PROPERTIES OF THE INTRINSIC FLAT DISTANCE

CHRISTINA SORMANI

Abstract. Here we explore the properties of Intrinsic Flat convergence, proving a number of theorems relating it to Gromov-Hausdorff convergence and proving new Gromov-Hausdorff and Intrinsic Flat compactness theorems including the Tetrahedral Compactness Theorem. We introduce the sliced filling volume and related notions which are continuous under intrinsic flat convergence and can be applied to prevent the disappearance of points in the limit. We prove a pair of Bolzano-Weierstrass theorems and three Arzela-Ascoli Theorems. Additional theorems are vaguely announced which may be added in future versions of this paper.

Additions and Changes in v2 of the manuscript:

1) Two new Bolzano Weierstrass Theorems have been added in Section 4.7 applying the updated theorems mentioned below. They were announced but unstated in v1 Section 4.7 These concern the existence of limits of subsequences of points $p_i \in M_i$ where $M_i \xrightarrow{\mathcal{F}} M_\infty$ where M_∞ is precompact.

2) Two new Arzela Ascoli Theorems have been added in a new Section 5 which concern sequences of functions $F_i : M_i \to M_i'$ where $M_i \xrightarrow{\mathcal{F}} M_\infty$ and $M_i' \xrightarrow{\mathcal{F}} M_\infty$. Theorem 5.4 deals with surjective local isometries (including covering maps) and may be useful to questions by Gromov [12] concerning results of Burago-Ivanov [2] (see Remark 5.7). Theorem 5.16 provides a far more generally applicable Arzela-Ascoli Theorem. The old Arzela Ascoli Theorem from v1 Section 4.8 has been moved to Section 5.1 and is now Theorem 5.1. It requires the target space to be constant but F_i need only have $\mathrm{Lip}(F_i) \le \lambda$.

3) The Sliced Filling Compactness Theorem (v1 Thm 4.30) and the Tetrahedral Compactness Theorem (v1 Thm 4.31) have been moved from Section 4.9 to a new Section 6. They are now Theorems 6.1 and 6.2 respectively. To clarify their proofs, the remark [v1 Remark 4.22] about how to apply the Continuity of Sliced Filling Volumes Theorem [Theorem 4.20 in v1 and v2] is now directly appended to that theorem's statement. Note that these compactness theorems did not directly cite Theorem 4.20, but instead cite Theorem 4.27 (same number in v1 but slightly restated) which in turn cites Theorem 4.20 (the proof is as before). Proposition 4.26 (same number as before) also cited Theorem 4.20 but is unchanged by the addition to Thm 4.20.

4) The old announced plans in Sections 4.10 - 4.11 are now announced in Sections 7 -8. It is possible they may be postponed to a future paper or completed in v3 depending upon how the author decides to publish the article.

The plan is to include clean statements of all properties of intrinsic flat convergence which may be useful to Riemannian Geometers. As the author is asked questions, new theorems will be added. All theorems stated here are proven here.

partially supported by NSF DMS 1006059 and a PSC CUNY Research Grant.

1. Introduction

The Intrinsic Flat distance between Riemannian manifolds has been applied to study the stability of the Positive Mass Theorem, the rectifiability of Gromov-Hausdorff limits of Riemannian manifolds and smooth convergence away from singular sets. Here we present properties of Riemannian manifolds which are conserved under intrinsic flat convergence. The initial notion of the intrinsic flat distance and lower semicontinuity of mass and the continuity of filling volumes of balls is joint work with Stefan Wenger appearing in [22] and [23]. Here we provide more detailed proofs of the properties described in [22] and prove additional properties as well as new compactness theorems.

The intrinsic flat distance is defined using Gromov's idea of isometrically embedding two Riemannian manifolds into a common metric space. Rather than measuring the Hausdorff distance between the images as Gromov did when defining the Gromov-Hausdorff distance in [11], one views the images as integral currents in the sense of Ambrosio-Kirchheim in [1] and take the flat distance between them. Wenger's compactness theorem states that a sequence of Riemannian manifolds with a uniform upper bound on volume, boundary volume and diameter has a converging subsequence [25]. The limit spaces are countably $\mathcal{H}^m$ rectifiable metric spaces called integral current spaces [23]. In Section 2, we review Gromov-Hausdorff convergence, intrinsic flat convergence and a few key theorems of Ambrosio-Kirchheim that were never reviewed and applied in [23]. We also review the most important results from [23] that will be applied here. Recall that under intrinsic flat convergence thin regions can disappear in the limit and so it is essential for us to understand what points disappear and which remain.

In Section 3 we explore the metric properties of integral current spaces. We describe the balls in the spaces, their isometric products, level sets of Lipschitz functions called slices, spheres, estimates on the masses of the slices and on the filling volumes of slices and spheres. We recall how in prior work of the author with Wenger in [22], the filling volumes of spheres were estimated using contractibility functions much in the style of [8] and [10]. This was applied in [22] to prevent the disappearance of points in the limit because the masses of balls in the limit would be bounded below by the filling volumes of spheres. Then we introduce the new notion of a *sliced filling volume* [Defn 3.20] and prove a new Gromov-Hausdorff compactness theorem [Theorem 3.23]. We then relate the sliced filling volumes to a new notion called the tetrahedral property [Defn 3.30] and the integral tetrahedral property [Defn 3.36]. This provides another Gromov-Hausdorff Compactness Theorem [Theorem 3.41]. Here we state the three dimensional version of this theorem:

Theorem 1.1. *Given $r_0 > 0, \beta \in (0,1), C > 0, V_0 > 0$, If a sequence of Riemannian manifolds, M^3_i, has $\operatorname{Vol}(M^3_j) \le V_0$, $\operatorname{Diam}(M^3_i) \le D_0$ and the C,β tetrahedral property for all balls, $B_p(r) \subset M^3_i$, of radius $r \le r_0$:*

$$\exists p_1, p_2 \in \partial B_p(r) \text{ such that } \forall t_1, t_2 \in [(1-\beta)r, (1+\beta)r] \text{ we have}$$

$$\inf\{d(x,y) :\; x \neq y,\; x, y \in \partial B_p(r) \cap \partial B_{p_1}(t_1) \cap \partial B_{p_2}(t_2)\} \in [Cr, \infty)$$

then a subsequence of the M_i converges in the Gromov-Hausdorff sense. See Figure 1.

In fact this is a noncollapsing compactness theorem, since the volumes of balls are bounded below. Later in the paper we will see that these sequences also have intrinsic flat limits, the intrinsic flat limits agree with the Gromov-Hausdorff limits and thus the limits are countable $\mathcal{H}^3$ rectifiable metric spaces. Section 3 closes with the definition of interval filling volumes [Defn 3.43], sliced interval filling volumes [Defn 3.45] and corresponding estimates on mass based upon these notions. Those studying the proof of the Tetrahedral

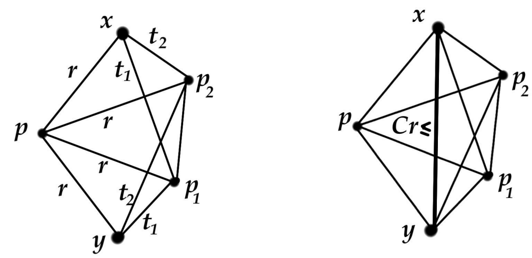

FIGURE 1. Tetrahedral Property

Compactness Theorem need to read all Sections except 3.2 and 3.12 before continuing to Section 4. Those studying the Bolzano-Weierstrass and Arzela-Ascoli Theorems need only read Sections 3.1 and 3.3-3.6 before continuing to Section 4.

In Section 4 we examine when sequences of points $p_i \in M_i$ disappear under intrinsic flat convergence, $M_i \xrightarrow{\mathcal{F}} M_\infty$. We begin by defining converging and Cauchy sequences of points $p_i \in M_j$ [Definitions 4.1 and 4.2]. We prove all points $p_\infty \in M_\infty$ are limits of converging sequences of points $p_i \in M_i$ and that the diameters are lower semicontinuous $\liminf_{i\to\infty} \mathrm{Diam}(M_i) \geq \mathrm{Diam}(M_\infty)$ [Theorems 4.3 and 4.5]. Then we prove Theorem 4.6, that M_∞ is the the Gromov-Hausdorff limit of regions, N_i within the M_i. If $\mathrm{Vol}(M_i) \to \mathrm{Vol}(M_\infty)$ then $\mathrm{Vol}(M_i \setminus N_i) \to 0$ [Remark 4.7].

In the second half of Section 4, we prove the continuity of the sliced filling volumes [Theorem 4.20], the interval filling volumes [Theorem 4.23] and the sliced interval filling volumes [Theorem 4.24]. These theorems are built upon the fact that Wenger and the author had shown the continuity of the filling volume in [22] (reproven here in more detail as Theorem 2.46). We first prove that spheres converge as well as slices of spheres and then obtain strong enough estimates on the fillings of the slices of the spheres to prove our theorems. We apply the sliced filling volumes to prevent the disappearance of Cauchy sequences of points [Theorem 4.27]. This leads to a pair of Bolzano-Weierstrass Theorems [Theorem 4.30 and Theorem 4.31]. The only material needed from Section 4 to follow the proof of the Tetrahedral Compactness Theorem are in Subsections 4.1, 4.3, 4.4 and 4.6.

In Section 5 we prove a few Arzela-Ascoli Theorems. The basic Theorem 5.1 applies to uniformly Lipschitz functions with a single target space. Theorem 5.4 applies to surjective uniformly local isometries with both domains and ranges converging in the intrinsic flat sense. This includes covering maps and has a variety of potential applications described in remarks below the statement. Finally there is the most general Theorem 5.16 which applies to uniformly open filling maps between spaces which are converging in the intrinsic flat sense.

In Section 6 we improve the Gromov-Hausdorff compactness theorems mentioned earlier stating that the Gromov-Hausdorff and Intrinsic Flat limit spaces agree and are thus countable$\mathcal{H}^m$ rectifiable spaces. The first is an (integral) sliced filling compactness theorem [Theorem 6.1]. The second is the Tetrahedral Compactness Theorem [Theorem 6.2]. See [19] for an announcement of these results. One may recall Wenger's Compactness Theorem: if a sequence of M_i has a uniform upper bound of diameter, volume and boundary volume, then a subsequence converges in the intrinsic flat sense to an integral current space possibly the zero space [25]. One contrast between Wenger's Compactness Theorem and these new compactness theorems, is that our limit spaces are never the zero space and so naturally are subjected to far stronger hypotheses. It should be noted that Wenger's Compactness Theorem is never applied to prove anything in this paper. Thus, those who might wish to find a new proof for his theorem, could use material found here. Naturally any proof of his compactness theorem would have to allow for sequences with disappearing points.

This paper is still in progress yet all theorems stated herein are proven in complete detail here. As the author is working with other mathematicians on applications of the intrinsic flat convergence to questions arising in General Relativity, new properties may also be seen to be necessary and they will be proven here. The goal is to include all theorems which require a deep understanding of the work of Ambrosio-Kirchheim in this single paper so that they may be verified and refereed by experts in that area. Then the applications to General Relativity can be explored without requiring everyone to master geometric measure theory. There are many remarks suggesting further results which may become theorems in future versions or in other papers. Sections 7 and 8 may be completed in a future version or in another paper.

Recommended Reading:

While it is not necessary, students reading this paper are encouraged to read Burago-Burago-Ivanov's textbook [3] which provides a thorough background in Gromov-Hausdorff convergence and also to read the authors joint paper with Wenger [23]. Those who would like to understand the Geometric Measure Theory more deeply should read Morgan's textbook [18] or Fanghua Lin's textbook [17] and then the work of Ambrosio-Kirccheim [1]. We review all theorems applied in this paper in the review section.

Acknowledgements:

I am grateful to Lars Andersson, Luigi Ambrosio, Toby Colding, Misha Gromov, Gerhardt Huisken, Tom Ilmanen, Jim Isenberg, Blaine Lawson, Paul Yang and Shing-Tung Yau for their interest in the Intrinsic Flat distance and invitations to visit their universities and to attend Oberwolfach. We discussed many interesting applications and I hope this paper provides the properties required to explore these possibilities. I appreciate Maria Gordina for assisting with some of the details regarding metric measure spaces and always being available when needed. I would especially like to thank the doctoral students Jorge Basilio (CUNY), Sajjad Lakzian(CUNY), Raquel Perales (SUNYSB) and Jim Portegies (NYU) for actively participating in the CUNY Geometric Measure Theory Reading Group while this paper was being written. Their careful presentations of Ambrosio-Kirchheim and deep questions lead to many interesting ideas.

2. Background

In this section we review the Gromov-Hausdorff distance introduced by Gromov in [11], then various topics from Ambrosio-Kirchheim's work in [1], then intrinsic flat convergence and integral current spaces from prior work of the author with Wenger in [23] and end with a review of filling volumes which are related to Gromov's notion from [10] but defined using the work of Ambrosio-Kirchheim.

2.1. Review of the Gromov-Hausdorff Distance.

First recall that $\varphi : X \to Y$ is an isometric embedding iff

$$d_Y(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2) \qquad \forall x_1, x_2 \in X. \tag{1}$$

This is referred to as a metric isometric embedding in [15] and it should be distinguished from a Riemannian isometric embedding.

Definition 2.1 (Gromov)**.** *The Gromov-Hausdorff distance between two compact metric spaces (X, d_X) and (Y, d_Y) is defined as*

$$d_{GH}(X, Y) := \inf d_H^Z(\varphi(X), \psi(Y)) \tag{2}$$

where Z is a complete metric space, and $\varphi : X \to Z$ and $\psi : Y \to Z$ are isometric embeddings and where the Hausdorff distance in Z is defined as

$$d_H^Z(A, B) = \inf\{\epsilon > 0 : A \subset T_\epsilon(B) \text{ and } B \subset T_\epsilon(A)\}. \tag{3}$$

Gromov proved that this is indeed a distance on compact metric spaces: $d_{GH}(X, Y) = 0$ iff there is an isometry between X and Y. When studying metric spaces which are only precompact, one may take their metric completions before studying the Gromov-Hausdorff distance between them.

Definition 2.2. *A collection of metric spaces is said to be equibounded or uniformly bounded if there is a uniform upper bound on the diameter of the spaces.*

Remark 2.3. *We will write $N(X, r)$ to denote the number of disjoint balls of radius r in a space X. Note that X can always be covered by $N(X, r)$ balls of radius $2r$.*

Note that Ilmanen's Example of [23] of a sequence of spheres with increasingly many splines is not equicompact, as the number of balls centered on the tips approaches infinity.

Definition 2.4. *A collection of spaces is said to be equicompact or uniformly compact if they are have a common upper bound $N(r)$ such that $N(X, r) \leq N(r)$ for all spaces X in the collection.*

Gromov's Compactness Theorem states that sequences of equibounded and equicompact metric spaces have a Gromov-Hausdorff converging subsequence [11]. In fact, Gromov proves a stronger version of this statement in [9]p 65: CHECK PAGE

Theorem 2.5 (Gromov's Compactness Theorem)**.** *If a sequence of compact metric spaces, X_j, is equibounded and equicompact, then a subsequence of the X_j converges to a compact metric space X_∞.*

Gromov also proved the following useful theorem:

Theorem 2.6. *If a sequence of compact metric spaces X_j converges to a compact metric space X_∞ then X_j are equibounded and equicompact. Furthermore, there is a compact metric spaces, Z, and isometric embeddings $\varphi_j : X_j \to Z$ such that*

$$d_H^Z\left(\varphi_j\left(X_{j_i}\right), \varphi_\infty(X_\infty)\right) \le 2d_{GH}(X_j, X_\infty) \to 0. \tag{4}$$

This theorem allows one to define converging sequences of points:

Definition 2.7. *We say that $x_j \in X_j$ converges to $x_\infty \in X_\infty$, if there is a common space Z as in Theorem 2.6 such that $\varphi_j(x_j) \to \varphi_\infty(x)$ as points in Z. If one discusses the limits of multiple sequences of points then one uses a common Z to determine the convergence to avoid difficulties arising from isometries in the limit space. Then one immediately has*

$$\lim_{j\to\infty} d_{X_j}(x_j, x'_j) = d_{X_\infty}(x_\infty, x'_\infty) \tag{5}$$

whenever $x_j \to x_\infty$ and $x'_j \to x'_\infty$ via a common Z.

Theorem 2.6 also allows one to extend the Arzela-Ascoli Theorem:

Definition 2.8. *A collection of functions, $f_j : X_j \to X'_j$ is said to be equicontinuous if for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ independent of j such that*

$$f_j(B_x(\delta_\epsilon)) \subset B_{f_j(x)}(\epsilon) \qquad \forall x \in X_j. \tag{6}$$

Theorem 2.9. *Gromov-Arz-Asc Suppose X_j and X'_j are compact metric spaces converging in the Gromov-Hausdorff sense to compact metric spaces X_∞ and X'_∞, and suppose $f_j : X_j \to X'_j$ are equicontinuous, then a subsequence of the f_j converge to a continuous function $f_\infty : X_\infty \to X'_\infty$ such that for any sequence $x_j \to x_\infty$ via a common Z we have $f_j(x_j) \to f_\infty(x_\infty)$.*

In particular, one can define limits of curves $C_i : [0, 1] \to X_i$ (parametrized proportional to arclength with a uniform upper bound on length) to obtain curves $C_\infty : [0, 1] \to X_\infty$. So that when X_i are compact length spaces whose distances are achieved by minimizing geodesics, so are the limit spaces X_∞.

One only needs uniform lower bounds on Ricci curvature and upper bounds on diameter to prove equicompactness on a sequence of Riemannian manifolds. This is a consequence of the Bishop-Gromov Volume Comparison Theorem [11]. Colding and Cheeger-Colding have studied the limits of such sequences of spaces proving volume convergence and eigenvalue convergence and many other interesting properties [6] [4]-[5]. One property of particular interest here, is that when the sequence of manifolds is noncollapsing (i.e. is assumed to have a uniform lower bound on volume), Cheeger-Colding prove that the limit space is countably $\mathcal{H}^m$ rectifiable with the same dimension as the sequence [4].

Greene-Petersen have shown that conditions on contractibility and uniform upper bounds on diameter also suffice to achieve compactness without any assumption on Ricci curvature or volume [8]. Sormani-Wenger have shown that if one assumes a uniform linear contractibility function on the sequence of manifolds then the limit spaces achieved in their setting are also countably $\mathcal{H}^m$ rectifiable with the same dimension as the sequence. Without the assumption of linearity, Schul-Wenger have provided an example where the Gromov-Hausdorff limit is not countably $\mathcal{H}^m$ rectifiable. [22]. The proofs here involve the Intrinsic Flat Convergence.

2.2. **Review of Ambrosio-Kirchheim Currents on Metric Spaces.** In [1], Ambrosio-Kirchheim extend Federer-Fleming's notion of integral currents using DiGeorgi's notion of tuples of functions. Here we review their ideas. Here Z denotes a complete metric space.

In Federer-Fleming currents were defined as linear functionals on differential forms [7]. This approach extends naturally to smooth manifolds but not to complete metric spaces which do not have differential forms. In the place of differential forms, Ambrosio-Kirchheim use DiGeorgi's $m+1$ tuples, $\omega \in \mathcal{D}^m(Z)$,

$$\omega = f\pi = (f, \pi_1 ... \pi_m) \in \mathcal{D}^m(Z) \tag{7}$$

where $f : X \to \mathbb{R}$ is a bounded Lipschitz function and $\pi_i : X \to \mathbb{R}$ are Lipschitz.

In [1] Definitions 2.1, 2.2, 2.6 and 3.1, an m dimensional current $T \in \mathbf{M}_m(Z)$ is defined. Here we combine them into a single definition:

Definition 2.10. *On a complete metric space, Z, an m dimensional* ***current****, denoted* $T \in \mathbf{M}_m(Z)$*, is a real valued* multilinear functional *on* $\mathcal{D}^m(Z)$*, with the following three required properties:*

i) Locality*:*

$$T(f, \pi_1, ...\pi_m) = 0 \textit{ if } \exists i \in \{1, ...m\} \textit{ s.t. } \pi_i \textit{ is constant on a nbd of } \{f \neq 0\}. \tag{8}$$

ii) Continuity*:*

$$\textit{Continuity of } T \textit{ with respect to the ptwise convergence of } \pi_i \textit{ such that } \operatorname{Lip}(\pi_i) \leq 1. \tag{9}$$

iii) Finite mass*:*

$$\exists \textit{ finite Borel } \mu \;\; s.t. \;\; |T(f, \pi_1, ...\pi_m)| \leq \prod_{i=1}^{m} \operatorname{Lip}(\pi_i) \int_Z |f| \, d\mu \;\; \forall (f, \pi_1, ...\pi_m) \in \mathcal{D}^m(Z). \tag{10}$$

In [1] Definition 2.6 Ambrosio-Kirchheim introduce their mass measure which is distinct from the masses used in work of Gromov [10] and Burago-Ivanov [2]. This definition is later used to define the notion of filling volume used in this paper.

Definition 2.11. *The mass measure* $\|T\|$ *of a current* $T \in \mathbf{M}_m(Z)$*, is the smallest Borel measure* μ *such that*

$$\left| T\,(f, \pi) \right| \leq \int_X |f| d\mu \qquad \forall \; (f, \pi) \;\; \textit{where} \;\; \operatorname{Lip}(\pi_i) \leq 1. \tag{11}$$

The mass of T is defined

$$M\,(T) = \|T\|\,(Z) = \int_Z d\|T\|. \tag{12}$$

In particular

$$\left| T(f, \pi_1, ...\pi_m) \right| \leq \mathbf{M}(T) |f|_\infty \operatorname{Lip}(\pi_1) \cdots \operatorname{Lip}(\pi_m). \tag{13}$$

Stronger versions of locality and continuity, as well as product and chain rules are proven in [1][Theorem 3.5]. In particular they define $T(f, \pi_1, ..., \pi_m)$ for f which are only Borel functions as limits of $T(f_j, \pi_1, ..., \pi_m)$ where f_j are bounded Lipschitz functions converging to f in $L^1(E, \|T\|)$. They also prove

$$T(f, \pi_\sigma(1), ...\pi_\sigma(m)) = \operatorname{sgn}(\sigma) T(f, \pi_1, ...\pi_m) \tag{14}$$

for any permutation, σ, of $\{1, 2, ...m\}$.

Definition 2.12. [1]*[Defn 2.5] The* restriction $T \llcorner \omega \in \mathbf{M}_m(Z)$ *of a current* $T \in M_{m+k}(Z)$ *by a* $k+1$ *tuple* $\omega = (g, \tau_1, ...\tau_k) \in \mathcal{D}^k(Z)$*:*

$$(T \llcorner \omega)(f, \pi_1, ...\pi_m) := T(f \cdot g, \tau_1, ...\tau_k, \pi_1, ...\pi_m). \tag{15}$$

Given a Borel set, A,

$$T \llcorner A := T \llcorner \omega \tag{16}$$

where $\omega = 1_A \in \mathcal{D}^0(Z)$ *is the indicator function of the set. In this case,*

$$\mathbf{M}(T \llcorner \omega) = \|T\|(A). \tag{17}$$

Ambrosio-Kirchheim then define:

Definition 2.13. *Given a Lipschitz map* $\varphi : Z \to Z'$*, the* push forward *of a current* $T \in \mathbf{M}_m(Z)$ *to a current* $\varphi_\# T \in \mathbf{M}_m(Z')$ *is given in* [1]*[Defn 2.4] by*

$$\varphi_\# T(f, \pi_1, ...\pi_m) := T(f \circ \varphi, \pi_1 \circ \varphi, ...\pi_m \circ \varphi) \tag{18}$$

exactly as in the smooth setting.

Remark 2.14. *One immediately sees that*

$$(\varphi_\# T) \llcorner (f, \pi_1, ...\pi_k)) = \varphi_\#(T \llcorner (f \circ \varphi, \pi_1 \circ \varphi, ...\pi_k \circ \varphi)) \tag{19}$$

and

$$(\varphi_\# T) \llcorner A = (\varphi_\# T) \llcorner (1_A) = \varphi_\#(T \llcorner (1_A \circ \varphi)) = \varphi_\#(T \llcorner \varphi^{-1}(A)). \tag{20}$$

In (2.4) [1], Ambrosio-Kirchheim show that

$$\|\varphi_\# T\| \le [\mathrm{Lip}(\varphi)]^m \varphi_\# \|T\|, \tag{21}$$

so that when φ is an isometric embedding

$$\|\varphi_\# T\| = \varphi_\# \|T\| \text{ and } \mathbf{M}(T) = \mathbf{M}(\varphi_\# T). \tag{22}$$

The simplest example of a current is:

Example 2.15. *If one has a bi-Lipschitz map,* $\varphi : \mathbb{R}^m \to Z$*, and a Lebesgue function* $h \in L^1(A, \mathbb{Z})$ *where* $A \in \mathbb{R}^m$ *is Borel, then* $\varphi_\#[h] \in \mathbf{M}_m(Z)$ *an m dimensional current in Z. Note that*

$$\varphi_\#[h](f, \pi_1, ...\pi_m) = \int_{A \subset \mathbb{R}^m} (h \circ \varphi)(f \circ \varphi)\, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_m \circ \varphi) \tag{23}$$

where $d(\pi_i \circ \varphi)$ *is well defined almost everywhere by Rademacher's Theorem. Here the mass measure is*

$$\|[h]\| = h d\mathcal{L}_m \tag{24}$$

and the mass is

$$\mathbf{M}([h]) = \int_A h d\mathcal{L}_m. \tag{25}$$

In [1][Theorem 4.6] Ambrosio-Kirchheim define a canonical set associated with any integer rectifiable current:

Definition 2.16. *The (canonical) set of a current, T, is the collection of points in Z with positive lower density:*

$$\mathrm{set}\,(\mathrm{T}) = \{\mathrm{p} \in \mathrm{Z} : \Theta_{*\mathrm{m}}\,(\|\mathrm{T}\|, \mathrm{p}) > 0\}, \tag{26}$$

where the definition of lower density is

$$\Theta_{*m}\,(\mu, p) = \liminf_{r\to 0} \frac{\mu(B_p(r))}{\omega_m r^m}. \tag{27}$$

In [1] Definition 4.2 and Theorems 4.5-4.6, an integer rectifiable current is defined using the Hausdorff measure, $\mathcal{H}^m$:

Definition 2.17. *Let $m \geq 1$. A current, $T \in \mathcal{D}_m(Z)$, is rectifiable if* set(T) *is countably $\mathcal{H}^m$ rectifiable and if $\|T\|(A) = 0$ for any set A such that $\mathcal{H}^k(A) = 0$. We write $T \in \mathcal{R}_m(Z)$.*

We say $T \in \mathcal{R}_m(Z)$ is integer rectifiable, denoted $T \in \mathcal{I}_m(Z)$, if for any $\varphi \in \mathrm{Lip}(Z, \mathbb{R}^m)$ and any open set $A \in Z$, we have

$$\exists\, \theta \in \mathcal{L}^1(\mathbb{R}^k, Z) \;\; s.t. \;\; \varphi_\#(T \llcorner A) = [\theta]. \tag{28}$$

In fact, $T \in \mathbf{I}_m(Z)$ iff it has a parametrization. A parametrization $(\{\varphi_i\}, \{\theta_i\})$ of an integer rectifiable current $T \in \mathcal{I}^m(Z)$ is a collection of bi-Lipschitz maps $\varphi_i : A_i \to Z$ with $A_i \subset \mathbb{R}^m$ precompact Borel measurable and with pairwise disjoint images and weight functions $\theta_i \in L^1(A_i, \mathbb{N})$ such that

$$T = \sum_{i=1}^{\infty} \varphi_{i\#}[\![\theta_i]\!] \quad and \quad \mathbf{M}\,(T) = \sum_{i=1}^{\infty} \mathbf{M}\,(\varphi_{i\#}[\![\theta_i]\!])\,. \tag{29}$$

A 0 dimensional rectifiable current is defined by the existence of countably many distinct points, $\{x_i\} \in Z$, weights $\theta_i \in \mathbb{R}^+$ and orientation, $\sigma_i \in \{-1, +1\}$ such that

$$T(f) = \sum_h \sigma_i \theta_i f(x_i) \qquad \forall f \in \mathcal{B}^\infty(Z). \tag{30}$$

where $\mathcal{B}^\infty(Z)$ is the class of bounded Borel functions on Z and where

$$\mathbf{M}(T) = \sum_h \theta_i < \infty \tag{31}$$

If T is integer rectifiable $\theta_i \in \mathbb{Z}^+$, so the sum must be finite.

