.

Example 6.1. For each $n \geq 1$, let M_n be the 2-torus $S^1 \times S^1_{1/n}$ in \mathbb{R}^4 , where $S^1_{1/n}$ is the circle of radius $1/n$. Then for any orientation on M_n , the corresponding integral currents $\llbracket M_n \rrbracket$ converge in mass to 0, that is $\mathbf{M}(\llbracket M_n \rrbracket) \rightarrow 0$, and thus converge weakly to 0. On the other hand, M_n converges in the Hausdorff sense to $S^1 \times \{(0, 0)\}$. In particular, $\text{spt } T$ is empty whereas the Hausdorff limit of the sequence $(\text{spt } T_n)$ is not.

Sometimes, the phenomenon appearing in the example above is called collapse (of mass). The next example shows that the limit can be 0 even if the mass of T_n is bounded away even locally.

Example 6.2. For each $n \geq 1$ let M_n be the ellipsoid in \mathbb{R}^3 given by

$$M_n := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + nz^2 = 1\}.$$

For any orientation on M_n the sequence of integral currents $\llbracket M_n \rrbracket$ converges weakly to 0. On the other hand, the sequence (M_n) converges in the Hausdorff sense to the flat disc $\{(x, y, 0) : x^2 + y^2 \leq 1\} \subset \mathbb{R}^3$. In particular, $\text{spt } T$ is empty whereas the Hausdorff limit of the sequence $(\text{spt } T_n)$ is not.

We call a phenomenon such as appearing in this example cancellation. Note that in both examples above, M_n does not carry the length metric but the induced metric from the ambient Euclidean space.

A more elaborate example of a sequence with cancellation is given as follows.

Example 6.3. Let M_0 be a closed oriented Riemannian manifold and fix $n \in \mathbb{N}$. Choose a collection of points,

$$\{p_1, \dots, p_{N_n}\} \subset M_0$$

such that $d(p_i, p_k) > 3/n$ and $M_0 = \bigcup_i B(p_i, 10/n)$. We choose any r_n such that $0 < r_n \leq \min\{1/n, \text{inj}(M_0)/2\}$, where $\text{inj}(M_0)$ denotes the injectivity radius of M_0 . We then construct a Riemannian manifold M_n by gluing $M_0 \setminus \bigcup_{i=1}^{N_n} B(p_i, r_n)$ to itself and smoothing the edges so that the metric on both copies of $M_0 \setminus \bigcup_{i=1}^{N_n} B(p_i, 2r_n)$ is preserved. M_n converges in the Gromov-Hausdorff sense to M_0 . Thus there exists a compact metric space Z and isometric embeddings $\varphi_n : M_n \hookrightarrow Z$ and $\varphi_0 : M_0 \hookrightarrow Z$ such that $\varphi_n(M_n)$ Hausdorff converges to $\varphi_0(M_0)$. However, the sequence $\varphi_{n\#} \llbracket M_n \rrbracket$ of integral m -currents in Z is easily seen to converge weakly to 0. Note that there are no sequences of points with uniform bounds on their λ local contractibility radius due to the increasingly dense topology on the M_n .

It is of course not difficult to construct sequences that exhibit partial collapse or partial cancellation.

7. W

R

Let (M_n) be a sequence of closed oriented Riemannian manifolds of dimension m with $\text{Ric}_{M_j} \geq 0$ and $\text{diam}(M_j) \leq D_0$ for all n . Denote by $T_n := \llbracket M_n \rrbracket$ the integral m -current in M_n induced by integration over M_n . Using the Bishop-Gromov volume comparison theorem it is not difficult to show that the sequence (M_n) is uniformly compact and thus, by Gromov's compactness theorem, a subsequence (M_{n_j}) converges in the Gromov-Hausdorff sense to a compact metric space X . More precisely, there exists a compact metric space Z and isometric embeddings $\varphi_n : M_n \hookrightarrow Z$; as in Remark 5.5 it follows that for some subsequence, which we denote again by (M_n) , $\varphi_n(M_n)$ converges in the Hausdorff distance to a compact subset $X \subset Z$ and $\varphi_{n\#} T_n$ converges weakly to some $T \in \mathbf{I}_m(Z)$.