In particular, the mass measure of $T \in \mathbf{I}_m(Z)$ satisfies

$$\|T\| = \sum_{i=1}^{\infty} \|\varphi_{i\#}[\![\theta_i]\!]\|. \tag{32}$$

Theorems 4.3 and 8.8 of [1] provide necessary and sufficient criteria for determining when a current is integer rectifiable.

Note that the current in Example 2.15 is an integer rectifiable current.

Example 2.18. *If one has a Riemannian manifold, M^m, and a biLipschitz map $\varphi : M^m \to Z$, then $T = \varphi_\#[1_M]$ is an integer rectifiable current of dimension m in Z. If φ is an isometric embedding, and $Z = M$ then $\mathbf{M}(T) = \mathrm{Vol}(M^m)$. Note further that* set(T) $= \varphi$(M).

If M has a conical singularity then set(T) $= \varphi$(M). *However, if M has a cusp singularity at a point $p \in M$ then* set(T) $= \varphi(\mathrm{M} \setminus \{\mathrm{p}\})$.

Definition 2.19. [1]*[Defn 2.3] The* boundary *of $T \in \mathbf{M}_m(Z)$ is defined*

$$\partial T(f, \pi_1, ...\pi_{m-1}) := T(1, f, \pi_1, ...\pi_{m-1}) \in M_{m-1}(Z) \tag{33}$$

When $m = 0$, we set $\partial T = 0$.

Note that $\varphi_\#(\partial T) = \partial(\varphi_\# T)$.

Definition 2.20. [1]*[Defn 3.4 and 4.2] An integer rectifiable current $T \in \mathcal{I}_m(Z)$ is called an integral current, denoted $T \in \mathbf{I}_m(Z)$, if ∂T defined as*

$$\partial T\,(f, \pi_1, ...\pi_{m-1}) := T\,(1, f, \pi_1, ...\pi_{m-1}) \tag{34}$$

has finite mass. The total mass of an integral current is

$$\mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T). \tag{35}$$

Observe that $\partial\partial T = 0$. In [1] Theorem 8.6, Ambrosio-Kirchheim prove that

$$\partial : \mathbf{I}_m(Z) \to \mathbf{I}_{m-1}(Z) \tag{36}$$

whenever $m \geq 1$. By (21) one can see that if $\varphi : Z_1 \to Z_2$ is Lipschitz, then

$$\varphi_\# : \mathbf{I}_m(Z_1) \to \mathbf{I}_m(Z_2). \tag{37}$$

However, the restriction of an integral current need not be an integral current except in special circumstances. For example, T might be integration over $[0,1]^2$ with the Euclidean metric and $A \subset [0,1]^2$ could have an infinitely long boundary, so that $T \llcorner A \notin \mathbf{I}_2([0,1]^2)$ because $\partial(T \llcorner A)$ has infinite mass.

Remark 2.21. *If T is an $\mathcal{H}^1$ integral current then ∂T is an $\mathcal{H}^0$ integer rectifiable current so $H = \mathrm{set}\partial\mathrm{T}$ must be finite and $\theta_{p_h} = \|\partial T\|(p_h) \in \mathbb{Z}^+$ for all $p \in H$ and*

$$\partial T(f) = \sum_{h\in H} \sigma_h \theta_h f(p_h) \qquad \forall f \in \mathcal{B}^\infty(Z). \tag{38}$$

as described above. In addition, we have

$$0 = T(1,1) = \partial T(1) = \sum_{h\in H} \sigma_h \theta_h. \tag{39}$$

Example 2.22. *If T is an $\mathcal{H}^1$ rectifiable current then*

$$T = \sum_{i=1}^{\infty} \sigma_i \theta_i \varphi_{i\#}[\chi_{A_i}] \tag{40}$$

where $\theta_i \in \mathbb{Z}^+$, $\sigma_i \in \{+1, -1\}$ and A_i is an interval with $\bar{A}_i = [a_i, b_i]$ because all Borel sets are unions of intervals and all integer valued Borel functions can be written up to Lebesgue measure 0 as a countable sum of characteristic functions of intervals. One might like to write:

$$\partial T(f) = \sum_i \sigma_i \theta_i \left(f(\varphi_i(b_i)) - f(\varphi_i(a_i))\right) \qquad \forall f \in \mathcal{B}^\infty(Z). \tag{41}$$

This works when the sum happens to be a finite sum. Yet if T is a infinite collection of circles based at a common point, $(0,0) \in \mathbb{R}^2$, defined with $\sigma_i = 1$ $\theta_i = 1$, $A_i = [0,\pi]$ and

$$\varphi_i(s) = (r_i \cos(s) + r_i, r_i \sin(s)) \text{ for } i \text{ odd and} \tag{42}$$

$$\varphi_i(s) = (r_i \cos(s+\pi) + r_i, r_i \sin(s+\pi)) \text{ for } i \text{ even} \tag{43}$$

where $r_{2i} = r_{2i-1} = 1/i$ then

$$\varphi_i(a_i) = (2r_i, 0) \quad \text{and} \quad \varphi_i(b_i) = (0,0) \text{ for } i \text{ odd and} \tag{44}$$

$$\varphi_i(a_i) = (0,0) \quad \text{and} \quad \varphi_i(b_i) = (2r_i, 0) \text{ for } i \text{ even.} \tag{45}$$

So when $f(0,0) = 1$, we end up with an infinite sum whose terms are all $+1$ and -1.

2.3. **Review of Ambrosio-Kirchheim Slicing Theorems.** As in Federer-Fleming, Ambrosio-Kirchheim consider the slices of currents:

Theorem 2.23. [Ambrosio-Kirchheim] [1]*[Theorems 5.6-5.7] Let Z be a complete metric space, $T \in \mathbf{I}_m Z$ and $f : Z \to \mathbb{R}$ a Lipschitz function. Let*

$$< T, f, s >:= \partial\left(T \llcorner f^{-1}(-\infty, s]\right) - (\partial T) \llcorner f^{-1}(-\infty, s]), \tag{46}$$

so that $\mathrm{set}(< \mathrm{T}, \mathrm{f}, \mathrm{s} >) \subset (\mathrm{set}(\mathrm{T}) \cup \mathrm{set}(\partial \mathrm{T})) \cap \mathrm{f}^{-1}(\mathrm{s})$,

$$\partial < T, f, s >=< -\partial T, f, s > \tag{47}$$

and $< T_1 + T_2, f, s >=< T_1, f, s > + < T_2, f, s >$. Then for almost every slice $s \in \mathbb{R}$, $< T, f, s >$ is an integral current and we can integrate the masses to obtain:

$$\int_{s\in\mathbb{R}} \mathbf{M}(< T, f, s >)\, ds = \mathbf{M}(T \llcorner df) \le \mathrm{Lip}(f)\, \mathbf{M}(T) \tag{48}$$

where

$$(T \llcorner df)(h, \pi_1, ..., \pi_{m-1}) = T(h, f, \pi_1, ...\pi_{m-1}). \tag{49}$$

In particular, for almost every $s > 0$ one has

$$T \llcorner f^{-1}(-\infty, s] \in \mathbf{I}_{m-1}(Z). \tag{50}$$

Furthermore for all Borel sets A we have

$$< T \llcorner A, f, s >=< T, f, S > \llcorner A \tag{51}$$

and

$$\int_{s\in\mathbb{R}} \| < T, f, s > \|(A)\, ds = \|T \llcorner df\|(A). \tag{52}$$

Remark 2.24. *Observe that for any $T \in \mathbf{I}_m(Z')$, and any Lipschitz functions, $\varphi : Z \to Z'$ and $f : Z' \to \mathbb{R}$ and any $s > 0$, we have*

$$\begin{aligned}
< \varphi_\# T, f, s > &= \partial\left((\varphi_\# T) \llcorner f^{-1}(-\infty, s]\right) - (\partial \varphi_\# T) \llcorner f^{-1}(-\infty, s]) && (53)\\
&= \partial\left(\varphi_\#(T \llcorner \varphi^{-1}(f^{-1}(-\infty, s]))\right) - (\varphi_\# \partial T) \llcorner f^{-1}(-\infty, s]) && (54)\\
&= \partial\left(\varphi_\#(T \llcorner (f\circ\varphi)^{-1}(-\infty, s])\right) - \varphi_\#\left(\partial T \llcorner \varphi^{-1}(f^{-1}(-\infty, s]))\right) && (55)\\
&= \left(\varphi_\# \partial(T \llcorner (f\circ\varphi)^{-1}(-\infty, s])\right) - \varphi_\#\left(\partial T \llcorner (f\circ\varphi)^{-1}(-\infty, s])\right) && (56)\\
&= \varphi_\# < T, (f \circ \varphi), s > && (57)
\end{aligned}$$

Remark 2.25. *Ambrosio-Kirchheim then iterate this definition, $f_i : Z \to \mathbb{R}$, $s_i \in \mathbb{R}$, to define iterated slices:*

$$< T, f_1, ..., f_k, s_1, ..., s_k >=<< T, f_1, ..., f_{k-1}, s_1, ..., s_{k-1} >, f_k, s_k >, \tag{58}$$

so that

$$< T_1 + T_2, f_1, ..., f_k, s_1, ..., s_k >=< T_1, f_1, ..., f_k, s_1, ..., s_k > + < T_2, f_1, ..., f_k, s_1, ..., s_k > . \tag{59}$$

In [1] *Lemma 5.9 they prove,*

$$< T, f_1, ..., f_k, s_1, ..., s_k >=<< T, f_1, ..., f_i, s_1, ..., s_i >, f_{i+1}, ..., f_k, s_{i+1}, ..., s_k > . \tag{60}$$

In [1] *(5.9) they prove,*

$$\int_{\mathbb{R}^k} \| < T, f_1, ..., f_k, s_1, ..., s_k > \| \, ds_1 ... ds_k = \|T \llcorner (1, f_1, ..., f_k)\|, \tag{61}$$

where

$$(T \llcorner df)(h, \pi_1, ..., \pi_{m-k}) = T(h, f_1, ..., f_k, \pi_1, ...\pi_{m-k}), \tag{62}$$

so

$$\int_{\mathbb{R}^k} \mathbf{M}(< T, f_1, ...f_k, s_1, ..., s_k >) \mathcal{L}^k = \mathbf{M}(T \llcorner df) \le \prod_{j=1}^{k} \mathrm{Lip}(f_j)\, \mathbf{M}(T). \tag{63}$$

In [1] *(5.15) they prove for any Borel set $A \subset Z$ and $\mathcal{L}^m$ almost every $(s_1, ...s_k) \in \mathbb{R}^k$,*

$$< T \llcorner A, f_1, ..., f_k, s_1, ..., s_k >=< t, f_1, ..., f_k, s_1, ..., s_k > \llcorner A. \tag{64}$$

and

$$\int_{s \in \mathbb{R}^k} \| < T, f_1, ..., f_k, s_1, ..., s_k > \|(A)\, ds = \|T \llcorner df\|(A). \tag{65}$$

By (66) one can easily prove by induction that

$$\partial < T, f_1, ..., f_k, s_1, ..., s_k >= (-1)^k < \partial T, f_1, ..., f_k, s_1, ..., s_k > . \tag{66}$$

In [1] *Theorem 5.7 they prove*

$$< T, f_1, ..., f_k, s_1, ..., s_k >\in \mathbf{I}_{m-k}(Z). \tag{67}$$

for $\mathcal{L}^k$ almost every $(s_1, ..., s_k) \in \mathbb{R}^k$. By Remark 2.24 one can prove inductively that

$$< \varphi_\# T, f_1, ...f_k, s_1, ...s_k >= \varphi_\# < T, f_1 \circ \varphi, ..., f_k \circ \varphi, s_1, ..., s_k > . \tag{68}$$

2.4. **Review of Convergence of Currents.** Ambrosio Kirchheim's Compactness Theorem, which extends Federer-Fleming's Flat Norm Compactness Theorem, is stated in terms of weak convergence of currents. See Definition 3.6 in [1] which extends Federer-Fleming's notion of weak convergence except that they do not require compact support.

Definition 2.26. *A sequence of integral currents $T_j \in \mathbf{I}_m(Z)$ is said to converge weakly to a current T iff the pointwise limits satisfy*

$$\lim_{j\to\infty} T_j(f, \pi_1, ...\pi_m) = T(f, \pi_1, ...\pi_m) \tag{69}$$

for all bounded Lipschitz $f : Z \to \mathbb{R}$ and Lipschitz $\pi_i : Z \to \mathbb{R}$. We write

$$T_j \to T \tag{70}$$

One sees immediately that $T_j \to T$ implies

$$\partial T_j \to \partial T, \tag{71}$$

$$\varphi_\# T_j \to \varphi_\# T \tag{72}$$

and

$$T_j \llcorner (f, \pi_1, ..., \pi_k) \to T \llcorner (f, \pi_1, ..., \pi_k). \tag{73}$$

However $T_j \llcorner A$ need not converge weakly to $T_j \llcorner A$ as seen in the following example:

Example 2.27. *Let $Z = \mathbb{R}^2$ with the Euclidean metric. Let $\varphi_j : [0, 1] \to Z$ be $\varphi_j(t) = (1/j, t)$ and $\varphi_\infty(t) = (0, t)$. Let $S \in \mathbf{I}_1([0, 1])$ be*

$$S(f, \pi_1) = \int_0^1 f\, d\pi_.1 \tag{74}$$

Let $T_j \in \mathbf{I}_1(Z)$ be defined $T_j = \varphi_{j\#}(S)$. Then $T_j \to T_\infty$. Taking $A = [0, 1] \times (0, 1)$, we see that $T_j \llcorner A = T_j$ but $T_\infty \llcorner A = 0$.

Immediately below the definition of weak convergence [1] Defn 3.6, Ambrosio-Kirchheim prove the lower semicontinuity of mass:

Remark 2.28. *If T_j converges weakly to T, then* $\liminf_{j\to\infty} \mathbf{M}(T_j) \geq \mathbf{M}(T)$.

Theorem 2.29 (Ambrosio-Kirchheim Compactness)**.** *Given any complete metric space Z, a compact set $K \subset Z$ and $A_0, V_0 > 0$. Given any sequence of integral currents $T_j \in \mathbf{I}_m(Z)$ satisfying*

$$\mathbf{M}(T_j) \leq V_0,\ \mathbf{M}(\partial T_j) \leq A_0 \text{ and } \operatorname{set}\left(\mathrm{T_j}\right) \subset \mathrm{K}, \tag{75}$$

there exists a subsequence, T_{j_i}, and a limit current $T \in \mathbf{I}_m(Z)$ such that T_{j_i} converges weakly to T.

2.5. **Review of Integral Current Spaces.** The notion of an integral current space was introduced by the author and Stefan Wenger in [23]:

Definition 2.30. *An m dimensional metric space (X, d, T) is called an integral current space if it has a integral current structure $T \in \mathbf{I}_m\left(\bar{X}\right)$ where $\bar{X}$ is the metric completion of X and* $\operatorname{set}(\mathrm{T}) = \mathrm{X}$. *Given an integral current space $M = (X, d, T)$ we will use* $\operatorname{set}(\mathrm{M})$ *or X_M to denote X, $d_M = d$ and $[\![M]\!] = T$.*

Note that $\operatorname{set}(\partial\mathrm{T}) \subset \bar{\mathrm{X}}$. *The boundary of (X, d, T) is then the integral current space:*

$$\partial\left(X, d_X, T\right) := \left(\operatorname{set}(\partial\mathrm{T}), \mathrm{d}_{\bar{\mathrm{X}}}, \partial\mathrm{T}\right). \tag{76}$$

If $\partial T = 0$ then we say (X, d, T) is an integral current without boundary.

Remark 2.31. *Note that any m dimensional integral current space is countably $\mathcal{H}^m$ rectifiable with orientated charts, φ_i and weights θ_i provided as in (29). A 0 dimensional integral current space is a finite collection of points with orientations, σ_i and weights θ_i provided as in (30). If this space is the boundary of a 1 dimensional integral current space, then as in Remark 2.21, the sum of the signed weights is 0.*

Example 2.32. *A compact oriented Riemannian manifold with boundary, M^m, is an integral current space, where $X = M^m$, d is the standard metric on M and T is integration over M. In this case $\mathbf{M}(M) = \operatorname{Vol}(M)$ and ∂M is the boundary manifold. When M has no boundary, $\partial M = 0$.*

Definition 2.33. *The space of $m \geq 0$ dimensional integral current spaces, $\mathcal{M}^m$, consists of all metric spaces which are integral current spaces with currents of dimension m as in Definition 2.30 as well as the $\mathbf{0}$ spaces. Then $\partial : \mathcal{M}^{m+1} \to \mathcal{M}^m$.*

Remark 2.34. *A 0 dimensional integral current space, $M = (X, d, T)$, is a finite collection of points, $\{p_1, ..., p_N\}$, with a metric $d_{i,j} = d(p_i, p_j)$ and a current structure defined by assigning a weight, $\theta_i \in \mathbb{Z}^+$, and an orientation, $\sigma_i \in \{+1, -1\}$ to each $p_i \in X$ and*

$$\mathbf{M}(M) = \sum_{i=1}^{N} \theta_i. \tag{77}$$

If M is the boundary of a 1 dimensional integral current space then, as in Remark 2.21, we have

$$\sum_{i=1}^{N} \sigma_i \theta_i = 0 \tag{78}$$

In particular $N \geq 2$ if $M \neq \mathbf{0}$.

2.6. **Review of the Intrinsic Flat distance.** The Intrinsic Flat distance was defined in work of the author and Stefan Wenger [23] as a new distance between Riemannian manifolds based upon the work of Ambrosio-Kirchheim reviewed above.

Recall that the flat distance between m dimensional integral currents $S, T \in \mathbf{I}_m(Z)$ is given by

$$d_F^Z(S,T) := \inf\{\mathbf{M}(U) + \mathbf{M}(V) : S - T = U + \partial V\} \tag{79}$$

where $U \in \mathbf{I}_m(Z)$ and $V \in \mathbf{I}_{m+1}(Z)$. This notion of a flat distance was first introduced by Whitney in [26] and later adapted to rectifiable currents by Federer-Fleming [7]. The flat distance between Ambrosio-Kirchheim's integral currents was studied by Wenger in [24]. In particular, Wengr proved that if $T_j \in \mathbf{I}_m(Z)$ has $\mathbf{M}(T_j) \le V_0$ and $\mathbf{M}(\partial T_j) \le A_0$ then

$$T_j \to T \text{ iff } d_F^Z(T_j, T) \to 0 \tag{80}$$

exactly as in Federer-Fleming.

The intrinsic flat distance between integral current spaces was first defined in [23][Defn 1.1]:

Definition 2.35. *For $M_1 = (X_1, d_1, T_1)$ and $M_2 = (X_2, d_2, T_2) \in \mathcal{M}^m$ let the intrinsic flat distance be defined:*

$$d_{\mathcal{F}}(M_1, M_2) := \inf d_F^Z(\varphi_{1\#}T_1, \varphi_{2\#}T_2), \tag{81}$$

where the infimum is taken over all complete metric spaces (Z, d) and isometric embeddings $\varphi_1 : \left(\bar{X}_1, d_1\right) \to (Z, d)$ and $\varphi_2 : \left(\bar{X}_2, d_2\right) \to (Z, d)$ and the flat norm d_F^Z is taken in Z. Here $\bar{X}_i$ denotes the metric completion of X_i and d_i is the extension of d_i on $\bar{X}_i$, while $\phi_\# T$ denotes the push forward of T.

In [23], it is observed that

$$d_{\mathcal{F}}(M_1, M_2) \le d_{\mathcal{F}}(M_1, 0) + d_{\mathcal{F}}(0, M_2) \le \mathbf{M}(M_1) + \mathbf{M}(M_2). \tag{82}$$

There it is also proven that $d_{\mathcal{F}}$ satisfies the triangle inequality [23][Thm 3.2] and is a distance:

Theorem 2.36. [23]*[Thm 3.27] Let M, N be precompact integral current spaces and suppose that $d_{\mathcal{F}}(M, N) = 0$. Then there is a current preserving isometry from M to N where an isometry $f : X_M \to X_N$ is called a current preserving isometry between M and N, if its extension $\bar{f} : \bar{X}_M \to \bar{X}_N$ pushes forward the current structure on M to the current structure on N: $\bar{f}_\# T_M = T_N$*

In [23] Theorem 3.23 it is also proven that

Theorem 2.37. [23]*[Thm 4.23] Given a pair of precompact integral current spaces, $M_1^m = (X_1, d_1, T_1)$ and $M_2^m = (X_2, d_2, T_2)$, there exists a compact metric space, (Z, d_Z), integral currents $U \in \mathbf{I}_m(Z)$ and $V \in \mathbf{I}_{m+1}(Z)$, and isometric embeddings $\varphi_1 : \bar{X}_1 \to Z$ and $\varphi_2 : \bar{X}_2 \to Z$ with*

$$\varphi_\# T_1 - \varphi'_\# T_2 = U + \partial V \tag{83}$$

such that

$$d_{\mathcal{F}}(M_1, M_2) = \mathbf{M}(U) + \mathbf{M}(V). \tag{84}$$

Remark 2.38. *The metric space Z in Theorem 2.37 has*

$$\mathrm{Diam}(Z) \le 3\,\mathrm{Diam}(X_1) + 3\,\mathrm{Diam}(X_2). \tag{85}$$

This is seen by consulting the proof of Theorem 3.23 in [23]*, where* Z *is constructed as the injective envelope of the Gromov-Hausdorff limit of a sequence of spaces* Z_n *with this same diameter bound.*

The following theorem in [23] is an immediate consequence of Gromov and Ambrosio-Kirchheim's Compactness Theorems:

Theorem 2.39. *Given a sequence of* m *dimensional integral current spaces* $M_j = \left(X_j, d_j, T_j\right)$ *such that* X_j *are equibounded and equicompact and with uniform upper bounds on mass and boundary mass. A subsequence converges in the Gromov-Hausdorff sense* $\left(X_{j_i}, d_{j_i}\right) \xrightarrow{GH} (Y, d_Y)$ *and in the intrinsic flat sense* $\left(X_{j_i}, d_{j_i}, T_{j_i}\right) \xrightarrow{\mathcal{F}} (X, d, T)$ *where either* (X, d, T) *is an* m *dimensional integral current space with* $X \subset Y$ *or it is the* $\mathbf{0}$ *current space.*

Immediately one notes that if Y has Hausdorff dimension less than m, then $(X, d, T) = \mathbf{0}$. In [23] Example A.7, there is an example where M_j are compact three dimensional Riemannian manifolds with positive scalar curvature that converge in the Gromov-Hausdorff sense to a standard three sphere but in the Intrinsic Flat sense to $\mathbf{0}$. It is proven in [22], that if (X_j, d_j, T_j) are compact Riemannian manifolds with nonnegative Ricci curvature or a uniform linear contractibility function, then the intrinsic flat and Gromov-Hausdorff limits agree.

There are many examples of sequences of Riemannian manifolds which have no Gromov-Hausdorff limit but have an intrinsic flat limit. The first is Ilmanen's Example of an increasingly hairy three sphere with positive scalar curvature described in [23] Example A.7. Other examples appear in work of the author with Dan Lee concerning the stability of the Positive Mass Theorem [15] [16] and in work of the author with Sajjad Lakzian concerning smooth convergence away from singular sets [14].

The following three theorems are proven in work of the author with Wenger [23]. Combining these theorems with the work of Ambrosio-Kirchheim reviewed earlier will lead to many of the properties of Intrinsic Flat Convergence described in this paper:

Theorem 2.40. [23]*[Thm 4.2] If a sequence of integral current spaces* $M_j = \left(X_j, d_j, T_j\right)$ *converges in the intrinsic flat sense to an integral current space,* $M_0 = (X_0, d_0, T_0)$*, then there is a separable complete metric space,* Z*, and isometric embeddings* $\varphi_j : X_j \to Z$ *such that* $\varphi_{j\#} T_j$ *flat converges to* $\varphi_{0\#} T_0$ *in* Z *and thus converge weakly as well.*

Theorem 2.41. [23]*[Thm 4.3] If a sequence of integral current spaces* $M_j = \left(X_j, d_j, T_j\right)$ *converges in the intrinsic flat sense to the zero integral current space,* $\mathbf{0}$*, then we may choose points* $x_j \in X_j$ *and a separable complete metric space,* Z*, and isometric embeddings* $\varphi_j : X_j \to Z$ *such that* $\varphi_j(x_j) = z_0 \in Z$ *and* $\varphi_{j\#} T_j$ *flat converges to* 0 *in* Z *and thus converges weakly as well.*

Theorem 2.42. *If a sequence of integral current spaces* M_j *converges in the intrinsic flat sense to a integral current space,* M_∞*, then*

$$\liminf_{i \to \infty} \mathbf{M}(M_i) \geq \mathbf{M}(M_\infty) \tag{86}$$

Proof. This follows from Theorems 2.40 and 2.41 combined with Ambrosio-Kirchheim's lower semicontinuity of mass [c.f. Remark 2.42]. □

Finally there is Wenger's Compactness Theorem [25]:

Theorem 2.43 (Wenger)**.** *Given $A_0, V_0, D_0 > 0$. If $M_j = (X_j, d_j, T_j)$ are integral current spaces such that*

$$\mathrm{Diam}(M_j) \le D_0 \qquad \mathbf{M}(M_j) \le V_0 \qquad \mathbf{M}(\partial(M_j)) \le A_0 \tag{87}$$

then a subsequence converges in the Intrinsic Flat Sense to an integral current space of the same dimension, possibly the **0** *space.*

Recall that this theorem applies to oriented Riemannian manifolds of the same dimension with a uniform upper bound on volume and a uniform upper bound on the volumes of the boundaries. One immediately sees that the conditions required to apply Wenger's Compactness Theorem are far weaker than the conditions required for Gromov's Compactness Theorem. The only difficulty lies in determining whether the limit space is **0** or not. Wenger's proof involves a thick thin decomposition, a study of filling volumes and uses the notion of an ultralimit.

It should be noted that Theorems 2.40-2.42 and all other theorems reviewed and proven within this paper are proven without applying Wenger's Compactness Theorem. Thus one may wish to attempt alternate proofs of Wenger's Compactness Theorem using the results in this paper.

2.7. **Review of Filling Volumes.** The notion of a filling volume was first introduced by Gromov in [10]. Wenger studied the filling volumes of integral currents in metric spaces in [24]

First we discuss the Plateau Problem on complete metric spaces. Given a integral current $T \in \mathbf{I}_m Z$, one may define the filling volume of ∂T within Z as

$$\mathrm{FillVol}_Z(T) = \inf\{\mathbf{M}(S) : \ S \in \mathbf{I}_m(Z)\, s.t.\ \partial S = \partial T\}. \tag{88}$$

This immediately provides an upper bound on the flat distance:

$$d_F^Z(\partial T, \mathbf{0}) \le \mathrm{FillVol}_{\mathbf{Z}}(\partial \mathbf{T}) \le \mathbf{M}(\mathbf{T}) \tag{89}$$

Ambrosio-Kirchheim proved this infimum is achieved on Banach spaces, Z [1] [Theorem10.2].

Wenger defined the absolute filling volume of $T \in \mathbf{I}_m Y$ to be

$$\mathrm{FillVol}_\infty(\partial T) = \inf\{\mathbf{M}(S) : \ S \in \mathbf{I}_m(Z)\, s.t.\ \partial S = \varphi_\# \partial T\} \tag{90}$$

where the infimum is taken over all isometric embeddings $\varphi : Y \to Z$, all complete metric spaces, Z, and all $S \in \mathbf{I}_m(Z)$ such that $\partial S = \varphi_\# T$. Clearly

$$\mathrm{FillVol}_\infty(\partial T) \le \mathrm{FillVol}_Y(\partial T). \tag{91}$$

He proved that this infimum is achieved and via the Kuratowski Embedding Theorem, this is achieved on a Banach space, Z [24].

Here we will use the following notion of a filling of an integral current space:

Definition 2.44. *Given an integral current space $M = (X, d, T) \in \mathcal{M}^m$ with $m \ge 1$ we define*

$$\mathrm{FillVol}(\partial M) := \inf\{\mathbf{M}(N) : \ N \in \mathcal{M}^{m+1} \text{ and } \partial N = \partial M\}. \tag{92}$$

That is we require that there exists a current preserving isometry from ∂N onto ∂M, where as usual, we have taken the metrics on the boundary spaces to be the restrictions of the metrics on the metric completions of N and M respectively.

It is easy to see that

$$\mathrm{FillVol}(\partial M) \le \mathbf{M}(M). \tag{93}$$

and

$$d_{\mathcal{F}}(\partial M, 0) \le \mathrm{FillVol}(\partial M) \le \mathbf{M}(M) \tag{94}$$

for any integral current space M.

Remark 2.45. *The infimum in the definition of the filling volume is achieved. This may be seen by imitating the proof that the infimum in the definition of the intrinsic flat norm is attained in* [23]. *Since the N achieving the infimum has $\partial N \neq 0$, the filling volume is positive.*

Any integral current space, $M = (X, d, T)$, is separable and so one can map the space into a Banach space, Z, via the Kuratowski Embedding theorem, $\iota : X \to Z$. By Ambrosio-Kirchheim's solution to the Plateau problem on Banach spaces [1][Prop 10.2],

$$\mathrm{FillVol}(\partial M) \le \mathrm{FillVol}_Z(\varphi_\#(\partial T) \le \mathrm{Diam}(X)\mathbf{M}(\partial T)\,\mathrm{Diam}(M)\mathbf{M}(\partial M). \tag{95}$$

Wenger proved that the filling volume is continuous with respect to weak convergence (and thus also intinsic flat convergence when applying Theorem 2.40). Here we provide a precise estimate which will be needed later in the paper:

Theorem 2.46. *For any pair of integral current spaces, M_i, we have*

$$\mathrm{FillVol}(\partial M_1) \le \mathrm{FillVol}(\partial M_2) + d_{\mathcal{F}}(M_1, M_2). \tag{96}$$

and

$$\mathrm{FillVol}(\partial M_1) \le \mathrm{FillVol}(\partial M_2) + (1 + 3\,\mathrm{Diam}(M_1) + 3\,\mathrm{Diam}(M_2))d_{\mathcal{F}}(\partial M_1, \partial M_2). \tag{97}$$

Proof. This can be seen because Theorem 2.37 and Remark 2.38 imply that there exists integral current spaces A, B such that and isometric embeddings, $\varphi_i : M_i \to Z$, such that

$$\varphi_{1\#}M_1 - \varphi_{2\#}M_2 = \partial B + A \tag{98}$$

where

$$d_{\mathcal{F}}(M_1, M_2) = \mathbf{M}(A) + \mathbf{M}(B) \tag{99}$$

In particular

$$\varphi_{1\#}\partial M_1 - \varphi_{2\#}\partial M_2 = \partial A \tag{100}$$

Next we need to fill in ∂M_2 and glue that integral current space to A. Add details.

Next observe that there exists integral current spaces M_3, M_4 and isometric embeddings, $\varphi_i : M_i \to Z'$, such that

$$\varphi_{1\#}\partial M_1 - \varphi_{2\#}\partial M_2 = \varphi_{3\#}\partial M_3 + \varphi_{4\#}M_4 \tag{101}$$

where

$$d_{\mathcal{F}}(\partial M_1, \partial M_2) = \mathbf{M}(M_3) + \mathbf{M}(M_4). \tag{102}$$

By Remark 2.38,

$$\mathrm{Diam}(M_3), \mathrm{Diam}(M_4) \le 3\,\mathrm{Diam}(M_1) + 3\,\mathrm{Diam}(M_2). \tag{103}$$

In [1] Theorem 10.2, it is proven that

$$\mathrm{FillVol}(M_4) \le \mathrm{Diam}(M_4)\mathbf{M}(M_4). \tag{104}$$

So if we glue the integral current spaces, N_i, which achieve the filling volume of M_i for $i = 2, 4$, to M_3 as needed, then we see that

$$\begin{aligned}
\mathrm{FillVol}(\partial M_1) &\leq \mathrm{FillVol}(\partial M_2) + \mathbf{M}(M_3) + \mathrm{FillVol}(M_4) \quad (105)\\
&\leq \mathrm{FillVol}(\partial M_2) + \mathbf{M}(M_3) + \mathrm{Diam}(M_4)\mathbf{M}(M_4) \quad (106)\\
&\leq \mathrm{FillVol}(\partial M_2) + (\mathrm{Diam}(M_4) + 1)(\mathbf{M}(M_3) + \mathbf{M}(M_4)) \quad (107)
\end{aligned}$$

and we have our second claim. □

Remark 2.47. *Gromov's Filling Volume in* [10] *is defined as in (92) where the infimum is taken over N^{n+1} that are Riemannian manifolds. Thus it is conceivable that the filling volume in Definition 2.44 might have a smaller value both because integral current spaces have integer weight and because we have a wider class of metrics to choose from, including metrics which are not length metrics.*

New Remark:

Remark 2.48. *Note also that the mass used in Definition 2.44 is Ambrosio-Kirchheim's mass* [1] *Definition 2.6 stated as Definition 2.11 here. Even when the weight is* 1 *and one has a Finsler manifold, the Ambrosio-Kirchheim mass has a different value than any of Gromov's masses* [10] *and the masses used by Burago-Ivanov* [2]. *We need Ambrosio-Kirchheim's mass to have continuity of the filling volumes under intrinsic flat convergence [Theorem 2.46] which is an essential tool in this paper.*

3. Metric Properties of Integral Current Spaces

In this section we prove a number of properties of integral current spaces as well as a new Gromov-Hausdorff Compactness Theorem. We describe the natural notions of balls, isometric products, slices, spheres and filling volumes in a rigorous way. We introduce the Sliced Filling Volume [Definition 3.20] and $\mathbf{SF}_k(p,r)$ [Definition 3.21]. Then we prove a new Gromov-Hausdorff Compactness Theorem [Theorem 3.23]. We explore the filling volumes of 0 dimensional spaces, apply them to bound the volumes of balls, and then introduce the Tetrahedral Property [Definition 3.30] and the Integral Tetrahedral Property [Definition 3.36]. We close this section with the notion of interval filling volumes in Definition 3.43 and Sliced Interval Filling Volumes in Definition 3.45. Those studying the proof of the Tetrahedral Compactness Theorem need to read all Sections except 3.2 and 3.12 before continuing to Section 4. Those studying the Bolzano-Weierstrass and Arzela-Ascoli Theorems need only read Sections 3.1 and 3.3-3.6 before continuing to Section 4.

3.1. Balls.

Many theorems in Riemannian geometry involve balls,

$$B(p,r) = \{x \in X : d_X(x,p) < r\} \qquad \bar{B}(p,r) = \{x \in X : d_X(x,p) \le r\}. \tag{108}$$

Lemma 3.1. *A ball in an integral current space, $M = (X, d, T)$, with the current restricted from the current structure of the Riemannian manifold is an integral current space itself,*

$$S(p,r) = (\mathrm{set}(T \llcorner B(p,r)), d, T \llcorner B(p,r)) \tag{109}$$

for almost every $r > 0$. Furthermore,

$$B(p,r) \subset \mathrm{set}(S(p,r)) \subset \bar{B}(p,r) \subset X. \tag{110}$$

Proof. We first show that $S(p,r) = T \llcorner B(p,r)$ is an integer rectifiable current. Let $\rho_p : \bar{X} \to \mathbb{R}$ be the distance function from p. Then by Ambrosio-Kirchheim's Slicing Theorem,

$$\begin{aligned}
\partial(T \llcorner B(p,r)) &= \partial(T \llcorner \rho_p^{-1}(-\infty, r)) && (111)\\
&= < T, \rho_p, r > + (\partial T) \llcorner \rho_p^{-1}(-\infty, r) && (112)\\
&= < T, \rho_p, r > + (\partial T) \llcorner B(p,r) && (113)
\end{aligned}$$

where the mass of the slice $< T, \rho_p, r >$ is bounded for almost every r. Thus

$$\begin{aligned}
\mathbf{M}(\partial(T \llcorner B(p,r))) &\le \mathbf{M}(< T, \rho_p, r >) + \mathbf{M}((\partial T) \llcorner B(p,r)) && (114)\\
&\le \mathbf{M}(< T, \rho_p, r >) + \mathbf{M}((\partial T) < \infty. && (115)
\end{aligned}$$

So $S(p,r)$ is an integral current in $\bar{X}$ for almost every r.

Next we prove (110). We know $x \in \mathrm{set}(S(p,r)) \subset \bar{X}$ iff

$$\begin{aligned}
0 &< \liminf_{s \to 0} \frac{\|S(p,r)\|(B(x,s))}{\omega_m s^m} && (116)\\
&= \liminf_{s \to 0} \frac{\|T\|(B(p,r) \cap B(x,s))}{\omega_m s^m} && (117)
\end{aligned}$$

If $x \in B(p,r) \subset X$, then eventually $B(x,s) \subset B(p,r)$ and the liminf is just the lower density of T at x. Since $x \in X = \mathrm{set}(T)$, this lower density is positive. If $x \in \bar{X} \setminus X$, then the liminf is 0 because it is smaller than the density of T at x, which is 0. If $x \notin \bar{B}(p,r)$, then the liminf is 0 because eventually the balls do not intersect. □

One may imagine that it is possible that a ball is cusp shaped when we are not in a length space and that some points in the closure of the ball that lie in X do not lie in the set of $S(p,r)$. In a manifold, the set of $S(p,r)$ is a closed ball:

Lemma 3.2. *When M is a Riemannian manifold with boundary*

$$S\,(p,r) = \left(\bar{B}\,(p,r)\,, d, T \llcorner B\,(p,r)\right) \tag{118}$$

is an integral current space for all $r > 0$.

Proof. In this case,

$$\begin{aligned}
\partial(T \llcorner B(p,r))(f, \pi_1, ..., \pi_m) &= (T \llcorner B(p,r))(1, f, \pi_1, ..., \pi_m) && (119)\\
&= T(\chi_{B(p,r)}, f, \pi_1, ..., \pi_m) && (120)\\
&= \int_M \chi_{B(p,r)} df \wedge d\pi_1 \wedge \cdots \wedge d\pi_m && (121)\\
&= \int_{B(p,r)} df \wedge d\pi_1 \wedge \cdots \wedge d\pi_m && (122)\\
&= \int_{B(p,r)} df \wedge d\pi_1 \wedge \cdots \wedge d\pi_m && (123)\\
&= \int_{\partial B(p,r)} f\, d\pi_1 \wedge \cdots \wedge d\pi_m && (124)
\end{aligned}$$

So $\mathbf{M}(\partial(T \llcorner B(p,r))) = \mathrm{Vol}_{m-1}(\partial B_p(r)) < \infty$.

To see that $\bar{B}(p,r) \subset M$ is set(S(p, r)), we refer to (116). If $d(x,p) = r$, then let $\gamma : [0,r] \to M$ be a curve parametrized by arclength running minimally from x to p. Then

$$B(\gamma(s/2), s/2) \subset B(x,s) \cap B(p,r). \tag{125}$$

and

$$\begin{aligned}
\liminf_{s\to 0} \frac{\|S\,(p,r)\|(B(x,s))}{\omega_m s^m} &= \liminf_{s\to 0} \frac{\|T\|(B(p,r)\cap B(x,s))}{\omega_m s^m} && (126)\\
&\geq \liminf_{s\to 0} \frac{\|T\|(B(\gamma(s/2), s/2)}{\omega_m s^m} && (127)\\
&\geq \liminf_{s\to 0} \frac{\mathrm{Vol}(B(\gamma(s/2), s/2)}{2^m \omega_m (s/2)^m} \geq \frac{1/2}{2^m} && (128)
\end{aligned}$$

because in a manifold with boundary, we eventually lie within a half plane chart where all tiny balls are either uniformly close to a Euclidean ball or half a Euclidean ball. □

Example 3.3. *There exist integral current spaces with balls that are not integral current spaces.*

Proof. Suppose we define an integral current space, (X, d, T) where $X = S^2$ with the following generalized metric

$$g = dr^2 + (\cos(r)/r^2)^2 d\theta^2 \qquad r \in [-\pi/2, \pi/2]. \tag{129}$$

The metric is defined as

$$d(p_1, p_2) = \inf\{L_g(\gamma) :\ \gamma(0) = p_1,\ \gamma(1) = p_2\} \tag{130}$$

where

$$L_g(\gamma) = \int_0^1 g(\gamma'(t), \gamma'(t))^{1/2}\, dt \tag{131}$$

as in a Riemannian manifold. In fact this metric space consists of two open isometric Riemannian manifolds diffeomorphic to disks whose metric completions are glued together along corresponding points. The current structure T is defined by

$$T(f, \pi_1, ..., \pi_m) = \int_{-\pi/2}^{\pi/2} \int_{S^1} f\, d\pi_1 \wedge \cdots \wedge d\pi_m \tag{132}$$

$$= \int_{-\pi/2}^{0} \int_{S^1} f\, d\pi_1 \wedge \cdots \wedge d\pi_m + \int_{0}^{\pi/2} \int_{S^1} f\, d\pi_1 \wedge \cdots \wedge d\pi_m \tag{133}$$

so that

$$\mathbf{M}(T) = \mathrm{Vol}_m(r^{-1}[-\pi/2, 0)) + \mathrm{Vol}_m(r^{-1}(0, \pi/2]) < \infty \tag{134}$$

and $\partial T = 0$

Setting p such that $r(p) = -\pi/2$ we see that $S(p, \pi/2)$ is a rectifiable current but its boundary does not have finite mass. This can be see by taking q such that $(r(q), \theta(q)) = (0, 0)$, setting $\pi_1 = \rho_q$ and $f = \rho_p = r + \pi/2$ and observing that

$$|\partial(S(p, \pi/2))(f, \pi_1)| = |S(p, \pi/2)(1, f, \pi_1)| \tag{135}$$

$$= \left| \int_{B(p,\pi/2)} df \wedge d\pi_1 \right| \tag{136}$$

$$\geq \left| \int_{B(p,\pi/2-\delta)} df \wedge d\pi_1 \right| \tag{137}$$

$$= \left| \int_{\partial B(p,\pi/2-\delta)} f\, d\pi_1 \right| \tag{138}$$

$$= \left| \int_{\theta=-\pi}^{\pi} (\pi/2 - \delta) \frac{d\pi_1}{d\theta}\, d\theta \right| \tag{139}$$

$$= \left| \int_{\theta=-\pi}^{\pi} (\pi/2 - \delta) \frac{\cos(r)}{r^2}\, d\theta \right| \tag{140}$$

$$\geq 2\pi(\pi/2 - \delta) \frac{\cos(-\delta)}{\delta^2} \tag{141}$$

which is unbounded as we decrease δ to 0. □

Remark 3.4. *Note that the outside of the ball, $(M \setminus B(p, r), d, T - S(p, r))$, is also an integral current space for almost every $r > 0$.*

3.2. **Isometric Products.** One of the most useful notions in Riemannian geometry is that of an isometric product $M \times I$ of a Riemannian manifold M with an interval, I. endowed with the metric

$$d_{M\times I}((p_1, t_1), (p_2, t_2)) = \sqrt{d_M(p_1, p_2)^2 + |t_1 - t_2|^2}. \tag{142}$$

We need to define the isometric product of an integral current space with an interval:

Definition 3.5. *The product of an integral current space, $M^m = (X, d_X, T)$, with an interval I_ϵ, denoted*

$$M \times I = (X \times I_\epsilon, d_{X\times I_\epsilon}, T \times I_\epsilon) \tag{143}$$

where $d_{X\times I_\epsilon}$ is defined as in (142) and

$$(T \times I_\epsilon)(f, \pi_1, ..., \pi_{m+1}) = \sum_{i=1}^{m+1} (-1)^{i+1} \int_0^\epsilon T\left(f_t \frac{\partial \pi_i}{\partial t}, , \pi_{1t}, ..., \hat{\pi}_{it}, ..., \pi_{(m+1)t} \right) dt \tag{144}$$

where $h_t : \bar{X} \to \mathbb{R}$ is defined $h_t(x) = h(x,t)$ for any $h : \bar{X} \times I_\epsilon \to \mathbb{R}$ and where

$$(\pi_{1t}, ..., \hat{\pi}_{it}, ..., \pi_{(m+1)t}) = (\pi_{1t}, \pi_{2t}, ..., \pi_{(i-1)t}, \pi_{(i+1)t}, ..., \pi_{(m+1)t}). \tag{145}$$

We prove this defines an integral current space in Proposition 3.7 below.

Remark 3.6. *This is closely related to the cone construction in Defn 10.1 of* [1]*, however our ambient metric space changes after taking the product and we do not contract to a point. Ambrosio-Kirchheim observe that (144) is well defined because for $\mathcal{L}^1$ almost every $t \in I_\epsilon$ the partial derivatives are defined for $\|T\|$ almost every $x \in X$. This is also true in our setting. The proof that their cone construction defines a current* [1] *Theorem 10.2, however, does not extend to our setting because our construction does not close up at a point as theirs does and our construction depends on ϵ but not on the size of a bounding ball.*

Proposition 3.7. *Given an integral current space $M = (X, d, T)$, the isometric product $M \times I_\epsilon$ is an integral current space such that*

$$\mathbf{M}(M \times I_\epsilon) = \epsilon \mathbf{M}(M) \tag{146}$$

and such that

$$\partial(T \times I_\epsilon) = (\partial T) \times I_\epsilon + T \times \partial I_\epsilon. \tag{147}$$

where

$$T \times \partial I := \psi_{\epsilon\#} T - \psi_{0\#} T \tag{148}$$

where $\psi_t : \bar{X} \to \bar{X} \times I_\epsilon$ is the isometric embedding $\psi_t(x) = (x, t)$.

Proof. First we must show $T \times I_\epsilon$ satisfies the three conditions of a current:

Multilinearity follows from the multilinearity of T and the use of the alternating sum in the definition of $T \times I$.

To see locality we suppose there is a π_i which is constant on a neighborhood of $\{f \neq 0\}$. Then $\partial\pi_i/\partial t = 0$ on a neighborhood of $\{f \neq 0\}$ so the i^{th} term in the sum is 0. Since for all $t \in I_\epsilon$, π_{it} is constant on a neighborhood of $\{f_t \neq 0\}$ the rest of the terms are 0 as well by the locality of T.

To prove continuity and finite mass, we will use the fact that T is integer rectifiable. In particular there exists a parametrization as $\varphi_i : A_i \subset \mathbb{R}^m \to \bar{X}$ and weight functions $\theta_i \in L^1(A_i, \mathbb{N})$ such that

$$T = \sum_{k=1}^{\infty} \varphi_{k\#}[\theta_k]. \tag{149}$$

So $(T \times I_\epsilon)(f, \pi_1, ..., \pi_{m+1}) =$

$$\begin{aligned}
&= \sum_{i=1}^{m+1} (-1)^{i+1} \int_0^\epsilon \sum_{k=1}^{\infty} \varphi_{k\#}[\theta_k] \left(f_t \frac{\partial \pi_{it}}{\partial t}, , \pi_{1t}, ..., \hat{\pi}_{it}, ..., \pi_{(m+1)t} \right) dt \\
&= \sum_{i=1}^{m+1} (-1)^{i+1} \sum_{k=1}^{\infty} \int_{t=0}^{\epsilon} \int_{A_k} \theta_k \, f_t \circ \varphi_k \, \frac{\partial \pi_{it}}{\partial t} \circ \varphi_k \, d(\pi_{1t} \circ \varphi_k) \wedge \cdots \wedge d\hat{\pi}_{it} \wedge \cdots \wedge d(\pi_{(m+1)t} \circ \varphi_k) \, dt \\
&= \sum_{k=1}^{\infty} \int_{A_k} \int_{t=0}^{\epsilon} \theta_k(x) f(\varphi_k(x), t) \left(\sum_{i=1}^{m+1} (-1)^{i+1} \frac{\partial \pi_{it}}{\partial t} \circ \varphi_k d(\pi_{1t} \circ \varphi_k) \wedge \cdots \wedge d\hat{\pi}_{it} \wedge \cdots \wedge d(\pi_{(m+1)t} \circ \varphi_k) \right) dt \\
&= \sum_{k=1}^{\infty} \int_{A_k \times I_\epsilon} \theta_k(x) f(\varphi_k(x), t) \, d(\pi_1 \circ \varphi) \wedge \cdots \wedge d(\pi_{m+1} \circ \varphi).
\end{aligned}$$

Thus

$$T \times I = \sum_{k=1}^{\infty} \varphi'_{k\#}[\theta'_k]. \tag{150}$$

where

$$\varphi'_k : A_k \times I_\epsilon \to \bar{X} \times I_\epsilon \text{ satisfies } \varphi'_k(x,t) = (\varphi_k(x), t) \tag{151}$$

and $\theta'_k \in L^1(A_k \times I_\epsilon, \mathbb{N})$ satisfies $\theta'_k(x,t) = \theta_k(x)$. Observe that the images of these charts are disjoint and that

$$\begin{aligned}
\mathbf{M}(T \times I_\epsilon) &= \sum_{k=1}^{\infty} \mathbf{M}(\varphi'_{k\#}[\theta'_k]) && (152)\\
&= \sum_{k=1}^{\infty} \int_{A_k \times I_\epsilon} |\theta'_k| \mathcal{L}^{m+1} && (153)\\
&= \sum_{k=1}^{\infty} \epsilon \int_{A_k} |\theta_k| \mathcal{L}^m && (154)\\
&= \sum_{k=1}^{\infty} \epsilon \mathbf{M}(\varphi_{k\#}[\theta_k]) = \epsilon \mathbf{M}(T). && (155)
\end{aligned}$$

The continuity of $T \times I_\epsilon$ now follows because all integer rectifiable currents defined by parametrizations are currents.

Observe also that if $A \subset \bar{X}$ and $(a_1, a_2) \subset I$ then

$$\begin{aligned}
\|T \times I_\epsilon\|(A \times (a_1, a_2)) &= \mathbf{M}((T \times I_\epsilon) \llcorner (A \times (a_1, a_2))) && (156)\\
&= \sum_{k=1}^{\infty} \mathbf{M}((\varphi'_{k\#}[\theta'_k]) \llcorner (A \times (a_1, a_2))) && (157)\\
&= \sum_{k=1}^{\infty} \int_{(A \cap A_k) \times (a_1, a_2)} |\theta'_k| \mathcal{L}^{m+1} && (158)\\
&= \sum_{k=1}^{\infty} (a_2 - a_1) \int_{A \cap A_k} |\theta_k| \mathcal{L}^m && (159)\\
&= \sum_{k=1}^{\infty} (a_2 - a_1) \mathbf{M}(\varphi_{k\#}[\theta_k] \llcorner A) && (160)\\
&= (a_2 - a_1) \mathbf{M}(T \llcorner A) = (a_2 - a_1) \|T\|(A). && (161)
\end{aligned}$$

Thus $\|T \times I_\epsilon\| = \|T\| \times \mathcal{L}^1$

To prove that $T \times I_\epsilon$ is an integral current, we need only verify that the current $\partial(T \times I_\epsilon)$ has finite mass. Applying the Chain Rule [1]Thm 3.5 and Lemma 3.8 (proven below), we have

$$\partial(T \times I_\epsilon)(f, \tau_1, ..., \tau_m) - ((\partial T) \times I_\epsilon)(f, \tau_1, ..., \tau_m) =$$

$$\begin{aligned}
&= (T\times I_\epsilon)(1,f,\tau_1,...,\tau_m) - \sum_{i=1}^m (-1)^i \int_0^\epsilon \partial T\left(f_t \frac{\partial \tau_i}{\partial t}, \tau_{1t},...,\hat{\tau}_{it},...,\tau_{mt}\right) dt\\
&= \int_0^\epsilon T\left(\frac{\partial f}{\partial t}, \tau_{1t},...,\tau_{mt}\right) dt + \sum_{i=1}^m (-1)^{i+1} \int_0^\epsilon T\left(\frac{\partial \tau_i}{\partial t}, f_t, \tau_{1t},...,\hat{\tau}_{it},...,\tau_{mt}\right) dt\\
&\qquad + \sum_{i=1}^m (-1)^{i+1} \int_0^\epsilon T\left(1, f_t\frac{\partial \tau_i}{\partial t}, \tau_{1t},...,\hat{\tau}_{it},...,\tau_{mt}\right) dt\\
&= \int_0^\epsilon T\left(\frac{\partial f}{\partial t}, \tau_{1t},...,\tau_{mt}\right) dt + \sum_{i=1}^m (-1)^{i} \int_0^\epsilon T\left(f_t, \frac{\partial \tau_i}{\partial t}, \tau_{1t},...,\hat{\tau}_{it},...,\tau_{mt}\right) dt\\
&= \int_0^\epsilon T\left(\frac{\partial f}{\partial t}, \tau_{1t},...,\tau_{mt}\right) dt + \sum_{i=1}^m \int_0^\epsilon T\left(f_t, \tau_{1t},...,\tau_{(i-1)t}, \frac{\partial \tau_i}{\partial t}, \tau_{(i+1)t},...,\tau_{mt}\right) dt\\
&= \int_0^\epsilon \frac{\partial}{\partial t} T\left(f_t, \tau_{1t},...,\tau_{mt}\right) dt\\
&= T(f_\epsilon, \tau_{1\epsilon},...,\tau_{m\epsilon}) - T(f_0,\tau_{10},...,\tau_{m0})\\
&= \psi_{\epsilon\#}T(f,\tau_1,...,\tau_m) - \psi_{0\#}T(f,\tau_1,...,\tau_m)\\
&= T\times \partial I(f,\tau_1,...,\tau_m).
\end{aligned}$$

Thus we have (147).