Theorem 7.1. *Let M_n, T_n, Z , and φ_n be as above. If $\text{Vol}(M_n) \geq V_0 > 0$ for all n then $\text{spt } T$ coincides with the (Gromov-)Hausdorff limit X of the sequence $(\varphi_n(M_n))$. Moreover, $\llbracket T \rrbracket$ has strictly positive m -dimensional lower density at every point $z \in \text{spt } T$ and, in particular, X is countably \mathcal{H}^m -rectifiable.*

Under the assumptions in the theorem, there do not exist, in general, $\lambda \geq 1$ and $r > 0$ such that M_n is λ -linearly locally m -connected up to scale r for all n , see [18]. As the

example in [16] shows, it is not even true that for every $z \in X$ there exist $\lambda \geq 1$, $r > 0$, and a sequence $z_n \in M_n$ with $\varphi_n(z_n) \rightarrow z$ such that $B(z_n, r)$ is λ -linearly locally m -connected in M_n for n large enough. In particular, Theorem 7.1 does not come as a direct consequence of either Theorem 1.5 or Theorem 1.3. It should also be noted that Cheeger-Colding [3] prove much stronger metric structure properties for the Gromov-Hausdorff limit X than the countable \mathcal{H}^m -rectifiability we exhibit here. Our proof uses results from Cheeger-Colding's [2] but not from [3].

Proof. Denote by $\mathcal{R} \subset X$ the subset of regular points in X in the sense of Cheeger-Colding, see [2, Definition 0.1]. Since $\text{Vol}(M_n) \geq V_0 > 0$ for all n , the sequence (M_n) is non-collapsed by the Bishop-Gromov volume comparison theorem. It thus follows from Theorem 5.11 in [2] that at each $z \in \mathcal{R}$, every tangent cone at z is isometric to \mathbb{R}^m , that is, z is an m -regular point in the sense of Cheeger-Colding. We claim that for every $z \in \mathcal{R}$ there exist $\lambda \geq 1$, $r_0 > 0$, and a sequence $z_n \in M_n$ with $\varphi_n(z_n) \rightarrow z$ and such that $B(z_n, r_0)$ is λ -linearly locally m -connected in M_n for n large enough. In order to do so, fix $z \in \mathcal{R}$ and let $\varepsilon > 0$ be sufficiently small (to be determined later). Given $\delta > 0$ there exists $r > 0$ such that

$$d_{GH}(B_{\mathbb{R}^m}(0, r), B_X(z, r)) < \delta r/2,$$

where $B_{\mathbb{R}^m}(0, r)$ and $B_X(z, r)$ denote balls in \mathbb{R}^m and X , respectively. Moreover, there is a sequence (z_n) with $z_n \in M_n$ and such that $\varphi_n(z_n) \rightarrow z$ and

$$d_{GH}(B_{M_n}(z_n, r), B_X(z, r)) < \delta r/2$$

for n large enough. It follows that

$$d_{GH}(B_{\mathbb{R}^m}(0, r), B_{M_n}(z_n, r)) < \delta r$$

for n large enough. It now follows from Colding's volume convergence theorem [4], more precisely from Corollary 2.19 in [4], that if δ was chosen sufficiently small then

$$\text{Vol}(B_{M_n}(z_n, r)) \geq (1 - \varepsilon)\omega_m r^m$$

for n large enough, where ω_m is the volume of the unit ball in \mathbb{R}^m . Finally, if $\varepsilon > 0$ was chosen small enough, Perelman's local contractibility theorem [17], see the Main Lemma therein and the remark following it, shows that $B_{M_n}(z_n, r_0)$ is 2-linearly locally contractible in M_n for all n large enough, where $r_0 := \nu r$ for some absolute constant $\nu \in (0, 1)$. This proves our claim. It thus follows from our main result, Theorem 1.3, that $\mathcal{R} \subset S_T$, where S_T is the set of points at which $\|T\|$ has strictly positive lower m -dimensional density. On the other hand, by [2, Theorems 5.9 and 2.1], we have $\mathcal{H}^m(X \setminus \mathcal{R}) = 0$ and, in particular, \mathcal{R} is dense in X . We conclude that $\text{spt } T = X$.