Observe that $T\times\partial I$ is an integral current because it is the sum of push forwards of integral currents and that

$$\mathbf{M}(T\times\partial I_\epsilon) = 2\mathbf{M}(T). \tag{162}$$

Since we know products are rectifiable, $(\partial T)\times I_\epsilon$ is rectifiable and has finite mass $\le \epsilon\mathbf{M}(\partial T)$. Thus applying (147) we see that

$$\mathbf{M}(\partial(T\times I_\epsilon)) \le \mathbf{M}((\partial T)\times I_\epsilon) + \mathbf{M}(T\times\partial I_\epsilon) \le \epsilon\mathbf{M}(\partial T) + 2\mathbf{M}(T). \tag{163}$$

Thus the current structure of $M\times I_\epsilon$ is an integral current.

Lastly we verify that

$$\mathrm{set(T\times I_\epsilon) = set(T)\times I_\epsilon}. \tag{164}$$

Given $(p,t)\in\bar{X}\times I_\epsilon$, then the following statements are equivalent:

$$(p,t) \in \mathrm{set(T\times I_\epsilon)}. \tag{165}$$

$$0 < \liminf_{r\to0} \frac{||T\times I_\epsilon||(B_{(p,t)}(r))}{r^{m+1}}. \tag{166}$$

$$0 < \liminf_{r\to0} \frac{||T\times I_\epsilon||(B_p(r)\times(t-r,t+r))}{r^{m+1}}. \tag{167}$$

$$0 < \liminf_{r\to0} \frac{2r||T||(B_p(r))}{r^{m+1}}. \tag{168}$$

$$0 < \liminf_{r\to0} \frac{||T||(B_p(r))}{r^{m}}. \tag{169}$$

$$p \in \mathrm{set(T)}. \tag{170}$$

The proposition follows. □

Lemma 3.8. *If π_{it} are Lipschitz in $Z \times I$, and $T \in \mathbf{I}_m(Z)$ then for almost every $t \in I$,*

$$\frac{\partial}{\partial t} T(\pi_{0t}, ..., \pi_{mt}) = \sum_{i=0}^{m} T\left(\pi_{0t}, ..., \pi_{(i-1)t}, \frac{\partial \pi_i}{\partial t}, \pi_{(i+1)t}, ..., \pi_{mt}\right) \tag{171}$$

Proof. This follows from the multilinearity of T, the usual expansion of the difference quotient as a sum of difference quotients in which one term changes at a time, the fact the T is continuous with respect to pointwise convergence and that the difference quotients have pointwise limits for almost every $t \in I$ because the π_i are Lipschitz in t. □

The following proposition will be applied later when studying limits under intrinsic flat and Gromov-Hausdorff convergence.

Proposition 3.9. *Suppose M_i^m are integral current spaces and $\epsilon > 0$, then*

$$d_{GH}(M_1^m \times I_\epsilon, M_2^m \times I_\epsilon) \le d_{GH}(M_1^m, M_2^m), \tag{172}$$

and

$$d_{\mathcal{F}}(M_1^m \times I_\epsilon, M_2^m \times I_\epsilon) \le (2+\epsilon) d_{\mathcal{F}}(M_1^m, M_2^m). \tag{173}$$

Proof. Since the intrinsic flat distance is achieved, there exists a metric space Z and isometric embeddings $\varphi_i : M_i \to Z$, and integral currents A, B on Z such that

$$\varphi_{1\#} M_1 - \varphi_{2\#} M_2 = A + \partial B \tag{174}$$

and

$$d_{\mathcal{F}}(M_1^m, M_2^m) = (\mathbf{M}(A) + \mathbf{M}(B)). \tag{175}$$

Setting $Z' = Z \times I_\epsilon$ endowed with the product metric, we have isometric embeddings $\varphi"_i : M_i \times I_\epsilon \to Z'$ and we have integral currents $A' = A \times I_\epsilon$ and $B' = B \times I_\epsilon$ such that

$$\begin{aligned}
\varphi'_{1\#}(M_1 \times I_\epsilon) - \varphi'_{2\#}(M_2 \times I_\epsilon) &= (\varphi'_{1\#} M_1) \times I_\epsilon - (\varphi_{2\#} M_2) \times I_\epsilon && (176)\\
&= (\varphi'_{1\#} M_1) - \varphi_{2\#} M_2) \times I_\epsilon && (177)\\
&= (A + \partial B) \times I_\epsilon && (178)\\
&= A \times I_\epsilon + \partial(B \times I_\epsilon) - B \times (\partial I_\epsilon). && (179)
\end{aligned}$$

Thus by Proposition 3.7 and (162) we have

$$\begin{aligned}
d_{\mathcal{F}}(M_1^m \times I_\epsilon, M_2^m \times I_\epsilon) &\le \mathbf{M}(A \times I_\epsilon) + \mathbf{M}(B \times I_\epsilon) + \mathbf{M}(B \times (\partial I_\epsilon)) && (180)\\
&\le \epsilon \mathbf{M}(A) + \epsilon \mathbf{M}(B) + 2\mathbf{M}(B) && (181)\\
&\le (2+\epsilon)\mathbf{M}(A) + (2+\epsilon)\mathbf{M}(B) && (182)\\
&= 3 d_{\mathcal{F}}(M_1^m, M_2^m). && (183)
\end{aligned}$$

To see the Gromov-Hausdorff estimate, one needs only observe that whenever $Y_1 \subset T_r(Y_2) \subset Z$, then

$$Y_1 \times I_\epsilon \subset T_r(Y_2 \times I_\epsilon) \subset Z \times I_\epsilon. \tag{184}$$

□

3.3. **Slices and Spheres.** While balls are a very natural object in metric spaces, a more important notion in integral current spaces is that of a slice. The following proposition follows immediately from the Ambrosio-Kirchheim Slicing Theorem (c.f. Theorem 2.23 and Remark 2.25):

Proposition 3.10. *Given an m dimensional integral current space $M = (X, d, T)$ and Lipschitz functions $F : X \to \mathbb{R}^k$ where $k < m$, then for almost every $t \in \mathbb{R}^k$, we can define an $m-k$ dimensional integral current space called the slice of (X, d, T):*

$$\mathrm{Slice}(M, F, t) = \mathrm{Slice}(F, t) = (\mathrm{set} < \mathrm{T}, \mathrm{F}, \mathrm{t} >, \mathrm{d}, < \mathrm{T}, \mathrm{F}, \mathrm{t} >) \tag{185}$$

where $< T, F, t > = < T, F_1, ..., F_k, t_1, ..., t_k >$ is an integral current on $\bar{X}$ defined using the Ambrosio-Kirchheim Slicing Theorem and $\mathrm{set} < \mathrm{T}, \mathrm{F}, \mathrm{t} > \subset \mathrm{F}^{-1}(\mathrm{t})$. *We can integrate the masses of slices to obtain lower bounds of the mass of the original space:*

$$\int_{t\in\mathbb{R}^k} \mathbf{M}(\mathrm{Slice}(M, F, t))\, \mathcal{L}^k \le \prod_{j=1}^{k} \mathrm{Lip}(F_j)\, \mathbf{M}(T). \tag{186}$$

and $\partial\, \mathrm{Slice}(M, F, t) = (-1)^k\, \mathrm{Slice}(\partial M, F, t)$.

Proof. This proposition follows immediately from the Ambrosio-Kirchheim Slicing Theorem 5.6 using the fact that F has a unique extension to $\bar{X}$ and Defn 2.5. The last part follows from Lemma 5.8. □

Example 3.11. *In Example 3.3, where $\partial T = \emptyset$ we see that taking $F : X \to \mathbb{R}$ to be the distance function from p the slice $\mathrm{Slice}(M, f, s) = \partial S(p, s)$ is an integral current space only when $s \neq \pi/2$.*

Lemma 3.12. *Given an m dimensional integral current space (X, d, T) and a point p then for almost every $r \in \mathbb{R}$, we can define an $m-1$ dimensional integral current space called the sphere about p of radius r:*

$$\mathrm{Sphere}(p, r) = \mathrm{Slice}(\rho_p, r) \tag{187}$$

On a Riemannian manifold with boundary,

$$\mathrm{Sphere}(p, r)(f, \pi_1, ...\pi_{m-1}) = \int_{\rho_p^{-1}(r)} f\, d\pi_1 \wedge \cdots \wedge d\pi_{m-1} \tag{188}$$

is an integral current space for all $r \in \mathbb{R}$.

Proof. This follows from Proposition 3.10 and the Ambrosio Kirchheim Slicing Theorem (c.f. Theorem 2.23 and the fact that $\mathrm{Lip}(\rho_p) = 1$. The Riemannian part follows from Stoke's Theorem and the fact that spheres of all radii in Riemannian manifolds have finite volume as can be seen either by applying the Ricatti equation or Jacobi fields. □

Observe the distinction between the sphere and the boundary of a ball in Lemma 3.12 when M has boundary. Next we examine the setting when we do not hit the boundary:

Lemma 3.13. *When* $\mathrm{set}(\partial\mathrm{T}) \cap \bar{\mathrm{B}}(\mathrm{p}, \mathrm{R}) \subset \bar{\mathrm{X}}$ *is empty then*

$$\mathrm{Sphere}(p, R) = \partial S(p, R). \tag{189}$$

Furthermore,

$$\int_0^R \mathbf{M}\left(\partial S\ (p, r)\right)\, d\mathcal{L}(r) \le \mathbf{M}\left(S\ (p, R)\right). \tag{190}$$

In particular, on an open Riemannian manifold, for any $p \in M$, there is a sufficiently small $R > 0$ such that this lemma holds. On a Riemannian manifold without boundary, these hold for all $R > 0$.

Proof. This follows from Proposition 3.10 and Theorem 2.23. □

Lemma 3.14. *Given an m dimensional integral current space (X, d, T) and a $\rho : X \to \mathbb{R}$ a Lipschitz function with $\mathrm{Lip}(\rho) \leq 1$ then for almost every $r \in \mathbb{R}$, we can define an $m-1$ dimensional integral current space, $\mathrm{Slice}(\rho, r)$ where*

$$\int_{-\infty}^{\infty} \mathbf{M}\left(\mathrm{Slice}(\rho, r)\right) \, d\mathcal{L}(r) \leq \mathbf{M}(T). \tag{191}$$

On a Riemannian manifold with boundary

$$\mathrm{Slice}(\rho, r)(f, \pi_1, ...\pi_{m-1}) = \int_{\rho^{-1}(r)} f \, d\pi_1 \wedge \cdots \wedge d\pi_{m-1} \tag{192}$$

is defined for all $r \in \mathbb{R}$.

Proof. This follows from Proposition 3.10 and Theorem 2.23. □

Lemma 3.15. *Given an m dimensional integral current space (X, d, T) and a $\rho : X \to \mathbb{R}^k$ have $\mathrm{Lip}(\rho_i) \leq 1$ then for almost every $r \in \mathbb{R}^k$, we can define an $m-k$ dimensional integral current space, $\mathrm{Slice}(\rho, r)$ where*

$$\int_{\mathbb{R}^k} \mathbf{M}\left(\mathrm{Slice}(\rho, r)\right) \, d\mathcal{L}(r) \leq \mathbf{M}(T). \tag{193}$$

Proof. This follows from Proposition 3.10 and Theorem 2.23. □

Remark 3.16. *On a Riemannian manifold with boundary*

$$\mathrm{Slice}(\rho, r)(f, \pi_1, ...\pi_{m-1}) = \int_{\rho^{-1}(r)} f \, d\pi_1 \wedge \cdots \wedge d\pi_{m-1} \tag{194}$$

is defined for all $r \in \mathbb{R}$ such that $\rho_p^{-1}(r)$ is $m-1$ dimensional. By the above lemma this will be true for almost every r. Note, however, that if ρ_i are distance functions from poorly chosen points, the slice may be the 0 space for almost every r because $\rho_p^{-1}(r) = \emptyset$. This occurs for example on the standard three dimensional sphere if we take ρ_1, ρ_2 to be distance functions from opposite poles.

3.4. **Filling Volumes of Spheres and Slices.**

The following lemmas were applied without proof in [22]. We may now easily prove them. First recall Definition 2.44 for the notion of filling volume used in this paper.

Lemma 3.17. *Given an integral current space, $M^m = (X, d, T)$,*

$$\mathbf{M}(S(p, r)) \geq \mathrm{FillVol}(\partial S(p, r)) \qquad \forall p \in \bar{X}. \tag{195}$$

Thus $p \in \bar{X}$ lies in $X = \mathrm{set(T)}$ if

$$\liminf_{r \to 0} \mathrm{FillVol}(\partial S(p, r))/r^m > 0. \tag{196}$$

Proof. This follows immediately from the definition of filling volume and the definition of set(T). □

Lemma 3.18. *Given an integral current space, $M = (X, d, T)$. If $B_p(r) \cap \partial M = \emptyset$ then*

$$\mathbf{M}(S(p,r)) \geq \mathrm{FillVol}(\mathrm{Sphere}(p,R)). \tag{197}$$

Thus if $\partial M = \emptyset$, we know that $p \in \bar{X}$ lies in $X = \mathrm{set(T)}$ if

$$\liminf_{r\to 0} \mathrm{FillVol}(\mathrm{Sphere}(p,r))/r^m > 0. \tag{198}$$

Proof. This follows immediately from Lemma 3.13 and Lemma 3.17. □

Theorem 4.1 of [22] can be stated as follows:

Theorem 3.19 (Sormani-Wenger)**.** *Suppose $M^m = (X, d, T)$ is a compact Riemannian manifold such that there exists $r_0 > 0, k > 0$ such that $\bar{B}(p, kr_0) \cap \partial M = \emptyset$ and every $B(x,r) \subset \bar{B}(p,r_0)$ is contractible within $B(x,kr) \subset \bar{B}(p,r_0)$ then $\exists C_k$ such that*

$$\mathrm{Vol}(\bar{B}(x,r)) = \|T\|(\bar{B}(x,r)\| \geq \mathrm{FillVol}(\partial S(p,r)) \geq C_k r^m. \tag{199}$$

This theorem essentially follows from a result of Greene-Petersen [8] combined with Lemma 3.18. The statement in [23] applies to a more general class of spaces and requires a much more subtle proof involving Lipschitz extensions.

3.5. **Sliced Filling Volumes of Balls.** Defn 3.21 has been moved into this subsection and more has been added to Lemma 3.22.

Spheres aren't the only slices whose filling volumes may be used to estimate the volumes of balls. We define the following new notions:

Definition 3.20. *Given an integral current space, $M^m = (X, d, T)$ and $F_1, F_2, ...F_k : M \to \mathbb{R}$ with $k \leq m-1$ are Lipschitz functions with Lipschtiz constant $\mathrm{Lip}(F_j) = \lambda_j$ then we define the sliced filling volume of $\partial S(p,r) \in \mathbf{I}_{m-1}(\bar{X})$, to be*

$$\mathbf{SF}(p, r, F_1, ..., F_k) = \int_{t\in A_r} \mathrm{FillVol}(\partial\, \mathrm{Slice}(S(p,r), F, t))\, \mathcal{L}^k. \tag{200}$$

where

$$A_r = [\min F_1, \max F_1] \times [\min F_2, \max F_2] \times \cdots \times [\min F_k, \max F_k] \tag{201}$$

where $\min F_j = \min\{F(x) : x \in \bar{B}_p(r)\}$ and $\max F_j = \max\{F(x) : x \in \bar{B}_p(r)\}$. Given $q_1, ..., q_k \in M$, we set,

$$\mathbf{SF}(p, r, q_1, ..., q_k) = \mathbf{SF}(p, r, \rho_1, ..., \rho_k) \text{ where } \rho_i(x) = d_X(q_i, x). \tag{202}$$

Definition 3.21. *Given an integral current space M^m and $p \in M^m$, then for almost every r, we can define the k^{th} sliced filling,*

$$\mathbf{SF}_k(p,r) = \sup\{\mathbf{SF}(p, r, q_1, ..., q_k) : q_i \in \partial B_p(r)\}. \tag{203}$$

In particular,

$$\mathbf{SF}_0(p,r) = \mathbf{SF}(p,r) = \mathrm{FillVol}(\partial S(p,r)). \tag{204}$$

Lemma 3.22. *Given an integral current space, $M^m = (X, d, T)$ and $F_1, F_2, ...F_k : M \to \mathbb{R}$ with $k \leq m-1$ are Lipschitz functions with Lipschitz constants, $\mathrm{Lip}(F_j) = \lambda_j > 0$, then*

$$\mathbf{M}(S(p,r)) \geq \prod_{j=1}^{k} \lambda_j^{-1} \mathbf{SF}(p, r, F_1, ..., F_k). \tag{205}$$

Thus $p \in \bar{X}$ lies in $X = \mathrm{set}(T)$ if there exists $F_i : M \to \mathbb{R}$ as above such that

$$\liminf_{r\to 0} \frac{1}{r^m} \mathbf{SF}(p, r, F_1, ..., F_k) > 0. \tag{206}$$

Applying (205) to $F_j = \rho_{q_{j,r}}$ where $q_{i,r}$ achieve the supremum in Definition 3.21, we see that

$$\mathbf{M}(S(p, r)) \geq \mathbf{SF}(p, r, q_{1,r}, ..., q_{k,r}) = \mathbf{SF}_k(p, r). \tag{207}$$

Thus $p \in \bar{X}$ lies in $X = \mathrm{set}(T)$ if

$$\liminf_{r\to 0} \frac{1}{r^m} \mathbf{SF}_k(p, r) > 0. \tag{208}$$

Conversely if $\partial S(p, r) \neq 0$ then for $k = 0$ we have

$$\mathbf{SF}_k(p, r) \neq 0. \tag{209}$$

It would be interesting to prove the converse for $k = 1, ..., m-k$ but this is not necessary for our applications.

Proof. By Proposition 3.10 we know

$$\mathbf{M}(S(p, r)) \geq \prod_{j=1}^{k} \lambda_j^{-1} \mathbf{M}(S(p, r) \llcorner dF) \tag{210}$$

$$\geq \prod_{j=1}^{k} \lambda_j^{-1} \int_{t\in\mathbb{R}^k} \mathbf{M}(\mathrm{Slice}(S(p, r), F, t))\, \mathcal{L}^k \tag{211}$$

$$= \prod_{j=1}^{k} \lambda_j^{-1} \int_{t\in A} \mathbf{M}(\mathrm{Slice}(S(p, r), F, t))\, \mathcal{L}^k. \tag{212}$$

Then (205) follows because $k \leq m-1$ implies each slice is at least 1 dimensional, combined with (93) and the fact that $\partial < S, F, t >= - < \partial S, F, t >$.

For converse follows because $\mathbf{SF}_0(p, r) = \mathrm{FillVol}(\partial S(p, r)) > 0$ when $S(p, r) \neq 0$. ◻

3.6. Uniform $\mathbf{SF}_k$ and Gromov-Hausdorff Compactness.

We now prove a new Gromov-Hausdorff Compactness Theorem:

Theorem 3.23. *If $M_i^m = (X_i, d_i, T_i)$ are integral current spaces with a uniform upper bound on $\mathrm{Vol}(M_i) \leq V_0$ and diameter $\mathrm{Diam}(M_i) \leq D_0$, and a uniform lower bound on the k^{th} sliced filling*

$$\mathbf{SF}_k(p, r) \geq C(r) > 0 \tag{213}$$

for all $p \in M_i$, for all i and for almost every $r \in \mathbb{R}$, then a subsequence (X_i, d_i) converges in the Gromov-Hausdorff sense to a limit space (Y, d_Y).

Later we will prove that the subsequence converges in the intrinsic flat sense to the same limit space when $C(r) \geq C_{SF} r^m > 0$ [Theorem 6.1].

Proof. For any p in any M_i, there exist $q_1, ..., q_k$ such that

$$\mathbf{SF}(p, r, q_1, ..., q_k) \geq C(r)/2 > 0. \tag{214}$$

So by Lemma 3.22, $\mathbf{M}(S(p, r)) \geq C(r)/2$. Thus the number of disjoint balls of radius r in M_i is $\leq 2V_0/C(r)$. So we may apply Gromov's Compactness Theorem. ◻

3.7. **Filling Volumes of 0 Dimensional Spaces.** Before proceeding we need the following lemma:

Lemma 3.24. *Let M be an integral current space. Suppose $S \in \mathbf{I}_1(M)$ such that $\partial S \neq \mathbf{0}$. Then* $\mathrm{set}(\partial S) = \{p_1, ..., p_N\}$ *with $N \geq 2$ and*

$$\partial S(f) = \sum_{i=1}^{N} \sigma_i \theta_i f(p_i) \tag{215}$$

where $\theta_i \in \mathbb{Z}^+$ and $|\sigma_i| = 1$ and

$$\mathrm{FillVol}(\partial S) \geq \max_{j=1..N} \left(|\theta_j| \min_{i \neq j} d_X(p_i, p_j) \right) > 0 \tag{216}$$

In particular

(217)

$$\mathrm{FillVol}(\partial S) \geq \inf \left\{ d_X(p_i, p_j) : \ i, j \in \{1, 2, ..., N\} \right\} \geq \inf\{d(x, y) : \ x \neq y, \ x, y \in \mathrm{set}(\partial S)\} > 0.$$

Proof. Recall that by Remark 2.34, ∂S satisfies (215) where $\sum_{i=1}^{N} \sigma_i \theta_i = 0$. So $N \geq 2$ when $\partial S \neq \mathbf{0}$.

Suppose $M' = (Y, d_Y, T)$ is any one dimensional integral current space with a current preserving isometry $\varphi : \mathrm{set}(\partial M') \to \mathrm{set}(\partial S) \subset \bar{X}$ so that

$$\varphi_\# \partial T = \partial S \in \mathbf{I}_0(M) \tag{218}$$

and $d_X(\varphi(y_1), \varphi(y_2)) = d_Y(y_1, y_2)$ for all $y_1, y_2 \in \mathrm{set}(T) \subset Y$. In particular there exist distinct points

$$p'_j = \varphi^{-1}(p_j) \in \bar{Y} \tag{219}$$

such that for any Lipschitz $f : \bar{Y} \to \mathbb{R}$ we have

$$T(1, f) = \partial(T)(f) = \sum_{i=1}^{N} \sigma_i \theta_i f(p'_i). \tag{220}$$

By (13) we have

$$|T(1, f)| \leq \mathrm{Lip}(f) \mathbf{M}(T). \tag{221}$$

Let $f_j(y) = \min_{i \neq j} d_Y(y, p'_i)$. Then we have, $\mathrm{Lip}(f_j) = 1$, so

$$\begin{aligned} \mathbf{M}(T) \quad &\geq \quad \left| \sum_{i=1}^{N} \sigma_i \theta_i f_j(p'_i) \right| && (222) \\ &\geq \quad \theta_j f_j(p'_j) = \theta_j \min_{i \neq j} d_Y(p'_i, p'_j) && (223) \\ &= \quad \theta_j \min_{i \neq j} d_X(p_i, p_j) && (224) \end{aligned}$$

Taking an infimum over all T, we have,

$$\mathrm{FillVol}(\partial S) \geq \theta_j \min_{i \neq j} d_X(p_i, p_j). \tag{225}$$

As this is true for all $j = 1..N$, we have (216). Since $\theta_j \in \mathbb{Z}^+$, we have the simpler lower bound given in (217). □

3.8. **Masses of Balls from Distances.** Here we provide a lower bound on the mass of a ball using a sliced filling volume and estimates on the filling volumes of 0 dimensional currents.

Theorem 3.25. *Given an integral current space, $M^m = (X, d, T)$ and points $p_1, ...p_{m-1} \in X$, then then if $\bar{B}_p(r) \cap \mathrm{set}(\partial \mathrm{T}) = \emptyset$ we have for almost every $r > 0$,*
(226)
$$\mathbf{M}(S(p,r)) \geq \mathbf{SF}(p, r, p_1, ..., p_{m-1}) \geq \int_{s_1-r}^{s_1+r} \cdots \int_{s_{m-1}-r}^{s_{m-1}+r} h(p, r, t_1, ..., t_{m-1})\, dt_1 dt_2 ... dt_{m-1}$$
where $s_i = d(p_i, p_0)$ and

$$h(p, r, t_1, ..., t_{m-1}) = \inf \{d(x, y) : \ x \neq y, \ x, y \in P(p, r, t_1, ..., t_{m-1})\} \tag{227}$$

when $P(p, r, t_1, ..., t_{m-1}) = \bar{B}_p(r) \cap \rho_{p_1}^{-1}(t_1) \cap \cdots \cap \rho_{p_{m-1}}^{-1}(t_{m-1})$ is a nonempty discrete set of points and 0 *otherwise.*

Thus $p \in \bar{X} \setminus Cl(\mathrm{set}(\partial \mathrm{T}))$ lies in $X = \mathrm{set}(\mathrm{T})$ if

$$\liminf_{r \to 0} (1/r^m) \int_{t_1=s_1-r}^{s_1+r} \cdots \int_{t_{m-1}=s_{m-1}-r}^{s_{m-1}+r} h(p, r, t_1, ..., t_{m-1})\, dt_1 dt_2 ... dt_{m-1} > 0 \tag{228}$$

Theorem 3.25 is in fact a special case of the following theorem:

Theorem 3.26. *Given an integral current space, $M^m = (X, d, T)$ and $F_1, F_2, ...F_{m-1} : M \to \mathbb{R}$ are Lipschitz functions with Lipschitz constants,* $\mathrm{Lip}(F_j) = \lambda_j$*, then for almost every $r > 0$*

$$\mathbf{M}(S(p,r)) \ \geq \ \mathbf{SF}(p, r, F_1, ..., F_k) \ \geq \ \prod_{j=1}^{k} \lambda_j^{-1} \ \int_{t \in A_r} h(p, r, F.t)\, d\mathcal{L}^k \tag{229}$$

where

$$h(p, r, F, t) = \inf\{d(x, y) : \ x \neq y, \ x, y \in \mathrm{set}(\partial\, \mathrm{Slice}(\mathrm{S(p, r), F, t)})\} > 0 \tag{230}$$

when $\partial\, \mathrm{Slice}(S(p,r), F, t) \in \mathbf{I}_0(\bar{X}) \setminus \{\mathbf{0}\}$ and $h(p, r, F, t) = 0$ otherwise, and where

$$A_r = [\min F_1, \max F_1] \times [\min F_2, \max F_2] \times \cdots \times [\min F_k, \max F_k] \tag{231}$$

with $\min F_j = \min\{F_j(x) : x \in \bar{B}_p(r)\}$ and $\max F_j = \max\{F_j(x) : x \in \bar{B}_p(r)\}$.