It remains to show that $\|T\|$ has strictly positive lower m -dimensional density at each point $z \in X \setminus \mathcal{R}$. For this, we note first that, by Lemma 9.2 and Theorem 9.5 in [1] and the fact that $\mathcal{H}^m(X \setminus \mathcal{R}) = 0$, we have

$$(9) \quad \|T\|(B(z, r)) \geq m^{-m/2} \mathcal{H}^m(S_T \cap B(z, r)) = m^{-m/2} \mathcal{H}^m(X \cap B(z, r))$$

for all $r > 0$. On the other hand, by Theorem 5.9 in [2] and the Bishop-Gromov volume comparison theorem,

$$(10) \quad \mathcal{H}^m(X \cap B(z, r)) \geq \frac{V_0 r^m}{(\text{diam } X)^m}$$

for all $0 < r < \text{diam } X$. From (9) and (10) it thus follows that $\|T\|$ has strictly positive m -dimensional lower density at z . This concludes the proof of the last statement. \square

We end this section by noting that if the Ricci curvature condition is replaced by a scalar curvature condition then Theorem 7.1 no longer holds even when the sequence of manifolds is known to converge in the Gromov-Hausdorff sense. There can be cancellation without collapse, as the following example shows.

Example 7.2. Let M_0 be a closed oriented m -dimensional Riemannian manifold with positive scalar curvature and $m \geq 3$. Construct M_n analogously to the M_n in Example 6.3, however using the Gromov-Lawson gluing construction [10] on tiny balls very close to the p_i (where M_0 is sufficiently flat). It can be achieved that M_n has positive scalar curvature and the metric on $M_0 \setminus \bigcup_{i=1}^{N_n} B(p_i, 2r_n)$ is preserved. Then these M_n still converge in the Gromov-Hausdorff sense to M_0 and the $\varphi_{n\#}[M_n]$ converge weakly to 0. A similar construction could be made using Schoen-Yau's surgery [20].

A A. L

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The purpose of this appendix is to prove the following theorem which shows that λ -linear local m -connectedness cannot be replaced by geometric contractibility in Corollary 1.8.

Theorem A.1. *For every $m \geq 2$ and every $\alpha \in (0, 1)$ there exist $C, r > 0$ and a sequence of Riemannian metrics g_n on the m -sphere S^m such that $M_n := (S^m, g_n)$ has*

$$\sup_n [\text{Vol}(M_n) + \text{diam } M_n] < \infty,$$

and satisfies the following properties: Each M_n has local geometric contractibility function $\varrho(s) := Cs^\alpha$, $s \in [0, r)$, and M_n converges in the Gromov-Hausdorff sense to a compact metric space X , homeomorphic to S^m , which satisfies $0 < \mathcal{H}^m(X) < \infty$ but which is not countably \mathcal{H}^m -rectifiable.

In the proof we will need the following: Let $N, m \geq 2$ and set $a := N^{-1/m}$. Define a metric d_∞ on $Z := \{1, \dots, N\}^{\mathbb{N}}$ such that for given $(z_l), (z'_l) \in Z$ we have $d_\infty((z_l), (z'_l)) := a^j$, where j is the smallest index, or 0, for which $z_j \neq z'_j$. Note that d_∞ is an ultrametric, that is,

$$(11) \quad d_\infty(z, z') \leq \max \{d_\infty(z, z''), d_\infty(z', z'')\}$$

for all $z, z', z'' \in Z$.

Lemma A.2. *The metric space (Z, d_∞) satisfies $0 < \mathcal{H}^m(Z) < \infty$, but is not countably \mathcal{H}^m -rectifiable.*

Proof. We first show that Z has finite \mathcal{H}^m -measure. For this, let $\delta > 0$ and choose n_0 large enough so that $2a^{n_0} < \delta$. Since Z can be covered by N^{n_0} closed balls of radius a^{n_0+1} we obtain

$$\mathcal{H}_\delta^m(Z) \leq N^{n_0} \omega_m a^{(n_0+1)m} = \omega_m a^m,$$

where ω_m is the volume of the unit ball in \mathbb{R}^m . This shows that Z has finite \mathcal{H}^m -measure. Next, we construct a Frostman measure on Z and use it to show that $\mathcal{H}^m(Z) > 0$. Let