Before presenting the proof we give two important examples:

Example 3.27. *On Euclidean space, $\mathbb{E}^m$, taking $F_i : \mathbb{E}^m \to \mathbb{R}$ to be a collection of perpendicular coordinate functions for $i = 1..m$, $F_i(x_1, ..., x_m) = x_i$, we have $\lambda_i = 1$ and*

$$h(p, r, F_1, ..., F_{m-1}, t_1, ..., t_{m-1}) = 2\sqrt{r^2 - (t_1^2 + \cdots + t_{m-1}^2)}. \tag{232}$$

So

$$\omega_m r^m = \mathbf{M}(S(p,r)) \geq \mathbf{SF}(p, r, F_1, ..., F_k) = \omega_m r^m. \tag{233}$$

Example 3.28. *On the standard sphere, S^2, taking $p_1 \in \partial B_p(\pi/2)$ and $r = \pi/2$ and $F_1(x) = d(p_1, x)$, then*

$$h(p, \pi/2, F_1, t) = \min\{2t, 2(\pi - t)\} \tag{234}$$

because the distances are shortest if one travels within the great circle, $\partial B_p(\pi/2)$. So

$$2\pi = \mathrm{Vol}(S^2_+) = \mathbf{M}(S(p, \pi/2)) \geq \mathbf{SF}(p, \pi/2, F_1) \tag{235}$$

with

$$\mathbf{SF}(p,\pi/2,F_1) = \int_0^{\pi} h(p,\pi/2,F_1,t)\,dt \tag{236}$$

$$= 2\int_0^{\pi/2} 2t\,dt = 2(\pi/2)^2 = \pi^2/2. \tag{237}$$

Proof. Theorem 3.25 follows from Theorem 3.26 taking $F(x) = (F_1(x), ..., F_{m-1}(x))$ where $F_i(x) = \rho_{p_i}(x)$. When $\bar{B}(p,r) \cap \mathrm{set}\partial \mathrm{T} = \emptyset$, then for almost every $r \in \mathbb{R}, t_1 \in \mathbb{R}, ..., t_{m-1} \in \mathbb{R}$ $\partial\,\mathrm{Slice}(S(p,r), F, t)) \in \mathbf{I}_0(\bar{X})$ and

$$\mathrm{set}(\partial\,\mathrm{Slice(S(p,r),F,t)}) = \rho_{\mathrm{p}}^{-1}(\mathrm{r}) \cap \mathrm{F}_1^{-1}(\mathrm{t}_1) \cap \cdots \cap \mathrm{F}_{\mathrm{m}-1}^{-1}(\mathrm{t}_{\mathrm{m}-1}). \tag{238}$$

so this set either has 0 points or at least two points. □

Proof. Theorem 3.26 is proven by applying Lemma 3.22 to F and then computing the filling volume of the 0 dimensional current, $\partial(\mathrm{Slice}(S(p,r),F,t))$, using Lemma 3.24 stated and proven below. Observe that if $t_i < s_i - r$ or $t_i > s_i + r$ then $h(p,r,t_1,...,t_{m-1}) = 0$ because $\rho_p^{-1}(r) \cap \rho_{p_i}^{-1}(t_i) = \emptyset$. □

Remark 3.29. *Naturally we could combine Theorem 3.26 with any other lower bound on the filling volumes of* 0 *dimensional sets, like, for example, (216).*

3.9. **Tetrahedral Property.**

Theorem 3.25 allows us to estimate the masses of balls using a tetrahedral property (see Figure 1).

Definition 3.30. *Given $C > 0$ and $\beta \in (0,1)$, a metric space X is said to have the m dimensional C,β-tetrahedral property at a point p for radius r if one can find points $p_1, ...p_{m-1} \subset \partial B_p(r) \subset \bar{X}$, such that*

$$h(p,r,t_1,...,t_{m-1}) \geq Cr \qquad \forall (t_1,...,t_{m-1}) \in [(1-\beta)r, (1+\beta)r]^m \tag{239}$$

where

$$h(p,r,t_1,...,t_{m-1}) = \inf\{d(x,y):\ \ x \neq y,\ \ x,y \in P(p,r,t_1,...,t_{m-1})\} \tag{240}$$

when

$$P(p,r,t_1,...,t_{m-1}) = \rho_p^{-1}(r) \cap \rho_{p_1}^{-1}(t_1) \cap \cdots \cap \rho_{p_{m-1}}^{-1}(t_{m-1}) \tag{241}$$

is nonempty and 0 *otherwise. In particular $P(p,r,t_1,...,t_{m-1})$ is a discrete set of points.*

Example 3.31. *On Euclidean space, $\mathbb{E}^3$, taking $p_1, p_2 \in \partial B(p,r)$ to such that $d(p_1,p_2) = r$, then there exists exactly two points $x, y \in P(p,r,r,r)$ each forming a tetrahedron with p, p_1, p_2. See Figure 1. As we vary $t_1, t_2 \in (r/2, 3r/2)$, we still have exactly two points in $P(p,r,t_1,t_2)$. By scaling we see that*

$$h(p,r,t_1,t_2) = rh(p,1,t_1/r,t_2/r) \geq C_{\mathbb{E}^3} r \tag{242}$$

where

$$C_{\mathbb{E}^3} = \inf\{h(p,1,s_1,s_2):\ s_i \in (1/2,3/2)\} > 0 \tag{243}$$

could be computed explicitly. So $\mathbb{E}^3$ satisfies the $C_{\mathbb{E}^3}$, $(1/2)$ tetrahedral property at p for all r.

Example 3.32. *On a torus, $M^3_\epsilon = S^1 \times S^1 \times S^1_\epsilon$ where S^1_ϵ has been scaled to have diameter ϵ instead of π, we see that M^3 satisfies the $C_{\mathbb{E}^3}, (1/2)$ tetrahedral property at p for all $r < \epsilon/4$. By taking $r < \epsilon/4$, we guarantee that the shortest paths between x and y stay within the ball $B(p, r)$ allowing us to use the Euclidean estimates. If r is too large, $P(p, r, t_1, t_2) = \emptyset$.*

Remark 3.33. *On a Riemannian manifold or an integral current space, we know that $P(p, r, t_1, ..., t_{m-1})$ is the set of a 0 current which is a boundary. So if it is not empty, it has at least two points, one with positive weight and one with negative weight.*

Remark 3.34. *It is not just a simple application of the triangle inequality to proceed from knowing $h(p, r, r, ..., r) \ge Cr$ to having $h(p, r, t_1, ..., t_{m-1}) \ge C_2 r$. There is the possibility that $P(p, r, t_1, ..., t_{m-1})$ is empty or has a closest pair of points both near a single point of $C(p, r, r, ..., r)$ even in a Riemannian manifold. However one expects the same type of curvature conditions that would lead to control of $h(p, r, ..., r)$ could be used to study $h(p, t_1, ..., t_{m-1})$.*

Remark 3.35. *On a manifold with sectional curvature bounded below, one should have the $C, 1/2$ tetrahedral property at any point p as long as $r < injrad(p)/4$ where C depends on the lower sectional curvature bound. This should be provable using the Toponogov Comparison Theorem. One would like to replace the condition on injectivity radius with radius depending upon a lower bound on volume. Work in this direction is under preparation by the author's doctoral students. Note that there is no uniform tetrahedral property on manifolds with positive scalar curvature even when the volume of the balls are uniformly bounded below by that of Euclidean balls [Remark 6.3]. With lower bounds on Ricci curvature one might expect to have the $C, 1/2$ tetrahedral property or an integral version of this property. Again a uniform lower bound on volume will be necessary as seen in the torus example above.*

3.10. **Integral Tetrahedral Property.** For our applications we need only the following property which is clearly holds at any point with the tetrahedral property:

Definition 3.36. *Given $C > 0$ and $\beta \in (0, 1)$, a metric space X is said to have the m dimensional integral C,β-tetrahedral property at a point p for radius r if one can find points $p_1, ...p_{m-1} \subset \partial B_p(r) \subset \bar{X}$, such that*

$$\int_{t_1=(1-\beta)r}^{(1+\beta)r} \cdots \int_{t_{m-1}=(1-\beta)r}^{(1+\beta)r} h(p, r, t_1, ...t_{m-1}))\, dt_1 dt_2 ... dt_{m-1} \ge C(2\beta)^{m-1} r^m. \tag{244}$$

Proposition 3.37. *If X is a metric space that satisfies the $C\beta$ tetrahedral property at p for radius r then it has the $C\beta$ integral tetrahedral property.*

Proof.

$$\begin{aligned}
\int_{t_1=(1-\beta)r}^{(1+\beta)r} \cdots \int_{t_{m-1}=(1-\beta)r}^{(1+\beta)r} \quad h(p, r, t_1, ...t_{m-1}) \quad dt_1 dt_2 ... dt_{m-1} \quad \ge & \\
\ge \quad & \int_{t_1=(1-\beta)r}^{(1+\beta)r} \cdots \int_{t_{m-1}=(1-\beta)r}^{(1+\beta)r} CR\, dt_1 dt_2 ... dt_{m-1} \\
\ge \quad & CR\ ((1+\beta)r - (1-\beta)r)^{m-1}
\end{aligned}$$

$\square$

3.11. **Tetrahedral Property and Masses of Balls.**

Theorem 3.38. *If (X, d, T) is an integral current space and $\bar{X}$ has the m dimensional (integral) C,β-tetrahedral property at a point p for radius r such that $\bar{B}_p(r) \cap \mathrm{set}\partial \mathrm{T} = 0$ then*

$$\mathbf{M}(S(p,r)) \ge \mathbf{SF}_{m-1}(p,r) \ge C(2\beta)^{m-1} r^m \tag{245}$$

Proof. By Theorem 3.25 with $s_i = r$ we have

$$\begin{aligned} \mathbf{M}(S(p,r)) &\ge \mathbf{SF}(p, r, p_1, ..., p_{m-1}) \\ &\ge \int_{t_1=s_1-r}^{s_1+r} \cdots \int_{t_{m-1}=s_{m-1}-r}^{s_{m-1}+r} h(P_{(r,t_1,...t_{m-1})})\, dt_1 dt_2 ... dt_{m-1} \\ &\ge \mathbf{SF}(p, r, p_1, ..., p_{m-1}) \\ &\ge \int_{t_1=0}^{2r} \cdots \int_{t_{m-1}=0}^{2r} h(P_{(r,t_1,...t_{m-1})})\, dt_1 dt_2 ... dt_{m-1} \\ &> \int_{t_1=(1-\beta)r}^{(1+\beta)r} \cdots \int_{t_{m-1}=(1-\beta)r}^{(1+\beta)r} h(p, r, t_1, ...t_{m-1}))\, dt_1 dt_2 ... dt_{m-1} \\ &> C(2\beta)^{m-1} r^m \end{aligned}$$

□

Theorem 3.39. *If p_0 lies in a Riemannian manifold with boundary and $B_{p_0}(r) \cap \partial M = \emptyset$, that has the m dimensional (integral) C,β-tetrahedral property at a point p for radius R such that $\bar{B}_p(r) \cap \mathrm{set}\partial\mathrm{T} = 0$ then*

$$\mathrm{Vol}(B(p,r)) \ge C(2\beta)^{m-1} r^m \tag{246}$$

Proof. This is an immediate consequence of Theorem 3.38. □

Remark 3.40. *In Example 3.32, as $\epsilon \to 0$, the $\mathrm{Vol}(B(p,r)) \le \mathrm{Vol}(M^3_\epsilon) \to 0$, so we could not have a uniform tetrahedral property on these spaces.*

Theorem 3.41. *Given $r_0 > 0, \beta \in (0,1), C > 0, V_0 > 0$. If a sequence of compact Riemannian manifolds, M^m, has $\mathrm{Vol}(M^m) \le V_0$, $\mathrm{Diam}(M^m) \le D_0$, and the C,β (integral) tetrahedral property for all balls of radius $\le r_0$, then a subsequence converges in the Gromov-Hausdorff sense. In particular they have a uniform upper bound on diameter depending only on these constants.*

The proof of this theorem strongly requires that the manifold have no boundary.

Proof. This follows immediately from Theorem 3.39 and Gromov's Compactness Theorem, using the fact that we can bound the number of disjoint balls of radius $\epsilon > 0$ in M^m. In a manifold, this provides an upper bound on the diameter of M^m, □

Later we will apply the following theorem to prove that the Gromov-Hausdorff limit is in fact an Intrinsic Flat limit and thus is countably $\mathcal{H}^m$ rectifiable [Theorem 6.2].

Theorem 3.42. *Given an integral current space (X, d, T) and a point $p_0 \in \bar{X} \setminus Cl(\mathrm{set}(\partial \mathrm{T})$ then $p_0 \in X = \mathrm{set(T)}$ if there exists a pair of constants $\beta \in (0,1)$ and $C > 0$ such that $\bar{X}$ has the tetrahedral property at p_0 for all sufficiently small $r > 0$.*

Proof. By Theorem 3.38 we have

$$\liminf_{r\to 0} \frac{\|T\|(B_p(r))}{r^m} \geq C(2\beta)^{m-1} > 0 \tag{247}$$

so $p \in \mathrm{set(T)}$. □

3.12. **Fillings, Slices and Intervals.** In the above sections, a key step consisted of estimating $\mathbf{M}(M) \geq \mathrm{FillVol}(\partial M)$. This is only a worthwhile estimate when $\partial M \neq 0$ or has a filling volume close to the mass.

A better estimate can be obtained using the following trick. Given a Riemannian manifold M,

$$\mathrm{Vol}(M) = \mathrm{Vol}(M \times I) \geq \mathrm{FillVol}(\partial(M \times I)) \tag{248}$$

where the metric on $M \times I$ is defined in (142). This has the advantage that $M \times I$ is always a manifold with boundary. It may also be worthwhile to use an interval, I_ϵ, of length ϵ, then

$$\mathrm{Vol}(M) = \frac{\mathrm{Vol}(M \times I_\epsilon)}{\epsilon} \geq \frac{\mathrm{FillVol}(\partial(M \times I_\epsilon))}{\epsilon}. \tag{249}$$

Intuitively it would seem that taking $\epsilon \to 0$ we converge to an equality.

We introduce the following notion made rigorous on arbitrary integral current spaces $M = (X, d_X, T)$ by applying Definition 3.5 and Proposition **??**.

Definition 3.43. *Given any $\epsilon > 0$, we define the ϵ interval filling volume,*

$$\mathbb{IFV}_\epsilon(M) = \mathrm{FillVol}(\partial(M \times I_\epsilon)). \tag{250}$$

Lemma 3.44. *Given an integral current space $M = (X, d, T)$,*

$$\mathbf{M}(M) = \epsilon^{-1}\mathbf{M}(M \times I_\epsilon) \geq \epsilon^{-1}\mathbb{IVF}_\epsilon(M). \tag{251}$$

Proof. This follows immediately from Proposition 3.7. □

Definition 3.45. *Given an integral current space, $M^m = (X, d, T)$ and $F_1, F_2, ...F_k : M \to \mathbb{R}$ with $k \leq m$ are Lipschitz functions with Lipschtiz constant $\mathrm{Lip}(F_j) = \lambda_j$ then for all $\epsilon > 0$ we can define the ϵ sliced interval filling volume of $\partial S(p,r) \in \mathbf{I}_{m-1}(\bar{X})$ to be*

$$\mathbb{SIF}_\epsilon(p, r, F_1, ..., F_k) = \prod_{j=1}^{k} \epsilon^{-1} \int_{t\in A_r} \mathrm{FillVol}(\partial(\mathrm{Slice}(S(p,r), F, t) \times I_\epsilon))\, \mathcal{L}^k \tag{252}$$

where

$$A_r = [\min F_1, \max F_1] \times [\min F_2, \max F_2] \times \cdots \times [\min F_k, \max F_k \tag{253}$$

where $\min F_j = \min\{F(x) : x \in \bar{B}_p(r)\}$ *and* $\max F_j = \max\{F(x) : x \in \bar{B}_p(r)\}$. *When the F_i are distance functions ρ_{p_i} we write,*

$$\mathbb{SIF}_\epsilon(p, r, p_i, ..., p_k) := \mathbb{SIF}_\epsilon(p, r, \rho_{p_i}, ..., \rho_{p_k}). \tag{254}$$

Proposition 3.46. *Given an integral current space, $M^m = (X, d, T)$ and $F_1, F_2, ...F_k : M \to \mathbb{R}$ with $k \leq m$ are Lipschitz functions with Lipschtiz constant $\mathrm{Lip}(F_j) = \lambda_j$ then for all $\epsilon > 0$ we can bound the mass of a ball in M as follows:*

$$\mathbf{M}(S(p,r)) \geq \prod_{j=1}^{k} \lambda_j^{-1}\epsilon^{-1}\mathbb{SIF}_\epsilon(p, r, F_1, ..., F_k). \tag{255}$$

Thus for any $p_1, ...p_k \in X$, we have

$$\mathbf{M}(S(p,r)) \geq \epsilon^{-1}\mathbb{SIF}_\epsilon(p,r,p_1,...,p_k). \tag{256}$$

Proof. By Proposition 3.10, Lemma 3.44 and Remark **??** we have

$$\begin{aligned}
\mathbf{M}(S(p,r)) &\geq \prod_{j=1}^{k} \lambda_j^{-1}\mathbf{M}(S(p,r) \llcorner dF) && (257)\\
&\geq \prod_{j=1}^{k} \lambda_j^{-1} \int_{t\in\mathbb{R}^k} \mathbf{M}(\text{Slice}(S(p,r),F,t))\,\mathcal{L}^k && (258)\\
&= \prod_{j=1}^{k} \lambda_j^{-1} \int_{t\in A} \mathbf{M}(\text{Slice}(S(p,r),F,t))\,\mathcal{L}^k && (259)\\
&\geq \prod_{j=1}^{k} \lambda_j^{-1}\epsilon^{-1} \int_{t\in A} \text{FillVol}(\text{Slice}(S(p,r),F,t)\times I_\epsilon)\,\mathcal{L}^k. && (260)
\end{aligned}$$

□

Corollary 3.47. *Thus $p \in \bar{X}$ lies in $X = \text{set(T)}$ iff there exists $\epsilon > 0$ such that*

$$\liminf_{r\to 0} \frac{1}{\epsilon r^m} \int_{t\in A_r} \text{FillVol}(\partial(\text{Slice}(S(p,r),F,t)\times I_\epsilon))\,\mathcal{L}^k > 0. \tag{261}$$

Corollary 3.48. *Also $p \in \bar{X}$ lies in $X = \text{set(T)}$ iff there exists $C > 0$ such that*

$$\liminf_{r\to 0} \frac{1}{Cr^{m+1}} \int_{t\in A_r} \text{FillVol}(\partial(\text{Slice}(S(p,r),F,t)\times I_{Cr}))\,\mathcal{L}^k > 0. \tag{262}$$

Corollary 3.49.

$$\mathbf{M}(M) \geq \prod_{j=1}^{k} \lambda_j^{-1}\epsilon^{-1} \int_{t\in\mathbb{R}^k} \text{FillVol}(\partial(\text{Slice}(M,F,t)\times I_\epsilon))\,\mathcal{L}^k \tag{263}$$

4. Convergence and Continuity

In this section we examine the limits of points in sequences of integral current spaces that converge in the intrinsic flat sense and prove various continuity theorems.

Before we begin recall that Theorem 2.40 which was proven in work of the author with Wenger in [23] states that a sequence of manifolds which converges in the intrinsic flat sense can be isometrically embedded into a common metric space. We use this theorem to define the notion of a converging sequence of points:

Definition 4.1. *If $M_i = (X_i, d_i, T_i) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$, then we say $x_i \in X_i$ are a converging sequence that converge to $x_\infty \in X_\infty$ if there exists a complete metric space Z and isometric embeddings $\varphi_i : M_i \to Z$ such that $\varphi_{i\#} T_i \xrightarrow{\mathcal{F}} \varphi_{\infty\#} T_\infty$ and $\varphi_i(x_i) \to \varphi_\infty(x_\infty)$. If we say collection of points, $\{p_{1,i}, p_{2,i}, ...p_{k,i}\}$, converges to a corresponding collection of points, $\{p_{1,\infty}, p_{2,\infty}, ...p_{k,\infty}\}$, if $\varphi_i(p_{j,i}) \to \varphi_\infty(p_{j,\infty})$ for $j = 1..k$.*

In the beginning of this section we prove every point $x \in X_\infty$ has a sequence $x_i \in X_i$ converging to x in this sense [Theorem 4.3]. This implies the lower semicontinuity of the diameter [Theorem 4.5]. We then prove that intrinsic flat limits of spaces are the Gromov-Hausdorff limits of regions within those spaces [Theorem **??**]. These basic theorems require almost no material from Section 3.

Unlike in Gromov-Hausdorff convergence, we have the possibility of disappearing sequences of points:

Definition 4.2. *If $M_i = (X_i, d_i, T_i) \xrightarrow{\mathcal{F}} M_\infty = (X_\infty, d_\infty, T_\infty)$, then we say $x_i \in X_i$ are Cauchy if there exists a complete metric space Z and isometric embeddings $\varphi_i : M_i \to Z$ such that $\varphi_{i\#} T_i \xrightarrow{\mathcal{F}} \varphi_{\infty\#} T_\infty$ and $\varphi_i(x_i) \to z_\infty \in Z$. We say the sequence is disappearing if $z_\infty \notin \varphi_\infty(X_\infty)$.*

Examples with disappearing splines from [23] demonstrate that there exist Cauchy sequences of points which disappear. In fact z_∞ may not even lie in the metric completion of the limit space, $\varphi_\infty(\bar{X}_\infty)$.. To avoid disappearance, we must show

$$z_\infty \in \mathrm{set}(\varphi_{\infty\#})(\mathrm{T}_\infty)) \tag{264}$$

which happens iff

$$\liminf_{r\to 0} \mathbf{M}(\varphi_{\infty\#})(T_\infty) \llcorner B(z_\infty, r))/r^m > 0. \tag{265}$$

In the prior subsections we gave many means for estimating these masses using filling volumes and sliced filling volumes.

Thus we need to prove continuity theorems. In [22], the author and Wenger stated without proof that balls converge. Here we provide the details of that proof here using the notion of converging and Cauchy sequences of points [Theorem 4.18]. We also prove that slices of the spaces converge [Proposition 4.19]. We then prove that the sliced filling volumes are continuous [Theorem 4.20]. Within this proof we provide the complete details proving that filling volumes of spheres converge (which was first described and applied in [22]). Finally we prove the continuity of the interval filling volumes [Theorem 4.23] and sliced interval filling volumes [Theorem 4.24].

Finally we apply these continuity theorems to prove two crucial theorems regarding the limits of sequences of points. Theorem 4.26 provides information on a sequence of points converging to a given limit point. Theorem 4.27 describes when a Cauchy sequence of points converges. This last theorem is a crucial step in the proof of the Tetrahedral Compactness Theorem.

We close the section with two Bolzano-Weiestrass Theorems. Theorem 4.30 concerns sequences of points $p_i \in M_i$ with lower bounds on the filling volumes of spheres around them and produces a subsequence which converges to a point in the intrinsic flat limit of the M_i. Theorem 4.31 assumes that the M_i also have a Gromov-Hausdorff limit and that the points have a lower bound on the sliced filling volumes of balls about them and obtains a converging subsequence as well.

Those interested in the proof of the Tetrahedral Compactness Theorem should read 4.1, 4.2, 4.4 and 4.6. Those interested in the Arzela-Ascoli Theorems need to read all subsections.

4.1. **Limit Points and Diameter Lower Semicontinuity.** Recall Definition 4.1.

Theorem 4.3. *If a sequence of integral current spaces, $M_i = (X_i, d_i, T_i) \in \mathcal{M}_0^m$, converges to a integral current space, $M = (X, d, T) \in \mathcal{M}_0^m$, in the intrinsic flat sense, then every point z in the limit space M is the limit of points $x_i \in M_i$. In fact there exists a sequence of maps $F_i : X \to X_i$ such that $x_i = F_i(x)$ converges to x and*

$$\lim_{i\to\infty} d_i(F_i(x), F_i(y)) = d(x, y). \tag{266}$$

This sequence of maps F_i are not uniquely defined and are not even unique up to isometry.

Proof. By Theorem 2.40 there exists a common metric space Z and isometric embeddings $\varphi_i : X_i \to Z$ and $\varphi : X \to Z$ such that

$$\varphi_\# T - \varphi_{i\#} T_i = U_i + \partial V_i \tag{267}$$

where $m_i = \mathbf{M}(U_i) + \mathbf{M}(V_i) \to 0$. So $\varphi_{i\#} T_i$ converges in the flat and the weak sense to $\varphi_\# T$.

Let ρ_x be the distance function from $\varphi(x)$. Applying the Ambrosio-Kirchheim Slicing Theorem we know

$$S_{i,\epsilon,x} := \varphi_{i\#} T_i \llcorner \rho_x^{-1}([0, \epsilon)) \in \mathbf{I}_m(Z) \tag{268}$$

$$T_{\epsilon,x} := \varphi_{i\#} T \llcorner \rho_x^{-1}([0, \epsilon)) \in \mathbf{I}_m(Z) \tag{269}$$

for almost every $\epsilon > 0$. Restricting ourselves to such an $\epsilon \in (0, 1/j)$, we have $S_{i,\epsilon,x}$ converges weakly to $T_{\epsilon,x}$. In particular, there exists $N_{\epsilon,x} \in \mathbb{N}$ large enough that $S_{i,\epsilon,x}$ is not zero for all $i \geq N_{\epsilon,x}$. So for all $x \in X$ and any $j \in \mathbb{N}$

$$\exists N_{j,x} \; s.t. \; \exists s_{i,j,x} \in \mathrm{set(S_i)} \cap \mathrm{B\,(x, 1/j)} \qquad \forall \mathrm{i} \geq \mathrm{N_{j,x}}. \tag{270}$$

Without loss of generality we assume $N_{j,x}$ is increasing in j. For $i \in \{1, ..., N_{1,x}$ take $j_i = 1$. Then for $i \in \{N_{j-1,x} + 1, ..., N_{j,x}\}$ let $j_i = j$. Thus $i \geq N_{j_x}$ Let

$$x_i = \varphi_i^{-1}(s_{i,j_i}). \tag{271}$$

Then $\varphi_i(x_i) \in B(x, 1/j_i)$ and $\varphi_i(x_i) \to \varphi(x)$.

Since this process can be completed for any $x \in X$, we have defined maps $F_i : X \to X_i$ such that

$$\varphi_i(F_i(x)) \to \varphi(x). \tag{272}$$

Finally

$$d_i(F_i(x), F_i(y)) = d_Z(\varphi_i(F_i(x)), \varphi_i(F_i(y))) \to d_Z(\varphi(x), \varphi(y)) = d(x, y). \tag{273}$$

□

Definition 4.4. *Like any metric space, one can define the diameter of an integral current space, $M = (X, d, T)$, to be*

$$\mathrm{Diam}(M) = \sup\{d_X(x, y) : \; x, y \in X\} \in [0, \infty]\}. \tag{274}$$

However, we explicitly define the diameter of the 0 *integral current space to be* 0*. A space is bounded if the diameter is finite.*

Theorem 4.5. *Suppose $M_i \xrightarrow{\mathcal{F}} M$ are integral current spaces then*

$$\mathrm{Diam}(M) \le \liminf_{i\to\infty} \mathrm{Diam}(M_i) \subset [0, \infty] \tag{275}$$

Proof. Note that by the definition, $\mathrm{Diam}(M_i) \ge 0$, so the liminf is always ≥ 0. Thus the inequality is trivial when M is the 0 space. Assuming M is not the 0 space, we have for any $\epsilon > 0$, there exists $x, y \in X$ such that

$$\mathrm{Diam}(M) \le d(x, y) + \epsilon. \tag{276}$$

By Theorem 4.3, we have $x_i, y_i \in X_i$ converging to $x, y \in X$ so that

$$\mathrm{Diam}(M) \le \lim_{i\to\infty} d_i(x_i, y_i) + \epsilon \le \liminf_{i\to\infty} \mathrm{Diam}(X_i) + \epsilon. \tag{277}$$

□

4.2. **Flat convergence to Gromov-Hausdorff Convergence.**

In this subsection, we prove Theorem 4.6:

Theorem 4.6. *If a sequence of precompact integral current spaces, $M_i = (X_i, d_i, T_i) \in \mathcal{M}_0^m$, converges to a precompact integral current space, $M = (X, d, T) \in \mathcal{M}_0^m$, in the intrinsic flat sense, then there exists $S_i \in \mathbf{I}_m\left(\bar{X}_i\right)$ such that $N_i = (\mathrm{set}\,(\mathrm{S_i})\,, \mathrm{d_i})$ converges to $\left(\bar{X}, d\right)$ in the Gromov-Hausdorff sense and*

$$\liminf_{i\to\infty} \mathbf{M}(S_i) \ge \mathbf{M}(M). \tag{278}$$

When the M_i are Riemannian manifolds, the N_i can be taken to be settled completions of open submanifolds of M_i.

Remark 4.7. *If in addition it is assumed that $\lim_{i\to\infty} \mathbf{M}(M_i) = \mathbf{M}(M)$, then by (278) we have $\mathbf{M}(\mathrm{set}(\mathrm{T_i} - \mathrm{S_i}), \mathrm{d_i}, \mathrm{T_i} - \mathrm{S_i}) = 0$. In the Riemannian setting, we have $\mathrm{Vol}(M_i \setminus N_i) \to 0$.*

Remark 4.8. *In Ilmanen's example* [23] *of a sphere with increasingly many splines, the S_i may be chosen to be integration over the spherical part of M_i with balls around the tips removed. Then* $\mathrm{set}(\mathrm{S_i})$ *are manifolds with boundary converging to the sphere in the Gromov-Hausdorff and intrinsic flat sense.*

Remark 4.9. *The precompactness of the limit integral current spaces is necessary in this theorem because a noncompact limit space can never be the Gromov-Hausdorff limit of precompact spaces, yet there are sequences of compact Riemannian manifolds, M_j, whose intrinsic flat limit is an unbounded complete Riemannian manifold of finite volume* [23]*[Ex A.10] and another example of such spaces whose Intrinsic Flat limit is a bounded noncompact integral current space* [23]*[Ex A.11].*

Remark 4.10. *Gromov's Compactness Theorem combined with Theorem 4.6 implies that that any sequence of $x_i \in N_i \subset M_i$ has a subsequence converging to a point x in the metric completion of M. Other points need not have limit points, as can be seen when the tips of thin splines disappear in the examples from* [23]. *We will prove a more general Bolzano-Weierstrass Theorem precisely identifying those points which do not disappear later in this section.*

We now prove Theorem 4.6:

Proof. By Theorem 2.40 there exists a common metric space Z and isometric embeddings $\varphi_i : X_i \to Z$ and $\varphi : X \to Z$ such that

$$\varphi_\# T - \varphi_{i\#} T_i = U_i + \partial V_i \tag{279}$$

where $m_i = \mathbf{M}(U_i) + \mathbf{M}(V_i) \to 0$. So $\varphi_{i\#} T_i$ converges in the flat and the weak sense to $\varphi_\# T$.

Since $M \in \mathcal{M}_0^m$, $\varphi(X)$ is precompact. Let $\rho : Z \to \mathbb{R}$ be the distance function from $\varphi(X)$.

By the Ambrosio-Kirchheim Slicing Theorem [Theorem 2.23], we know that

$$S_{i,\epsilon} := \varphi_{i\#} T_i \llcorner \rho^{-1}([0,\epsilon)) \in \mathbf{I}_m(Z) \tag{280}$$

for almost every $\epsilon > 0$. Fix any such ϵ.

Note that $\varphi_\# T = \varphi_\# T \llcorner \rho^{-1}[0,\epsilon)$. So both $\varphi_{i\#}(T_i)$ and $S_{i,\epsilon}$ converges weakly to $\varphi_\# T$ for fixed $\epsilon > 0$. In particular, by lower semicontinuity of mass we have

$$\forall \epsilon > 0 \ \exists M_\epsilon \in \mathbb{N} \text{ such that } \mathbf{M}(S_{i,\epsilon}) \ge \mathbf{M}(T) - \epsilon \ \forall i \ge M_\epsilon. \tag{281}$$

By definition

$$\mathrm{set}(\mathrm{S}_{i,\epsilon}) \subset \bar{\mathrm{T}}_\epsilon(\varphi(\mathrm{X})) \subset \mathrm{T}_{2\epsilon}(\varphi(\mathrm{X})). \tag{282}$$

To prove the Hausdorff distance between $S_{i,\epsilon}$ and X is small we will prove

$$\varphi(X) \subset T_{2\epsilon}(\mathrm{set}(\mathrm{S}_{i,\epsilon})) \qquad \forall i \ge N_\epsilon. \tag{283}$$

To prove (283), we first note that for any $x \in X$, we can let ρ_x be the distance function from $\varphi(x)$, and applying the slicing theorem again we know

$$S_{i,\epsilon,x} := \varphi_{i\#} T_i \llcorner \rho_x^{-1}([0,\epsilon)) \in \mathbf{I}_m(Z) \tag{284}$$

$$T_{\epsilon,x} := \varphi_{i\#} T \llcorner \rho_x^{-1}([0,\epsilon)) \in \mathbf{I}_m(Z) \tag{285}$$

for almost every $\epsilon > 0$. Restricting ourselves to such an $\epsilon > 0$, we have $S_{i,\epsilon,x}$ converges weakly to $T_{\epsilon,x}$. In particular, there exists $N_{\epsilon,x} \in \mathbb{N}$ large enough that $S_{i,\epsilon,x}$ is not zero for all $i \ge N_{\epsilon,x}$. So for all $x \in X$ and almost every $\epsilon > 0$

$$\exists N_{\epsilon,x} \ s.t. \ \exists s_{i,\epsilon,x} \in \mathrm{set}(\mathrm{S}_\mathrm{i}) \cap \mathrm{B}(\mathrm{x},\epsilon) \qquad \forall \mathrm{i} \ge \mathrm{N}_{\epsilon,\mathrm{x}}. \tag{286}$$

Note that for all $\epsilon > 0$ and all $x \in X$ we have (286).

By the precompactness of X, there is a finite ϵ net, $X_\epsilon = \{x_1, ...x_N\}$ on $\varphi(X)$ (i.e. the union of $B(x_i, \epsilon)$ contains X_ϵ) and choosing $N_\epsilon = \max\{N_{\epsilon,x_j} : x_j \in X_\epsilon\}$, we know every $x \in X$, there exists $x_j \in X_\epsilon$ such that for all $i \ge N_\epsilon$ there exists

$$s_x := s_{i,\epsilon,x_j} \in \mathrm{set}(\mathrm{S}_{i,\epsilon}) \ \text{ s.t. } \mathrm{d_Z}(\mathrm{s_x}, \varphi(\mathrm{x})) < 2\epsilon \tag{287}$$

so we have proven (283). Combining this with (282) we have the Hausdorff distance

$$d_H^Z(\mathrm{set}(\mathrm{S}_{i,\epsilon}), \varphi(\mathrm{X})) \le 2\epsilon \qquad \forall i \ge N_\epsilon. \tag{288}$$

We may choose $N_\epsilon \ge M_\epsilon$ of (281).

Let $\epsilon_k \to 0$ be a decreasing sequence of ϵ for which all these currents are defined. Let $N_k := N_{\epsilon_k}$. Let

$$S_i = T_i \in \mathbf{I}_m(X_i) \text{ for } i = 1 \text{ to } N_1 \tag{289}$$

$$S_i = \varphi_{i\#}^{-1} S_{i,\epsilon_1} \in \mathbf{I}_m(X_i) \text{ for } i = N_1 + 1 \text{ to } N_2 \tag{290}$$

and so on:

$$S_i = \varphi_{i\#}^{-1} S_{i,\epsilon_j} \in \mathbf{I}_m(X_i) \text{ for } i = N_j + 1 \text{ to } N_{j+1} \tag{291}$$

Then by (288) we have Gromov-Hausdorff convergence:

$$d_{GH}^Z(\mathrm{set}(\mathrm{S_i}), \varphi(\mathrm{X})) \le 2\epsilon_i. \tag{292}$$

By (281) we also have

$$\mathbf{M}(S_i) \ge \mathbf{M}(T) - \epsilon_i \tag{293}$$

which gives us (279). □

Remark 4.11. *One could construct a common metric space Z for Examples A.10 and A.11 of* [23] *and find $S_{i,\epsilon}$ as in the above proof satisfying (282). However, in that example, (283) will fail to hold. This is where the precompactness of the limit space is essential in the proof.*

Remark 4.12. *Examples in* [23] *demonstrate that the metric space of a current space need not be a length space. In general, when a sequence of Riemannian manifolds converges in the flat sense to a current space it need not be a geodesic length space. Here we see that if the* set($\mathrm{S_i}$) *are length spaces or approximately length spaces, then the limit current space is in fact a length space. This occurs for example in Ilmanen's example of* [23]. *It also occurs whenever the Gromov-Hausdorff limits and flat limits of length spaces agree which we will examine further below. It might be interesting to develop a notion of an approximate length space that suffices to give a geodesic limit space. What properties must hold on M_i to guarantee that their limit is a geodesic length space?*

Remark 4.13. *It is not immediately clear whether the integral current spaces, N_i, constructed in the proof of Theorem 4.6 actually converge in the intrinsic flat sense to M. One expects an extra assumption on total mass would be needed to interchange between flat and weak convergence, but even so it is not completely clear. One would need to uniformly control the masses of ∂N_i using a common upper bound on $\mathbf{M}(N)$ which can be done using theorems in Section 5 of* [1]*, but is highly technical. It is worth investigating.*

4.3. **Limits of Slices and Balls.** Here we prove that well chosen slices and balls about converging points converge.

Proposition 4.14. *Given an integral current space, $M = (X, d, T)$ and Lipschitz functions, $\rho : X \to \mathbb{R}$ and $f : X \to \mathbb{R}$ such that*

$$|f(x) - \rho_p(x)| < \delta \qquad \forall x \in X, \tag{294}$$

then for almost every $r \in \mathbb{R}$

$$d_{\mathcal{F}}(\mathrm{Slice}(M, \rho, r), \mathrm{Slice}(M, f, r)) \le \|T\|(\rho^{-1}(r - \delta, r + \delta)) + \|\partial T\|(\rho^{-1}(r - \delta, r + \delta)). \tag{295}$$

Proof. First observe that by the definition of intrinsic flat distance,

$$d_{\mathcal{F}}(\text{Sphere}(p,r),\text{Slice}(M,f,r)) \le d_F^{\bar{X}}(< T,\rho,r >,< T,f,r >) \le \mathbf{M}(B)+\mathbf{M}(A)$$

where

$$B = T\llcorner\rho^{-1}(-\infty,r]) - T\llcorner f^{-1}(-\infty,r]) \tag{296}$$
$$A = (\partial T)\llcorner f^{-1}(-\infty,r] - (\partial T)\llcorner\rho^{-1}(-\infty,r]. \tag{297}$$

Next note that for any pair of sets $U,V\subset X$,

$$\mathbf{M}(T\llcorner U - T\llcorner V) = \mathbf{M}(T\llcorner(\chi_U-\chi_V)) = \mathbf{M}(T\llcorner(U\setminus V)) + \mathbf{M}(T\llcorner(V\setminus U)) \tag{298}$$

and the same holds for ∂T. Since

$$\rho^{-1}(-\infty,r]\setminus f^{-1}(-\infty,r]\subset\rho^{-1}(r-\delta,r+\delta) \tag{299}$$

and

$$f^{-1}(-\infty,r]\setminus\rho^{-1}(-\infty,r]\subset\rho^{-1}(r-\delta,r+\delta) \tag{300}$$

we have

$$\mathbf{M}(B) \le \mathbf{M}(T\llcorner(\rho^{-1}(r-\delta,r+\delta)) \tag{301}$$
$$\mathbf{M}(A) \le \mathbf{M}(\partial T\llcorner(\rho^{-1}(r-\delta,r+\delta)). \tag{302}$$

□

Proposition 4.15. *Given two integral current spaces, $M_i=(X_i,d_i,T_i)$, and isometric embeddings $\varphi_i: X_i\to Z$, and points $p_i\in X_i$, then*

$$d_{\mathcal{F}}(S(p_1,r),S(p_2,r)) \le d_F^Z(\varphi_{1\#}T_1,\varphi_{2\#}T_2) \tag{303}$$
$$+\|T_2\|\left(\rho_{p_2}^{-1}(r-\delta,r+\delta)\right)+\|\partial T_2\|\left(\rho_{p_2}^{-1}(r-\delta,r+\delta)\right) \tag{304}$$

where $\delta=d_Z(\varphi_1(p_1),\varphi_2(p_2))$. More generally, if we have points $z_i\in Z$ then

$$d_F^Z(\varphi_{1\#}\,\text{Slice}(M_1,\rho_{z_1}\circ\varphi_1,r)\varphi_{2\#}(\text{Slice}(M_2,\rho_{z_2}\circ\varphi_2,r)\le \tag{305}$$

$$\le d_F^Z(\varphi_{1\#}T_1,\varphi_{2\#}T_2)+\|T_2\|\left(\varphi_2^{-1}\rho_{z_2}^{-1}(r-\delta,r+\delta)\right)+\|\partial T_2\|\left(\varphi_2^{-1}\rho_{z_2}^{-1}(r-\delta,r+\delta)\right). \tag{306}$$

where $\delta=d_Z(z_1,z_2)$.

Proof. By (68), if $F:Z\to\mathbb{R}^k$ is Lipschitz, then for almost every $t\in\mathbb{R}^k$,

$$< \varphi_\# T,F,t > = \varphi_\# < T,F\circ\varphi,t > . \tag{307}$$

By (59), we know that, if

$$< \varphi_{1\#}T_1,F,t > - < \varphi_{2\#}T_2,F,t > = < \partial B,F,t > + < A,F,t > \tag{308}$$
$$= (-1)^k\partial < B,F,t > + < A,F,t > \tag{309}$$

Combining these two facts, we have

$$\begin{aligned} d_F^Z(\varphi_{1\#} < T_1,F\circ\varphi_1,t >,\varphi_{2\#} < T_2,F\circ\varphi_2,t >) &\le \mathbf{M}(< B,F,t >)+\mathbf{M}(< A,F,t >)\\ &\le \prod_{j=1}^k \text{Lip}(F_j)(\mathbf{M}(B)+\mathbf{M}(A))\\ &\le \prod_{j=1}^k \text{Lip}(F_j)(d_{\mathcal{F}}(M_1,M_2)). \end{aligned}$$

Now we apply this to the function $F(z) = \rho_{z_1}(z)$. Then setting $f = \rho_{z_1} \circ \varphi_2$ we have

$$d_F^Z(< T_1, \rho_{z_1} \circ \varphi_1, t >, < T_2, f, t >) \le d_{\mathcal{F}}(M_1, M_2). \tag{310}$$

By the triangle inequality on Z, the function $f : X_2 \to \mathbb{R}$ satisfies (294) for $M = M_2$ and $\rho = \rho_{z_2} \circ \varphi_2$. Thus we have (295). Combining this with (310) gives (305).

Finally (305) implies (303) by taking $z_i = \varphi_1(p_i)$ and noting that $\rho_{z_i} \circ \varphi_i = \rho_{p_i}$. □

Lemma 4.16. *Given an integral current space, (X, d, T), and any Lipschitz function, $f : X \to \mathbb{R}$, we have*

$$\lim_{\delta\to 0} \|T\| \left(f^{-1}(r-\delta, r+\delta)\right) = 0 \tag{311}$$

Proof. In fact, given any bounded Borel measure μ on a complete metric space Z such that $\mu(Z) < \infty$ and any Lipschitz function $f : Z \to \mathbb{R}$, we have for almost every $r \in \mathbb{R}$:

$$\lim_{\delta\to 0} \mu(f^{-1}(r-\delta, r+\delta)) = 0. \tag{312}$$

Assume not. Then there exists $A \subset \mathbb{R}$, such that $\mathcal{L}^1(A) > 0$ where (312) fails. Since A is the countable union of sets A_k where the limit in (312) is greater than $1/k$, we know there exists k such that $\mathcal{L}^1(A_k)$. Let $N = 2k\mu(Z)$, then there exists $r_1 < r_2 < \cdots < r_N$ all in A_k and $\delta_j > 0$ such that

$$\mu(f^{-1}(r-\delta, r+\delta)) > 1/k \qquad \forall \delta < \delta_j. \tag{313}$$

Then taking $\delta'_j < \delta_j$ such that the intervals $(r_j - \delta'_j, r_j + \delta'_j)$ are disjoint, we have

$$\mu(Z) \ge \sum_{j=1}^{N} \mu\left(f^{-1}(r_j - \delta'_j, r_j + \delta'_j)\right) \tag{314}$$

$$\ge \sum_{j=1}^{N} (1/k) = N(1/k) = 2\mu(Z) \tag{315}$$

which is a contradiction. □

Proposition 4.17. *Suppose we have a sequence of integral current spaces, $M_i = (X_i, d_i, T_i)$ and isometric embeddings $\varphi_i : X_i \to Z$, such that*

$$\lim_{i\to\infty} d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) = 0 \tag{316}$$

and points $z_i \in Z$ such that $\delta_i = d_Z(z_i, z_\infty)$, then setting

$$S_i = \varphi_{i\#}\,\mathrm{Slice}(M_i, \rho_{z_i} \circ \varphi_i, r) \tag{317}$$

then for almost every $r \in \mathbb{R}$ we have

$$\begin{aligned} d_F^Z(S_i, S_\infty) \quad &\le \quad d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) && (318)\\ &+ \|T_\infty\|\left(f^{-1}(r-\delta_i, r+\delta_i)\right) + \|\partial T_\infty\|\left(f^{-1}(r-\delta_i, r+\delta_i)\right) && (319)\\ & && (320) \end{aligned}$$

where $f(x) = \rho_{z_\infty}(\varphi_\infty(x))$. If $\delta_i \to 0$, then for almost every $r \in \mathbb{R}$,

$$\lim_{i\to\infty} d_F^Z(S_i, S_\infty) = 0. \tag{321}$$

In our applications this proposition will be applied to $z_i = \varphi_i(p_i)$ where $p_i \in X_i$ are converge (or are Cauchy) in which case $\rho_{z_i}(\varphi_i(x)) = \rho_{p_i}(x)$. Thus, spheres

$$d_{\mathcal{F}}(\text{Sphere}(p_i, r), \text{Sphere}(p_\infty, r)) \to 0 \text{ for almost every } r \in \mathbb{R} \tag{322}$$

if $p_i \in X_i$ converge to $p_\infty \in X_\infty$. This convergence of spheres was applied without an explicit proof in [22]. The precise bound obtained here is needed to study limits of sliced filling volumes.

Proof. We know that for almost every $r \in \mathbb{R}$, the slices are well defined. First we may apply Proposition 4.15 with $M_i = M_1$ and $M_\infty = M_2$. Note that

$$\rho_{z_\infty}(\varphi(x)) = \rho_{p_\infty}(x) \qquad \forall x \in X_\infty. \tag{323}$$

The rest follows from Lemma 4.16. □

Proposition 4.18. *Suppose we have a sequence of integral current spaces, $M_i = (X_i, d_i, T_i)$ and isometric embeddings $\varphi_i : X_i \to Z$, such that*

$$d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) = 0 \tag{324}$$

and points $z_i \in Z$ such that $\delta_i = d_Z(z_i, z_\infty)$, then for almost every $r \in \mathbb{R}$ the balls, $S_i = \varphi_{i\#}T_i \llcorner B(z_i, r)$, satisfy

$$\begin{aligned} d_F^Z(S_i, S_\infty) \quad &\leq \quad 2d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) && (325)\\ &\quad + \|T_\infty\|\left(f^{-1}(r-\delta_i, r+\delta_i)\right) && (326)\end{aligned}$$

where $f(x) = \rho_{z_\infty}(\varphi_\infty(x))$. If $\delta_i \to 0$, then for almost every $r \in \mathbb{R}$,

$$\lim_{i\to\infty} d_F^Z(S_i, S_\infty) = 0. \tag{327}$$

In our applications this proposition will be applied to $z_i = \varphi_i(p_i)$ where $p_i \in X_i$ are converge (or are Cauchy) in which case $\rho_{z_i}(\varphi_i(x)) = \rho_{p_i}(x)$. Thus, balls

$$d_{\mathcal{F}}(S(p_i, r), S(p_\infty, r)) \to 0 \text{ for almost every } r \in \mathbb{R} \tag{328}$$

if $p_i \in X_i$ converge to $p_\infty \in X_\infty$. This convergence of balls was applied without an explicit proof in [22].

Proof. There exists integral currents A_i, B_i in Z, such that

$$\varphi_{i\#}T_i - \varphi_{\infty\#}T_\infty = A_i + \partial B_i \tag{329}$$

and

$$d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) = \mathbf{M}(A_i) + \mathbf{M}(B_i). \tag{330}$$

For almost every r, the restrictions of these spaces to balls, $B(z_i, r)$, are integral current spaces such that

$$\begin{aligned} (\varphi_{i\#}T_i - \varphi_{\infty\#}T_\infty) \llcorner \bar{B}(z_i, r) \quad &= \quad A_i \llcorner \bar{B}(z_i, r) + (\partial B_i) \llcorner \bar{B}(z_i, r) && (331)\\ &= \quad A_i \llcorner \bar{B}(z_i, r) + < B_i, \rho_i, r > + \partial(B_i \llcorner \bar{B}(z_i, r)) && (332)\end{aligned}$$

Since

$$\mathbf{M}(A_i \llcorner \bar{B}(z_i, r) + < B_i, \rho_i, r >) \leq \mathbf{M}(A_i) + \mathbf{M}(B_i), \tag{333}$$

$$\mathbf{M}(B_i \llcorner \bar{B}(z_i, r)) \leq \mathbf{M}(B_i) \tag{334}$$

and

$$\mathbf{M}\left(\varphi_{\infty\#}T_\infty) \llcorner \bar{B}(z_i, r) - \varphi_{\infty\#}T_\infty) \llcorner \bar{B}(z_\infty, r)\right) \leq \|\varphi_{\infty\#}T_\infty\| Ann_{z_\infty}(r - \delta_i, r + \delta_i) \tag{335}$$

we have our first claim. The rest follows from Lemma 4.16. □

Proposition 4.19. *Suppose we have a sequence of m dimensional integral current spaces, $M_i = (X_i, d_i, T_i)$ and isometric embeddings $\varphi_i : X_i \to Z$, such that*

$$\lim_{i\to\infty} d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) = 0 \tag{336}$$

and points $z_{j,i} \in Z$ such that $\delta_{j,i} = d_Z(z_{j,i}, z_{j,\infty})$ for $j = 1..k \le m$ then

$$d_F^Z(\varphi_{i\#}\operatorname{Slice}(M_i,\rho_i,r), \varphi_{\infty\#}(\operatorname{Slice}(M_\infty,\rho_\infty,r) \le d_F^Z(\varphi_{i\#}T_i,\varphi_{\infty\#}T_\infty)+\sum_{j=1}^k (a_j(r_1,...,r_k)+b_j(r_1,...,r_k)) \tag{337}$$

where

$$a_j(r_1,...,r_k) = \|(\operatorname{Slice}(T_\infty, f_1,...,f_j,r_1,...,r_j)\|\left(\bigcap_{n=1}^{j}\left(f_n^{-1}(r-\delta_{n,i}, r+\delta_{n,i})\right)\right) \tag{338}$$

and

$$b_j(r_1,...,r_k) = \|(\operatorname{Slice}(\partial T_\infty, f_1,...,f_j,r_1,...,r_j)\|\left(\bigcap_{n=1}^{j}\left(f_i^{-1}(r-\delta_{n,i}, r+\delta_{n,i})\right)\right) \tag{339}$$

where $f_i(x) = \rho_{z_\infty,i}(\varphi_\infty(x))$. If $\lim_{i\to\infty}\delta_{j,i} = 0$ then for almost every $r \in \mathbb{R}^k$,

$$\lim_{i\to\infty} d_F^Z(\varphi_{i\#}\operatorname{Slice}(M_i,\rho_i,r), \varphi_{\infty\#}(\operatorname{Slice}(M_\infty,\rho_\infty,r) = 0 \tag{340}$$

where $\rho_i : M_i \to \mathbb{R}^k$ is defined by

$$\rho_i(x) = (\rho_{z_{1,i}}(\varphi_i(x)), \rho_{z_{2,i}}(\varphi_i(x)), ..., \rho_{z_{k,i}}(\varphi_i(x))). \tag{341}$$

In our applications this proposition will be applied to $z_{j,i} = \varphi_i(p_{j,i})$ where $p_{j,i} \in X_i$ in which case $\rho_{z_{j,i}}(\varphi_i(x)) = \rho_{p_{j,i}}(x)$.