$\pi_n : Z \rightarrow \{1, \dots, N\}$ be the projection onto the n -th coordinate. Let μ be the unique Borel probability measure on Z such that for all $n \geq 1$ and all $z_1, \dots, z_n \in \{1, \dots, N\}$

$$\mu(\{\pi_i = z_i : i = 1, \dots, n\}) = \frac{1}{N^n}.$$

We claim that for all $z \in Z$ and all $r \in (0, a)$ we have $\mu(\bar{B}(z, r)) \leq a^{-2m}r^m$. Indeed, let $n \in \mathbb{N}$ be such that $a^{n+1} < r \leq a^n$. Then every $z' \in \bar{B}(z, r)$ satisfies $\pi_i(z') = \pi_i(z)$ for $i = 1, \dots, n-1$ and hence $\mu(\bar{B}(z, r)) \leq N^{-n+1} \leq a^{-2m}r^m$. Now, let $\varepsilon > 0$ and let (B_i) be a covering of Z by sets of diameter less than $a/2$ and such that

$$\sum_{i=1}^{\infty} \omega_m \left(\frac{\text{diam } B_i}{2} \right)^m \leq \mathcal{H}^m(Z) + \varepsilon.$$

For each i set $r_i := \text{diam } B_i$ and choose $z_i \in B_i$ arbitrary. Then

$$\sum_{i=1}^{\infty} r_i^m \geq 2^{-m} a^{2m} \sum_{i=1}^{\infty} \mu(\bar{B}(z_i, r_i)) \geq 2^{-m} a^{2m} \mu(Z) = 2^{-m} a^{2m}.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\mathcal{H}^m(Z) \geq 2^{-m} a^{2m} \omega_m > 0$.

We finally show that for any $K \subset \mathbb{R}^m$ Borel and any Lipschitz map $\varphi : K \rightarrow Z$ we have $\mathcal{H}^m(\varphi(K)) = 0$. Indeed, if this is not the case, then it follows from [12] that there exists $K' \subset K$ of strictly positive Lebesgue measure, a norm $\|\cdot\|$ on \mathbb{R}^m , and $x \in K'$ a Lebesgue density point of K' such that the map $\varphi|_{K'}$ is an approximate isometry around x when viewed as a map from $(K', \|\cdot\|)$ to Z . Now, for any $\varepsilon > 0$ there exist $x', x'' \in K'$ with $\|x - x'\| < \varepsilon$ and $\|x - x''\| \leq \frac{3}{4}\|x - x'\|$ and $\|x' - x''\| \leq \frac{3}{4}\|x - x'\|$. On the other hand, if k is such that $d_{\infty}(\varphi(x), \varphi(x')) = a^k$ then, by (11), either $d_{\infty}(\varphi(x), \varphi(x'')) \geq a^k$ or $d_{\infty}(\varphi(x'), \varphi(x'')) \geq a^k$. Since $\varphi|_{K'}$ is an approximate isometry around x , this yields a contradiction. This completes the proof. \square

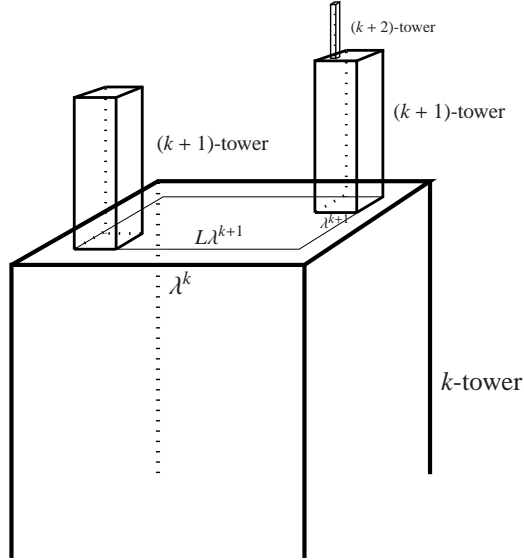
We use the above lemma to prove the theorem.