Proof. We will prove this by induction on k. The base step has been proven in Proposition 4.17. By the induction hypothesis, for almost every $r \in \mathbb{R}$, we have a sequence of the integral current spaces $N_i = \operatorname{Slice}(M_i, \rho_i, s)$ such that

$$d_F^Z(\varphi_{1\#}N_i, \varphi_{\infty\#}N_\infty) = d_{F,i,k} \tag{342}$$

We introduce two new points $z_{k+1,i} \in Z$ and set

$$\delta_{k+1,i} = d_Z(z_{k+1,i}), z_{k+1,\infty}). \tag{343}$$

Applying Proposition 4.17, we see that for almost every $r_{k+1} \in \mathbb{R}$ we have

$$d_{F,i,(k+1)} \le d_{F,i,k} + a_{k+1} + b_{k+1} \tag{344}$$

The first inequality in the proposition follows from

$$\operatorname{Slice}(N, \rho_{k+1}, r_{k+1}) = \operatorname{Slice}(\operatorname{Slice}(M, \rho_1, ..., \rho_k, r_1, ..., r_k), \rho_{k+1}, r_{k+1}) \tag{345}$$

$$= \operatorname{Slice}(M, \rho_1, ..., \rho_{k+1}, r_1, ..., r_{k+1}). \tag{346}$$

For almost every $r \in \mathbb{R}^k$, the k^{th} slice of T_∞ is an integral current space, so we may apply Lemma 4.16, to obtain the limit. □

4.4. **Continuity of Sliced Filling Volumes.** Recall that Theorem 2.46 implies the continuity of filling volume in the following sense:

$$M_i \xrightarrow{\mathcal{F}} M_\infty \quad \Longrightarrow \quad \operatorname{FillVol}(\partial M) \to \operatorname{FillVol}(\partial M) \tag{347}$$

where the filling volume is defined as in Definition 2.44. In this section we combine Theorem 2.46 in combination with the convergence of slices proven in Proposition 4.17 and Proposition 4.19. An immediate consequence of these results is that the filling volumes of slices converge. In particular the filling volumes of spheres converge to the filling volumes spheres, as stated in [23]. Finally we prove the sliced filling volumes are continuous via an application of the Dominated Convergence Theorem of Real Analysis.

We have moved a remark from v1 into the statement of the theorem so the proof of the Tetrahedral Compactness Theorem is clearer.

Theorem 4.20. *Suppose $M_i \xrightarrow{\mathcal{F}} M$ and $\operatorname{Diam}(M_i) \le D_0$. If $p_i \in M_i$ converge to $p_\infty \in M_\infty$, and $q_{j,i} \in M_\infty$ converge to $q_{j,\infty} \in M_\infty$ for $j = 1..k$ then*

$$\lim_{i\to\infty} \mathbf{SF}(p_i, r, q_{1,i}, ...q_{k,i}) = \mathbf{SF}(p_\infty, r, q_{1,\infty}, ...q_{k,\infty}) \tag{348}$$

Thus for $k \in \{1, ..., m-1\}$,

$$\liminf_{i\to\infty} \mathbf{SF}_k(p_i, r) \ge \mathbf{SF}_k(p_\infty, r). \tag{349}$$

while by continuity of the filling volumes we have,

$$\lim_{i\to\infty} \mathbf{SF}_0(p_i, r) = \mathbf{SF}_0(p_\infty, r) \le \mathbf{M}(S(p_\infty, r)), \tag{350}$$

Furthermore if M_i have a Gromov-Hausdorff limit (which is not necessarily M_∞) and $\mathbf{M}(M_i) + \mathbf{M}(\partial M_i) \le C$ then

$$\mathbf{M}(S(p_\infty, r)) \ge \liminf_{i\to\infty} \mathbf{SF}_k(p_i, r). \tag{351}$$

Before proving this theorem we first state and prove the following technical proposition:

Proposition 4.21. *Suppose we have a sequence of m dimensional integral current spaces, $M_i = (X_i, d_i, T_i)$ and isometric embeddings $\varphi_i : X_i \to Z$, such that*

$$d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) = 0 \tag{352}$$

and points $z_{j,i} \in Z$ such that $\delta_{j,i} = d_Z(z_{j,i}, z_{j,\infty})$ for $j = 1..k$ for some $k \in \{0, .., m-1\}$ and $p_i \in X_i$, such that $\delta_i = d_Z(\varphi_i(p_i), \varphi(p))$, then for almost every $t \in \mathbb{R}^k$

$$|\operatorname{FillVol}(\partial \operatorname{Slice}(M_i, \rho_i, t)) - \operatorname{FillVol}(\partial \operatorname{Slice}(M_\infty, \rho_\infty, t))| \le d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) + \sum_{j=1}^k (a_j(t) + b_j(t)) \tag{353}$$

where a_j, b_j are defined in (338)-(339) and ρ_i is defined in (341). If $\lim_{i\to\infty} \delta_{j,i} = 0$ then for almost every $r \in \mathbb{R}$ and $r \in \mathbb{R}^k$, the filling volumes of slices converge,

$$\lim_{i\to\infty} \operatorname{FillVol}(\partial \operatorname{Slice}(M_i, \rho_i, t)) = \operatorname{FillVol}(\partial \operatorname{Slice}(M_\infty, \rho_\infty, t)) \tag{354}$$

while the masses satisfy

$$\liminf_{i\to\infty} \mathbf{M}(\operatorname{Slice}(M_i, \rho_i, t)) \ge \mathbf{M}(\operatorname{Slice}(M_\infty, \rho_\infty, t)). \tag{355}$$

Furthermore, if $\operatorname{Diam}(M_i) \le D_0$ then

$$\lim_{i\to\infty} \int_{t\in[-R,R]^k} \operatorname{FillVol}(\partial \operatorname{Slice}(M_i, \rho_i, t))\, \mathcal{L}^k = \int_{t\in[-R,R]^k} \operatorname{FillVol}(\partial \operatorname{Slice}(M_\infty, \rho_\infty, t))\, \mathcal{L}^k. \tag{356}$$

where

$$R = 2D_0 + \sup\{\delta_{j,i} : \; i = 1..\infty, \; j = 1..k\} < \infty\} \tag{357}$$

Proof. Our first claim and the limits of the filling volumes of slices and the lower semicontinuity of the mass follows immediately from Proposition 4.19 combined with Theorem 2.46 and Theorem 2.42 respectively. In fact, we have (353). Since

$$\begin{aligned}
\int_{[-R,R]^k} \operatorname{FillVol}(\partial \operatorname{Slice}(M_\infty, \rho_i, t)) \; \mathcal{L}^k &\le \int_{[-R,R]^k} \mathbf{M}(\operatorname{Slice}(M_\infty, \rho_i, t))\mathcal{L}^k \\
&\le \mathbf{M}(M_\infty) \\
\int_{[-R,R]^k} a_j(t)\mathcal{L}^k &\le \int_{[-R,R]^k} \mathbf{M}(\operatorname{Slice}(T_\infty, \rho_\infty, t))\mathcal{L}^k \\
&\le \mathbf{M}(M_\infty) \\
\int_{[-R,R]^k} b_j(t)\mathcal{L}^k &\le \int_{[-R,R]^k} \mathbf{M}(\operatorname{Slice}(\partial T_\infty, \rho_\infty, t))\mathcal{L}^k \\
&\le \mathbf{M}(\partial M_\infty).
\end{aligned}$$

Thus we may apply the Dominated Convergence Theorem of Real Analysis to obtain the limit of the integrals. □

We may now prove Theorem 4.20:

Proof. The convergence of the Sliced Filling Volumes with respect to converging points follows from Proposition 4.18 and Definition 3.20 and Theorem 4.21 stated and proven below. The lower semicontinuity of the k^{th} sliced filling then follows taking $q_{j,\infty}$ close to achieving the supremum in the definition of $\mathbf{SF}_k(p_\infty, r)$ and $q_{j,i} \to q_{j,\infty}$ so that so that

$$\liminf_{i\to\infty} \mathbf{SF}_k(p_i, r) \ge \lim_{i\to\infty} \mathbf{SF}(p_i, r, q_{j,i}) = \mathbf{SF}(p_\infty, r, q_{j,\infty}) \ge \mathbf{SF}_k(p_\infty, r) - \epsilon \tag{358}$$

and then taking $\epsilon \to 0$.

Note that we only need a subsequence satisfying (351) to obtain (351) by the definition of liminf. We are now assuming M_i has a uniform bound on $\mathbf{M}(M_i) + \mathbf{M}(\partial M_i)$ and it has a Gromov-Hausdorff limit. In that case, applying Gromov's Embedding Theorem and Ambrosio-Kirchheim's Compactness Theorem, a subsequence of the M_i isometrically embed into a compact common space Z with flat convergence in that compact space to an image of M_∞ by uniqueness of intrinsic flat limits.

Now take $q_{j,i}$ close to achieving the supremum in the definition of $\mathbf{SF}_k(p_i, r)$ so that

$$\lim_{i\to\infty} |\mathbf{SF}_k(p_i, r) - \mathbf{SF}(p_i, r, q_{j,i})| = 0. \tag{359}$$

When $\varphi_i : M_i \to Z$ where Z is compact, a subsequence of the $\varphi_i(q_{j,i}) \to z_{j,\infty}$ for $j = 1, ..., k$. Thus

$$\liminf_{i\to\infty} \mathbf{SF}_k(p_i, r) = \liminf_{i\to\infty} \mathbf{SF}(p_i, r, q_{j,i}) \tag{360}$$

$$= \liminf_{i\to\infty} \mathbf{SF}(\varphi(p_i), r, \varphi_i(q_{j,i})) \tag{361}$$

$$= \mathbf{SF}(\varphi_\infty(p_\infty), r, z_{j,\infty}) \tag{362}$$

$$\le \mathbf{M}(S(p_\infty, r)). \tag{363}$$

□

Remark 4.22. *Note that the extra hypothesis used to achieve (351 is used only to obtain limits of the $q_{i,j}$ close to achieving the suprema in the definition of $\mathbf{SF}_k(p_i, r)$ and is clearly not necessary if one has other means of proving they have limit points or if one can prove* *the Lipschitz functions $\rho_{\varphi(p_i)} : Z \to \mathbb{R}$ converge strongly enough. It may be possible to prove Theorem 4.20 without that extra hypothesis.*

4.5. **Continuity of Interval Filling Volumes.** Recall the definition of the interval filling volume of a manifold or integral current space in Definition 3.43,

$$\mathbf{IFV}_\epsilon(M) = \mathrm{FillVol}(\partial(M \times I_\epsilon)) \le \epsilon \mathbf{M}(M). \tag{364}$$

This notion was particularly useful for M without boundary. In this section we prove the interval filling volume is continuous with respect intrinsic flat convergence [Theorem 4.23]. Taking more precise estimates we prove the sliced interval filling volumes are continuous as well [Theorem 4.24].

Theorem 4.23. *Suppose we have m dimensional integral current spaces, $M_i = (X_i, d_i, T_i)$, such that $M_i \xrightarrow{\mathcal{F}} M_\infty$, then for any fixed $\epsilon > 0$, their interval filling volumes converge,*

$$\lim_{i\to\infty} \mathbf{IFV}_\epsilon(M_i) = \mathbf{IFV}_\epsilon(M_\infty) \tag{365}$$

.

Proof. By Proposition 3.9, we see that $M_i \times I_\epsilon \xrightarrow{\mathcal{F}} M_\infty \times I_\epsilon$. Thus we have continuity applying Theorem 2.46. □

We now prove the continuity of the sliced interval filling volume defined in Definition 3.45.

Theorem 4.24. *Suppose $M_i \xrightarrow{\mathcal{F}} M$ and $\mathrm{Diam}(M_i) \le D_0$. If $p_i \in M_i$ converge to $p_\infty \in M_\infty$, and $q_{j,i} \in M_\infty$ converge to $q_{j,\infty} \in M_\infty$ for $j = 1..k$ then for any fixed $\epsilon > 0$,*

$$\lim_{i\to\infty} \mathbf{SIF}_\epsilon(p_i, r, q_{1,i}, ...q_{k,i}) = \mathbf{SIF}_\epsilon(p_\infty, r, q_{1,\infty}, ...q_{k,\infty}). \tag{366}$$

Proof. This theorem is a consequence of Proposition 4.25 stated and proven immediately below, combined with Proposition 4.18 and Definition 3.45. □

Proposition 4.25. *Suppose we have a sequence of m dimensional integral current spaces, $M_i = (X_i, d_i, T_i)$ and isometric embeddings $\varphi_i : X_i \to Z$, such that*

$$\lim_{i\to\infty} d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty) = 0 \tag{367}$$

and points $z_{j,i} \in Z$ such that $\delta_{j,i} = d_Z(z_{j,i}, z_{j,\infty})$ for $j = 1..k$ for some $k \in \{0, .., m-1\}$ and $p_i \in X_i$, such that $\delta_i = d_Z(\varphi_i(p_i), \varphi(p))$, then for any fixed $\epsilon > 0$ and almost every $t \in \mathbb{R}^k$

$$|\mathbf{IFV}_\epsilon(\mathrm{Slice}(M_i, \rho_i, r)) - \mathbf{IFV}_\epsilon(\mathrm{Slice}(M_\infty, \rho_\infty, r))| \le (2+\epsilon)\left(D_i + + \sum_{j=1}^{k}(a_j + b_j)\right) \tag{368}$$

where a_j, b_j are defined in (338)-(339) and ρ_i is defined in (341). If $\lim_{i\to\infty} \delta_{j,i} = 0$ then for almost every $r \in \mathbb{R}$ and $r \in \mathbb{R}^k$, the filling volumes of slices converge,

$$\lim_{i\to\infty} \mathbf{IFV}_\epsilon(\mathrm{Slice}(M_i, \rho_i, t)) = \mathbf{IFV}_\epsilon(\mathrm{Slice}(M_\infty, \rho_\infty, t)). \tag{369}$$

Furthermore, if $\mathrm{Diam}(M_i) \le D_0$ then

$$\lim_{i\to\infty} \int_{t\in[-R,R]^k} \mathbf{IFV}_\epsilon(\mathrm{Slice}(M_i, \rho_i, t))\, \mathcal{L}^k = \int_{t\in[-R,R]^k} \mathbf{IFV}_\epsilon(\mathrm{Slice}(M_\infty, \rho_\infty, t))\, \mathcal{L}^k. \tag{370}$$

where

$$R = 2D_0 + \sup\{\delta_{j,i} : i = 1..\infty,\ j = 1..k\} < \infty \tag{371}$$

Proof. Let $D_i = d_F^Z(\varphi_{i\#}T_i, \varphi_{\infty\#}T_\infty)$ so $D_i \to 0$. By Proposition 4.19 we have

$$d_{\mathcal{F}}(\mathrm{Slice}(M_i,\rho_i,r), \mathrm{Slice}(M_\infty,\rho_\infty,r)) \le D_i + \sum_{j=1}^k (a_j + b_j). \tag{372}$$

Next applying Proposition 3.9, we have

$$d_{\mathcal{F}}(\mathrm{Slice}(M_i,\rho_i,r)\times I_\epsilon, \mathrm{Slice}(M_\infty,\rho_\infty,r)\times I_\epsilon)) \le (2+\epsilon)\left(D_i + + \sum_{j=1}^k (a_j + b_j)\right). \tag{373}$$

By Theorem 2.46 we then have

$$|\mathrm{FillVol}(\partial N_i) - \mathrm{FillVol}(\partial N_\infty)| \le d_{\mathcal{F}}(N_1, N_2). \tag{374}$$

Taking $N_i = \mathrm{Slice}(M_i,\rho_i,r)\times I_\epsilon$, we have (368). which gives us (369). We cannot just integrate and obtain a limit as in (370) because a_j and b_j are functions. So we observe that

$$\begin{aligned}
\int_{[-R,R]^k} \mathbf{IFV}_\epsilon(\mathrm{Slice}(M_\infty,\rho_i,t))\ \mathcal{L}^k &\le \int_{[-R,R]^k} \mathbf{M}(\mathrm{Slice}(M_\infty,\rho_i,t))\mathcal{L}^k \\
&\le \mathbf{M}(M_\infty) \\
\int_{[-R,R]^k} a_j(t)\mathcal{L}^k &\le \int_{[-R,R]^k} \mathbf{M}(\mathrm{Slice}(T_\infty,\rho_\infty,t))\mathcal{L}^k \\
&\le \mathbf{M}(M_\infty) \\
\int_{[-R,R]^k} b_j(t)\mathcal{L}^k &\le \int_{[-R,R]^k} \mathbf{M}(\mathrm{Slice}(\partial T_\infty,\rho_\infty,t))\mathcal{L}^k \\
&\le \mathbf{M}(\partial M_\infty).
\end{aligned}$$

Thus we may apply the Dominated Convergence Theorem of Real Analysis to obtain the limit of the integrals in (370). □

4.6. **Limits of Points.** In this section we prove two statements about Cauchy and converging sequences of points. Recall also Definition 4.1 and Definition 4.2. Proposition 4.26 assumes one has a limit point either in the limit space or the metric completion of the limit space and draws consequences about the Cauchy sequence converging to that point. Theorem 4.27 assumes one has a Cauchy sequence and determines when the Cauchy sequence has a limit point in the limit space or in the metric completion of the limit space.

Recall that given an integral current space (X,d,T), then any isometric embedding $\varphi : X \to Z$ with Z complete maps X isometrically onto set($\varphi_\#$T) and extends to $\varphi : \bar{X} \to Z$ which maps isometrically onto spt($\varphi_\# T$). Recall that a point $z \in \mathrm{spt}(S)$ iff for every $r > 0$, $T \llcorner \bar{B}(p,r) \ne 0$.

Recall Definition 3.21 of $\mathbf{SF}_k(p,r)$ and that $\mathbf{SF}_0(p,r) = \mathrm{FillVol}(\partial S(p,r))$.

Proposition 4.26. *If M_i are integral current spaces and $d_{\mathcal{F}}(M_i,M_\infty) \to 0$ and $p \in \bar{M}_\infty$, then there exists $p_i \in M_i$ and $r_p > 0$ such that p_i converge to p and for any such converging sequence of p_i,*

$$\mathbf{M}(S(p_i,r)) \ge \mathbf{SF}_k(p_i,r) \ge C_{k,r} > 0 \tag{375}$$

for $k = 0$ and for almost every $r < r_p$.

If $p \in M_\infty$, then there exists $p_i \in M_i$ and $r_p > 0$ and $C_p > 0$ such that p_i converge to p and any such converging sequence has

$$\mathbf{M}(S(p_i,r)) > C_p r^m \tag{376}$$

for almost every $r < r_p$.

Note that (376) does not claim that $\mathbf{SF}_k(p_i, r) > C_p r^k$ even for $k = 0$ because converse part of Lemma 3.22 is not powerful enough to give this level of control. It would be nice to prove (375) for any $k \le m-1$ but again Lemma 3.22 is not yet powerful enough to do this.

Proof. If $p \in \bar{M}_\infty$, then for all $r > 0$, $B_p(r) \cap M \neq \emptyset$. So for almost every $r > 0$, $S(p, r) \neq 0$. So by the converse part of Lemma 3.22 $\mathbf{SF}_k(p, r) = C_{k,r} > 0$ for $k = 0$. Taking $z_\infty = \varphi_\infty(p)$, we know $\varphi_{\infty\#}T_\infty \cap B(z_\infty, r)$ is a nonzero integral current and since $\varphi_{i\#}T_i$ converge to $\varphi_{\infty\#}T_\infty$ in Z, we know that for almost every $r > 0$, there exists N_r sufficiently large that $S_i = \varphi_{i\#}T_i \cap B(z_\infty, r)$ is a nonzero integral current. In particular there is a point

$$z_i \in \mathrm{set}(\varphi_{i\#}\mathrm{T_i}) \cap \mathrm{B}(z_\infty, \mathrm{r}). \tag{377}$$

Since $\varphi_i : X_i \to \mathrm{set}(\varphi_{i\#}\mathrm{T_i})$ is an isometry, there exists $p_i \in X_i$, such that $\varphi_i(p_i) = z_i \to z_\infty$.

By Theorem 4.20, we have

$$\liminf_{i\to\infty} \mathbf{SF}_k(p_i, r) \ge \mathbf{SF}_k(p, r) \tag{378}$$

and

$$\liminf_{i\to\infty} \mathbf{M}(S(p_i, r)) \ge \mathbf{M}(S(p, r)) \tag{379}$$

which is nonzero when $p \in \bar{M}_\infty$ and is $\ge C_p r^m$ when $p = p_\infty \in M_\infty$. The theorem is proven. □

The next theorem concerns Cauchy sequences of points in the sense of Definition 4.2.

Theorem 4.27. *Suppose M_i are integral current spaces of dimension m with M_i converge to M_∞ in the intrinsic flat sense. Let $k \in \{0, 1, 2..., m-1\}$. If $k \neq 0$ assume that $\mathbf{M}(M_i) + \mathbf{M}(\partial M_i) \le C$ and and M_i also have a Gromov-Hausdorff limit not necessarily isometric to M_∞ . Suppose that $p_i \in M_i$ are Cauchy. If there is a uniform constant $r_0 > 0$ such that we have*

$$S(p_i, r) \neq 0 \text{ and } \mathbf{SF}_k(B(p_i, r)) \ge C_{k,r} > 0 \text{ for a.e. } r < r_0, \tag{380}$$

then p_i converge to a point $p_\infty \in \bar{M}_\infty$. If we have a uniform constant $C > 0$ and

$$\mathbf{SF}_k(p_i, r) \ge C_{SF} r^m > 0 \qquad \text{for a.e. } r < r_0, \tag{381}$$

then p_i converge to a point $p_\infty \in M_\infty$.

This theorem statement has been updated in v2 but the proof is as before.

Proof. By the definition of a Cauchy sequences of points $p_i \in M_i$ we have the hypothesis of Theorem 4.20 with $z_i = \varphi_i(p_i)$ and a point $z_\infty = \lim_{i\to\infty} z_i$.

Thus applying Theorem 4.20 to balls $S(z_i, r)$ which converge to $S(z_\infty, r)$ for almost every $r \in (0, r_0)$ as in Proposition 4.18, we have

$$\mathbf{M}(S(z_\infty, r)) \ge \liminf \mathbf{SF}_k(p_i, r) \ge C > 0 \tag{382}$$

By Lemma 3.22 we have

$$\mathbf{M}(\varphi_{\infty\#}T_\infty \llcorner \bar{B}(z, r)) = \mathbf{M}(S(z_\infty, r)) \ge C_{k,r} > 0 \tag{383}$$

and so $\varphi_{i\#}T_i \cap B(z_\infty, r) \neq 0$. So $z_\infty \in \mathrm{spt}(T_\infty) = Cl(\mathrm{set}(\mathrm{T}_\infty))$. So there exists $p_\infty \in \bar{X}_\infty$ such that $\varphi_\infty(p_\infty) = z_\infty$ Thus p_i have a limit point $p_\infty \in \bar{M}_\infty$.

Once again applying Theorem 4.20 to balls $S(z_i, r)$ which converge to $S(z, r)$ as in Proposition 4.18, we have

$$\mathbf{M}(S(z_\infty, r)) \geq \liminf \mathbf{SF}_k(p_i, r) \geq C_{SF} r^m. \tag{384}$$

By Lemma 3.22 we have

$$\liminf_{r\to 0} \frac{\mathbf{M}(\varphi_{\infty\#} T_\infty \llcorner \bar{B}(z, r))}{r^m} = \liminf_{r\to 0} \frac{\mathbf{M}(S(z_\infty, r))}{r^m} \geq C_{SF} > 0 \tag{385}$$

and so $z_\infty \in \mathrm{set}(\varphi_{\infty\#} \mathrm{T}_\infty)$, and so there exists $p_\infty \in X_\infty$ such that $\varphi_\infty(p_\infty) = z_\infty$ Thus p_i have a limit point $p_\infty \in M_\infty$. □

Example 4.28. *It is quite possible for a Cauchy sequence of points to have more than one limit as can be seen simply by taking the constant sequence of integral current spaces, S^1, and noting that due to the isometries, any point may be set up as the limit of a Cauchy sequence of points. One may also use isometries of S^1 to relocate a Cauchy sequence so that the images are no longer Cauchy in Z. This is also true of converging sequences in the theory of Gromov-Hausdorff convergence.*

Remark 4.29. *One can remove the extra hypothesis for $k \neq 0$ in Theorem 4.27, if one can remove that extra hypothesis in Theorem 4.20. See Remark 4.22.*

4.7. **Bolzano-Weierstrass Theorems.** This subsection which was vaguely announced in v1 is completed here.

When one has a sequence of compact metric spaces converging in the Gromov-Hausdorff sense to a compact metric space, and one has a sequence of points in those metric spaces, then a subsequence converges to a point in the Gromov-Hausdorff limit. This is the Gromov-Hausdorff Bolzano-Weierstrass Theorem and is an immediate consequence of Gromov's Embedding Theorem which provides a common metric space which is compact. The immediate restatement of the Gromov-Hausdorff Bolzano-Weierstrass Theorem is not true when the spaces converge in the intrinsic flat sense instead of the Gromov-Hausdorff sense. This can be seen in Ilmanen's Example with disappearing tips. The key difficulty lies in the fact that, unlike Gromov's Embedding Theorem, Theorem 2.40 does not provide a compact common metric space.

Nevertheless we are able to prove the following Bolzano-Weierstrass Theorems by assuming the limit space is compact and preventing the points in the sequence from disappearing. This theorem is strong enough for many applications. It applies to sequences of complete oriented Riemannian manifolds M_i^m with boundary such that $\mathrm{Vol}(M_i) \leq V_0$ and $\mathrm{Vol}(\partial M_i) \leq A_0$. Recall our notion of filling volume in Definition 2.44.

Theorem 4.30. *Suppose $M_i^m = (X_i, d_i, T_i)$ are integral current spaces with a uniform upper bound on mass, $\mathbf{M}(M_i^m) \leq V_0$ and on boundary mass, $\mathbf{M}(\partial M_i^m) \leq A_0$. Suppose M_i^m converge in the intrinsic flat sense to a precompact nonzero limit integral current space $M_\infty^m = (X_\infty, d_\infty, T_\infty)$. Suppose there exists $r_0 > 0$ and a sequence $p_i \in M_i$ such that*

$$\mathrm{FillVol}(\partial S(p_i, r)) \geq C_{0,r} > 0 \qquad \text{for a.e. } r \in (0, r_0]. \tag{386}$$

Then there exists a subsequence, i_j, such that p_{i_j} converges to $p_\infty \in \bar{M}_\infty^m$. If in addition, there is a uniform constant $C > 0$ such that for some k we have

$$\mathrm{FillVol}(\partial S(p_i, r)) \geq C r^m \qquad \text{for a.e. } r \in (0, r_0]. \tag{387}$$

then the subsequence converges to a point $p_\infty \in M_\infty^m$.