Proof of Theorem A.1. Let $m, L \geq 2$ and $\alpha \in (0, 1)$, set $a := L^{-1}$ and $N := L^m$, and let $\lambda \in (0, a)$ such that $\alpha \log \lambda = \log a$. For each $n \geq 1$, let P_n be the boundary of the $(m+1)$ -dimensional rectangle $[0, \lambda^n]^m \times [0, a^n]$. We call $[0, \lambda^n]^m \times \{0\}$ the base of P_n and $[0, \lambda^n]^m \times \{a^n\}$ the roof of P_n . Let $X_0 := [0, L\lambda]^m$ have the Euclidean metric, divide X_0 into N equal cubes of edge length λ each, and glue to each cube a copy of P_1 along its base (and ‘delete’ the interior of the base after gluing). Endow the so obtained space with the length metric and call it X_1 . We call the copies of P_1 in X_1 the 1-towers of X_1 . Suppose now that we have constructed X_n for some $n \geq 1$. In order to construct X_{n+1} from X_n , center a cube of edge length $L\lambda^{n+1}$ on the roof of each n -tower in X_n , note for this that $L\lambda^{n+1} < \lambda^n$, then divide the cube into N equal cubes of edge length λ^{n+1} and glue a copy of P_{n+1} along its base to each cube (and ‘delete’ the interior of the base after gluing). Endow the new space with the length metric and call it X_{n+1} . The copies of P_{n+1} in X_{n+1} are called $(n+1)$ -towers in X_{n+1} . Figure 1 below shows how towers stand on top of each other.

Clearly, each X_n is biLipschitz homeomorphic to the cube X_0 and satisfies

$$\mathcal{H}^m(X_n) = (L\lambda)^m + 2m \sum_{k=1}^n \mu^k \leq (L\lambda)^m + \frac{2m}{1-\mu},$$

where $\mu := aN\lambda^{m-1} < 1$. For $k < n$, the distance d from any point at the base of a given k -tower in X_n to any point on its roof satisfies $a^k \leq d \leq (m+1)a^k$. It follows easily that all X_n have uniformly bounded diameter. Now, each X_n has local geometric contractibility function $\varrho : [0, \lambda/2) \rightarrow [0, \infty)$ given by $\varrho(s) := Cs^\alpha$, where $C = 2^\alpha(m+2-a)(1-a)^{-1}L$.

F 1. Some of the towers in X_n

Indeed, let $z \in X_n$ and $s \in (0, \lambda/2)$. Let $k \in \mathbb{N}$ be such that $\lambda^{k+1} \leq 2s < \lambda^k$. Suppose that $B(z, s)$ intersects a p -tower but no $(p-1)$ -tower in X_n . If $p < k$ then $B(z, s)$ cannot intersect an l -tower with $l > p+1$ since otherwise $2s \geq a^{p+1}$ and hence $p \geq k$, a contradiction. Since $2s < \lambda^{p+1}$ it thus follows that $B(z, s)$ is contractible within itself. Now, if $p \geq k$ then $B(z, s)$ is contractible within the ball $B(z, s')$ where

$$s' = \lambda^k + \frac{a^p(m+1)}{1-a}.$$

It is trivial to check that $s' \leq \varrho(s)$. This shows that ϱ is indeed a geometric contractibility function for X_n .

Let now X_∞ be the Gromov-Hausdorff limit of the sequence X_n . We first show that X_∞ has finite \mathcal{H}^m -measure. For this, let $\delta > 0$ and choose n_0 so large that $2(m+1)(1-a)^{-1}a^{n_0} < \delta$. Fix an n_0 -tower in X_∞ and a point x at its base. Then the closed ball of radius $(m+1)(1-a)^{-1}a^{n_0}$ around x contains the given n_0 -tower and all the towers on its roofs (and the towers on their roofs, and so on). Since X_∞ contains exactly N^{n_0} n_0 -towers, we see that

$$\mathcal{H}_\delta^m(X_\infty) \leq \mathcal{H}^m(X_{n_0-1}) + N^{n_0} \omega_m \left[\frac{(m+1)a^{n_0}}{1-a} \right]^m \leq (L\lambda)^m + \frac{2m}{1-\mu} + \omega_m \left[\frac{m+1}{1-a} \right]^m.$$

Since $\delta > 0$ was arbitrary, this shows that indeed $\mathcal{H}^m(X_\infty) < \infty$. Next, the metric space Z from the lemma above admits a biLipschitz embedding into X_∞ . Since Z is not countably \mathcal{H}^m -rectifiable by Lemma A.2, it follows that X_∞ is not countably \mathcal{H}^m -rectifiable either.

Finally, in order to obtain a closed oriented Riemannian manifold M_n we glue X_n along its boundary to the boundary of an m -cube and then smooth it. In this way, we can also achieve that each M_n is biLipschitz homeomorphic to the standard m -sphere S^m and that the Gromov-Hausdorff limit is homeomorphic to S^m . This concludes the proof of our theorem. \square

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