This theorem is a special case of the following more general Bolzano-Weierstrass Theorem which only requires controls on the sliced filling volumes because the $k = 0$ sliced filling is not sliced: $\mathbf{SF}_0(p, r) = \mathrm{FillVol}(\partial S(p, r))$.

Theorem 4.31. *Suppose $M_i^m = (X_i, d_i, T_i)$ are integral current spaces with a uniform upper bound on mass, $\mathbf{M}(M_i^m) \le V_0$ and on boundary mass, $\mathbf{M}(\partial M_i^m) \le A_0$. Suppose M_i^m converge in the intrinsic flat sense to a precompact nonzero limit integral current space $M_\infty^m = (X_\infty, d_\infty, T_\infty)$ Suppose there exists $k \in \{0, 1, ..., (m-1)\}$, $r_0 > 0$ and a sequence $p_i \in M_i$ such that*

$$S(p_i, r) \neq 0 \text{ and } \mathbf{SF}_k(p_i, r) \ge C_{k,r} > 0 \qquad \text{for a.e. } r \in (0, r_0]. \tag{388}$$

If $k \neq 0$ suppose that the sequence M_i^m also has a Gromov-Hausdorff limit which may not be isometric to M_∞^m.

Then there exists a subsequence, i_j, such that p_{i_j} converges to $p_\infty \in \bar{M}_\infty^m$. If in addition, there is a uniform constant $C > 0$ such that for some k we have

$$\mathbf{SF}_k(p_i, r) \ge C r^m \qquad \text{for a.e. } r \in (0, r_0]. \tag{389}$$

then the subsequence converges to a point $p_\infty \in M_\infty^m$. This theorem holds without the compactness of the common space Z when $k = 0$ and we are controlling filling volumes of spheres.

Proof. By Theorem 2.40 we have $\varphi_i : M_i \to Z$ such that $d_F^Z(\varphi_{i\#} T_i, \varphi_\infty T_\infty) \to 0$ where Z is a complete metric space. Thus $\varphi_{i\#} T_i$ converges weakly to $\varphi_\infty T_\infty$ in Z.

Since M_∞ is precompact, $K = \varphi_\infty(\bar{M}_\infty) \subset Z$, is compact. Let $f : Z \to [0, \infty)$ be the Lipschitz function

$$f(z) = -d(z, K). \tag{390}$$

By Ambrosio-Kirchheim's Slicing Theorem [Theorem 2.23], for almost every $s \in \mathbb{R}$ we have

$$\varphi_{i\#} T_i \llcorner f^{-1}(-\infty, s]) \in \mathbf{I}_m(Z). \tag{391}$$

Since $\varphi_{i\#} T_i$ converges weakly to $\varphi_\infty T_\infty$ their restrictions to $f^{-1}(-\infty, s])$ converge weakly as well. Note that for $s < 0$,

$$\varphi_{\infty\#} T_\infty \llcorner f^{-1}(-\infty, s]) = 0. \tag{392}$$

Thus these exists I_s sufficiently large (depending on s), such that for all $i \ge I_s$

$$\varphi_{i\#} T_i \llcorner f^{-1}(-\infty, s]) \text{ converges weakly to } 0. \tag{393}$$

Let $z_i = \varphi_i(p_i)$. Let $q_i \in K$ be the point closest to z_i lying in K. Since K is compact, there exists a subsequence i_j and a limit point $q_\infty \in K$ such that $q_{i_j} \to q_\infty$.

We claim that z_{i_j} converges to q_∞ in Z. Once this claim is proven, we know p_{i_j} is Cauchy and the rest of the proof follows from Theorem 4.27.

Suppose on the contrary that z_{i_j} does not converge to q_∞. Then there exists $\delta > 0$ such that $z_{i_j} \notin B(q_\infty, \delta)$ for all $j \in \mathbb{N}$. By the choice of q_i, taking j sufficiently large we see that

$$z_{i_j} \notin T_{\delta/2}(K). \tag{394}$$

Take any $r < \min\{\delta/4, r_0\}$ such that (388) holds. Then

$$\varphi_{i_j}(\bar{B}(z_i, r)) \cap T_{\delta/4}(K) = \emptyset. \tag{395}$$

Taking $s = -\delta/4$, we see that

$$\mathrm{spt}(\varphi_{i_j,\#} S(p_{i_j}, r)) \subset \mathrm{spt}(\varphi_{i_j\#} T_{i_j} \llcorner f^{-1}(-\infty, s])). \tag{396}$$

In fact

$$\varphi_{i_j,\#}S(p_{i_j}, r) = \varphi_{i_j\#}T_{i_j} \llcorner f^{-1}(-\infty, s]) \llcorner \bar{B}(z_{i_j}, r), \tag{397}$$

which converges weakly to 0 by (393). This is true for any $r < \min\{\delta/4, r_0\}$ such that (388) holds. Since $\mathbf{M}M_i \le V_0$ and $\mathbf{M}(\partial M_i) \le A_0$ for all i, we can apply the Slicing Theorem over the countable collection to see that there exists $V_1 > 0$ and $A_1 > 0$ and a set $R \in (0, r)$ of measure $\ge r/2$ such that for all $r' \in R$ we have

$$\mathbf{M}(S(p_i, r')) \le V_1 \text{ and } \mathbf{M}(\partial S(p_i, r')) \le A_1. \tag{398}$$

Thus we can apply Wengers's Weak Implies Flat Convergence Theorem [24]. This implies that

$$d_{\mathcal{F}}(S(p_{i_j}, r'), \mathbf{0}) \le \mathbf{d_F^Z}(\boldsymbol{\varphi_{i_j,\#}S(p_{i_j}, r), 0}) \to \mathbf{0}. \tag{399}$$

Viewing $S(p_{i_j}, r')$ as our sequence of spaces and observing that $\mathrm{Diam}(S(p_{i_j}, r')) \le 2r'$ we can apply Theorem 4.20 to obtain

$$\liminf_{i\to\infty} \mathbf{SF}_k(p_{i_j}, r') \le 0 \tag{400}$$

which contradicts $\mathbf{SF}_k(p_{i_j}, r') \ge C_{k,r'} > 0$. □

Remark 4.32. *The additional hypothesis in Theorem 4.31 for k $\neq 0$ is only need to apply Theorem 4.20. This additional assumption may not be necessary (see Remark 4.29). The proof above follows exactly without that hypothesis if it can be removed from Theorem 4.20.*

5. Arzela-Ascoli Theorems

This new section was created to keep the Arzela-Ascoli Theorems together

In this section we prove three Arzela-Ascoli Theorems. Recall that the Gromov-Hausdorff Arzela-Ascoli Theorem states that if one has compact metric spaces $X_i \xrightarrow{\mathrm{GH}} X_\infty$ and $W_i \xrightarrow{\mathrm{GH}} W_\infty$ and $F_i : X_i \to W_i$ have $\mathrm{Lip}(F_i) \le \lambda$ then there is a limiting continuous function $F_\infty : X_\infty \to W_\infty$ (c.f. [8]). We cannot simply extend this theorem by replacing the Gromov-Hausdorff convergence by intrinsic flat convergence (see Remark 5.2 within).

Theorem 5.1 extends this theorem assuming the domains converge in the intrinsic flat sense and the ranges are fixed.

Theorem 5.4 allows both the domains and ranges to change but assumes the functions are surjective local isometries. It includes covering maps. Possible applications are listed in remarks below the statement including applications related to work of Gromov [12] and Burago-Ivanov [2], as well as the author with Wei [20] [21]. Examples are provided to show the surjectivity and uniform local hypotheses are necessary.

Theorem 5.16 concerns sequences of open filling functions which are defined in Definition 5.12. It is a very general theorem and possible applications possibly to uniformly biLipschitz functions are described in remarks below its statement.

This section is not needed for those interested only in the Tetrahedral Compactness Theorem.

5.1. An Arzela-Ascoli Theorem.

This subsection is exactly Subsection 4.8 from v1 with no changes

In this section we prove our first Arzela-Ascoli Theorem. This basic theorem is proven using only Theorem 2.40 and Theorem 4.3.

Theorem 5.1. *Suppose $M_i = (X_i, d_i, T_i)$ are integral current spaces and M_i converge in the intrinsic flat sense to M_∞ and $F_i : X_i \to W$ are Lipschitz maps into a compact metric space W with $\mathrm{Lip}(F_i) \le K$, then a subsequence converges to a Lipschitz map $F_\infty : X_\infty \to W$ with $\mathrm{Lip}(F_\infty) \le K$. More specifically, there exists isometric embeddings of the subsequence, $\varphi_i : X_i \to Z$, such that $d_F^Z(\varphi_{i\#}T_i, \varphi_\infty T_\infty) \to 0$ and for any sequence $p_i \in X_i$ converging to $p \in X_\infty$ (i.e. $d_Z(\varphi_i(p_i), \varphi_\infty(p)) \to 0$, we have $\lim_{i\to\infty} F_i(p_i) = F_\infty(p_\infty)$.*

Remark 5.2. *Recall that the corresponding Gromov-Hausdorff Arzela-Ascoli Theorem allows the target spaces to vary as well, as long as they converge in the Gromov-Hausdorff sense to a compact limit space. See for example Grove-Petersen* [13]. *Here we cannot allow the target space to vary and converge only in the intrinsic flat sense even if the domain is fixed. For example, one may have a sequence of compact connected manifolds, W_i, which converge in the intrinsic flat sense to a compact metric space, W, that is not connected* [23]. *In that setting one has a sequence of Lipschitz maps which are unit speed geodesics, $F_i : [0,1] \to W_i$ where $W_i \xrightarrow{\mathcal{F}} W$ with no limiting function $F : [0,1] \to \bar{W}_\infty$.*

Proof. By Theorem 2.40 we have $\varphi_i : M_i \to Z$ such that $d_F^Z(\varphi_{i\#}T_i, \varphi_\infty T_\infty) \to 0$. Take any $p_\infty \in X_\infty$. By Theorem 4.3, there exists $p_i \in X_i$ such that $\lim_{i\to\infty} \varphi_i(p_i) = \varphi_\infty(p_\infty)$. Their images $F_i(p_i) \in W$ have a subsequence which converges to some $w \in W$. We set $F_\infty(p_\infty) = w$. Recall that integral current spaces are seperable. So we have a countable dense subset $X_0 \subset X_\infty$. So we may repeat this process creating subsequences of subsequences for a countable dense collection of $p \in X_0 = X_\infty$. Diagonalizing, we have defined a function $F_\infty : X_0 \subset X_\infty \to K$.

Observe that for all $p, q \in X_0$ we have p_i and q_i converging to them such that

$$\begin{aligned}
d_W(F_\infty(p), F_\infty(q)) &= \lim_{i\to\infty} d_W(F_i(p_i), F_i(q_i)) && (401)\\
&\le \lim_{i\to\infty} K d_{X_i}(p_i, q_i) && (402)\\
&\le \lim_{i\to\infty} K d_Z(\varphi_i(p_i), \varphi_i(q_i)) && (403)\\
&\le K d_Z(\varphi_\infty(p), \varphi_\infty(q)) && (404)\\
&\le K d_{X_\infty}(p, q). && (405)
\end{aligned}$$

Thus we may extend F_∞ continuously to $F_\infty : X_\infty \to W$ and $\mathrm{Lip}(F_\infty) \le K$.

Now suppose we have an arbitrary sequence $p_i \in X_i$ such that $d_Z(\varphi_i(p_i), \varphi_\infty(p)) \to 0$ then there exists $q_j \in X_0$ converging to $p_\infty \in X$ with

$$d_W(F_\infty(p_\infty), F_\infty(q_j)) \le K d_{X_\infty}(p_\infty, q_j) \tag{406}$$

By the definition of F_∞ we have have $q_{j,i} \in X_i$ with $d_Z(\varphi_i(q_{j,i}), \varphi_\infty(q_j)) \to 0$ and

$$\lim_{i\to\infty} d_W(F_i(q_{j,i}), F_\infty(q_j)) = 0. \tag{407}$$

Observe that

$$\begin{aligned}
\lim_{i\to\infty} d_W(F_i(p_i), F_i(q_{j,i})) &\le \lim_{i\to\infty} K d_{X_i}(p_i, q_{j,i}) && (408)\\
&\le \lim_{i\to\infty} K d_Z(\varphi_i(p_i), \varphi_i(q_{j,i})) && (409)\\
&\le K d_Z(\varphi_\infty(p_\infty), \varphi_\infty(q_j)) && (410)\\
&= K d_{X_\infty}(p_\infty, q_j) && (411)
\end{aligned}$$

Combining these we have

$$\begin{aligned}
\lim_{i\to\infty} d_W(F_i(p_i), F_\infty(p_\infty)) &\le \lim_{i\to\infty} d_W(F_i(p_i), F_i(q_{j,i})) && (412)\\
&\quad + \lim_{i\to\infty} d_W(F_i(q_{j,i}), F_\infty(q_j)) + d_W(F_\infty(p_\infty), F_\infty(q_j)) && (413)\\
&\le K2 d_Z(\varphi_\infty(p_\infty), \varphi_\infty(q_j)) && (414)
\end{aligned}$$

Taking $j \to \infty$ we have our claim. □

Remark 5.3. *It should be possible to extend Theorem 5.1 to sequences $F_i : X_i \to W_i$ of Lipschitz maps into compact metric spaces W_i with $\mathrm{Lip}(F_i) \le K$ where $W_i \xrightarrow{GH} W$ and $X_i \xrightarrow{\mathcal{F}} X$ using Gromov's Embedding Theorem. No applications are known for such a theorem at this time so we do not prove this here at this time.*

5.2. **Limits of Surjective Uniformly Local Isometries.** This subsection which was announced vaguely in v1 is completed here.

We now prove an Arzela-Ascoli Theorem which allows both the domain and the target spaces to converge in the intrinsic flat sense as long as they have uniform bounds on total mass and the functions are surjective uniformly local current preserving isometries. See the examples and remarks below the statement of the theorem concerning the necessity of these hypothesis and possible applications of the theorem.

Theorem 5.4. *Suppose $M_i = (X_i, d_i, T_i)$ and $M_i' = (X_i', d_i', T_i)$ are integral current spaces with*

$$\mathbf{M}(M_i) \le V_0 \text{ and } \mathbf{M}(\partial M_i) \le A_0 \tag{415}$$

such that M_i, M_i' converge in the intrinsic flat sense to nonzero precompact integral current spaces M_∞, M_∞' respectively. Let $F_i : M_i \to M_i'$ be surjective uniformly local current preserving isometries:

$$\exists \delta > 0 \text{ such that } \forall x \in X_i \;\; F_i : \bar{B}(x_i, \delta) \to \bar{B}(F_i(x_i), \delta) \text{ is a current preserving isometry.} \tag{416}$$

Then a subsequence, also denoted F_i, which converges to a local isometry $F_\infty : \bar{X}_\infty \to \bar{X}_\infty'$ satisfying (416). More specifically, there exists isometric embeddings of the subsequence $\varphi_i : X_i \to Z$, $\varphi_i' : X_i' \to Z'$, such that $d_F^Z(\varphi_{i\#}T_i, \varphi_\infty T_\infty) \to 0$ and $d_F^{Z'}(\varphi_{i\#}' T_i', \varphi_\infty' T_\infty') \to 0$ and for any sequence $p_i \in X_i$ converging to $p \in X_\infty$ (i.e. $d_Z(\varphi_i(p_i), \varphi_\infty(p)) \to 0$), we have

$$\lim_{i\to\infty} \varphi_i'(F_i(p_i)) = \varphi_\infty'(F_\infty(p_\infty)). \tag{417}$$

In particular, if $M_\infty \neq \mathbf{0}$, then $M_\infty' \neq 0$.

Example 5.5. *The surjectivity is necessary in this theorem. For example one may take M_i' to be a sequence in the example of the author and Wenger in* [22]. *These M_i' are three dimensional manifolds created by joining to two standard three spheres with increasingly dense increasingly tiny Gromov-Lawson tunnels:*

$$M_i' \quad (418)\Big(\S^3 \setminus \bigcup_{j=1}^{i} B(x_j, \epsilon_i)\Big)$$

$$(419) \sqcup \partial B(x_1, \epsilon_i) \times [0, \delta_i] \sqcup \partial B(x_2, \epsilon_i) \times [0, \delta_i] \sqcup \cdots \sqcup \partial B(x_i, \epsilon_i) \times [0, \delta_i] \sqcup \Big(\S^3 \setminus \bigcup_{j=1}^{i} B_{x_j}(\epsilon_i)\Big)$$

glued along the $\partial B(x_j, \epsilon_i)$ and endowed with a Riemannian metric tensor of positive scalar curvature. In [22] *it was shown that M_i' converge in the intrinsic flat sense to the 0 space. Now let M_i be just a single sphere with many balls removed that lies isometrically within M_i':*

$$M_i' \quad = \quad \Big(\S^3 \setminus \bigcup_{j=1}^{i} B(x_j, \epsilon_i)\Big) \tag{420}$$

Then $M_i \xrightarrow{\mathcal{F}} S^3$.

Example 5.6. *The hypothesis that a uniform $\delta > 0$ exists such that (416) holds is necessary. This can be seen by taking M_i to be standard flat 1×1 tori and M_i' to be flat $1 \times (1/i)$ tori. Let $F_i : M_i \to M_i'$ be the i fold covering maps which are surjective local isometries on balls of radius $\delta_i = 1/(2i)$. Then M_i converges in the intrinsic flat sense to a standard flat torus while M_i' converges in the intrinsic flat sense to the $\mathbf{0}$ integral current space. Thus there cannot be any limit map F_∞.*

Remark 5.7. *In* [2], *Burago and Ivanov prove that the volume growth of the universal cover of a Riemannian manifold homeomorphic to a torus is at least that of Euclidean space. If it is exactly equal, then they have a rigidity theorem stating that the Riemannian manifold is flat. Theorem 5.4 may be useful in the study of questions arising in Gromov's work* [12] *analyzing the almost rigidity of Burago-Ivanov's Theorem (where the volume growth is close to that of Euclidean space).*

Remark 5.8. *Theorem 5.4 should be useful when wishing to study limits of covering maps and analyze the existence of a universal cover of an intrinsic flat limit as in the work of the author with Guofang Wei in* [20].

Remark 5.9. *Theorem 5.4 should also be useful when studying how covering spectra behave under intrinsic flat convergence.*

We now prove Theorem 5.4:

Proof. We begin with the construction of the limit function $F_\infty : P \to X'_\infty$ satisfying (417) for all $p \in P$ where P is a countably dense collection of points in X_∞.

By Theorem 2.40 we have $\varphi_i : M_i \to Z$ such that $d_F^Z(\varphi_{i\#}T_i, \varphi_\infty T_\infty) \to 0$ and $\varphi_i : M_i \to Z$ such that $d_F^Z(\varphi_{i\#}T_i, \varphi_\infty T_\infty) \to 0$. Take any $p \in X_\infty$. By Proposition 4.26, there exists $r_p > 0$, $C_p > 0$ and $p_i \in X_i$ such that $\lim_{i\to\infty} \varphi_i(p_i) = \varphi_\infty(p)$ and such that for $k = 0$ we have

$$\mathbf{M}(S(p_i, r)) \geq \mathbf{SF}_k(p_i, r) > C_{p,r} > 0 \text{ for all } r < r_p. \tag{421}$$

By (416) and the surjectivity of F_i, we have

$$\bar{B}(q_i, r) \text{ is isometric to } \bar{B}(p_i, r) \text{ for all } r \leq \delta \tag{422}$$

where $q_i = F_i(p_i)$. Thus

$$F_{i\#}(S(p_i, r)) = S(q_i, r) \text{ for a.e. } r \leq \delta \tag{423}$$

and for $k = 0$ and $\eta = 1$ we have

$$\mathbf{M}(S(q_i, r)) \geq \mathbf{SF}_k(q_i, r) > C_{p,r} \text{ for a.e } r < \min\{r_p, \delta\}. \tag{424}$$

By (415) and the fact that $F_i : M_i \to M'_i$ are surjective and locally current preserving, we have

$$\mathbf{M}(M'_i) \leq \mathbf{M}(M_i) \leq V_0 \text{ and } \mathbf{M}(\partial M'_i) \leq \mathbf{M}(\partial M_i) \leq A_0 \tag{425}$$

and M'_∞ is precompact, so we may apply Theorem 4.31 to the sequence $q_i \in M'_i$ with $k = 0$. Thus there is a subsequence i_j such that q_{i_j} converge to $q_\infty \in \bar{X}'_\infty$.

We can apply this process to the countable collection $P \in \bar{X}'_\infty$ and diagonalize the subsequences, to obtain the desired subsequence in the statement of the theorem, which we denote simply as F_i to avoid writing subscripts of subscripts. This subsequence has the property that for all $p \in P$, there exists $p_i \in X_i$, such that $q_i = F_i(p_i) \in X'_i$ converge to $q_\infty \in X'_\infty$. We set $F_\infty(p) = q_\infty$. Thus we have completed the construction of the limit map $F_\infty : P \to \bar{X}'_\infty$ satisfying (417) for all $p \in P$.

It is easy to see that F_∞ is a local isometry onto its image. Thus we can extend F_∞ to the metric completion of P obtaining a local isometry $F_\infty : \bar{X}_\infty \to \bar{X}'_\infty$ satisfying (417) for all $p \in \bar{X}_\infty$.

We claim that $F_\infty : \bar{X}_\infty \to \bar{X}'_\infty$ is surjective. Take $q \in \bar{X}'_\infty$, then by Proposition 4.26, there exists $r_q > 0$, $C_q > 0$ and $q_i \in X'_i$ converging to $q \in X'$ such that for $k = 0$

$$\mathbf{M}(S(q_i, r)) \geq \mathbf{SF}_k(q_i, r) > C_{q,r} > 0 \text{ for all } r < r_q. \tag{426}$$

Since F_i are surjective, there exists $p_i \in X_i$ such that $F(p_i) = q_i$ Again we have (423), so we have

$$\mathbf{M}(S(p_i, r)) \geq \mathbf{SF}_0(p_i, r) > C_{p,r} \text{ for a.e } r < \min\{r_p, \delta\}. \tag{427}$$

By (415) and the fact that M_∞ is precompact we may apply Theorem 4.31 to the sequence $p_i \in M_i$ with $k = 0$. Thus there is a subsequence i_j such that p_{i_j} converge to $p \in \bar{X}'_\infty$. By (417), $F(p) = q$. □

Remark 5.10. *One expects that the limit F_∞ is in fact locally a current preserving isometry. That is, for all $p \in \bar{X}_\infty$*

$$F_{\infty\#}(S(p, r)) = S(F_\infty(q), r) \textit{ for a.e. } r \leq \delta. \tag{428}$$

If so, then

$$\mathbf{M}(S(p,r)) = \mathbf{M}(S(F_\infty(p),r)) \text{ for a.e. } r \le \delta \tag{429}$$

which implies $p \in X_\infty$ iff $F(p) \in X'_\infty$. Thus $F_\infty(X_\infty) = X'_\infty$ and our limit is a surjective uniformly local current preserving isometry. Note that the claim in (428) up to a sign is an immediate consequence of being a local isometry if M_∞ and M'_∞ are Riemannian manifolds. In general the proof is somewhat technical but the author can provide a suggested outline of a possible proof to anyone who might need this stronger result.

Remark 5.11. *It may be possible to prove a similar theorem replacing the surjective uniformly local isometries with surjective uniformly local uniformly bi-Lipschitz maps but the proof would be fairly technical and there is no immediate application for this at this time.*

5.3. **Limits of Uniformly Open Filling Functions.** This is a new section in v2.

In this section we prove an Arzela Ascoli Theorem for the following class of functions:

Definition 5.12. *Let $C > 0$ $\eta > 0$ and $r_0 > 0$. A function $F : M \to M'$ between integral current spaces is called a C, η, r_0 uniformly open filling function if the following holds*

$$\mathrm{FillVol}(\partial S(F(p),r)) \ge C \cdot \mathrm{FillVol}(\partial S(p,\eta r)) \qquad \forall r \le r_0 \tag{430}$$

where we use the notion of filling volume given in Definition 2.44.

Note uniform local isometries are in this class with $C = \eta = 1$. So our Arzela-Ascoli theorem here applies to uniform local isometries which are not surjective.

Remark 5.13. *Do λ bi-Lipschitz functions belong in this class for some values of C and η? Proving this would have many applications as an Arzela-Ascoli theorem for biLipschitz functions would be very useful.*

Remark 5.14. *Recall that filling volumes can be estimated using contractibility functions. Is there some sort of contractibility notion that can be assigned to a function that would then place it into this class?*

Note also that this is the $k = 0$ version of the following notion:

Definition 5.15. *A function $F : M \to M'$ between integral current spaces is called a C, η, r_0 uniformly k-sliced open filling function if*

$$\mathbf{SF}_k(F(p),r)) \ge C \cdot \mathbf{SF}_k(p,\eta r) \qquad \forall r \le r_0. \tag{431}$$

We now state and prove our Arzela-Ascoli Theorem for Uniformly Open Filling Functions:

Theorem 5.16. *Suppose $M_i = (X_i, d_i, T_i)$ and $M'_i = (X'_i, d'_i, T_i)$ are integral current spaces with*

$$\mathbf{M}(M_i) \le V_0 \text{ and } \mathbf{M}(\partial M_i) \le A_0 \tag{432}$$

such that M_i, M'_i converge in the intrinsic flat sense to nonzero precompact integral current spaces M_∞, M'_∞ respectively. Let $\lambda > 0$, $C > 0$, $\eta > 0$ and $r_0 > 0$. Let $F_i : X_i \to X'_i$ be C, η, r_0 uniformly open filling functions with $\mathrm{Lip}(F_i) \le \lambda$.

Then a subsequence converges to a Lipschitz function $F_\infty : \bar{X}_\infty \to \bar{X}'_\infty$ in the sense that (417) holds.

Proof. We begin again with the construction of the limit function $F_\infty : P \to X'_\infty$ satisfying (417) for all $p \in P$ where P is a countably dense collection of points in X_∞.

By Theorem 2.40 we have $\varphi_i : M_i \to Z$ such that $d_F^Z(\varphi_{i\#}T_i, \varphi_\infty T_\infty) \to 0$ and $\varphi_i : M_i \to Z$ such that $d_F^Z(\varphi_{i\#}T_i, \varphi_\infty T_\infty) \to 0$. Take any $p \in X_\infty$. By Proposition 4.26, there exists $r_p > 0$, $C_p > 0$ and $p_i \in X_i$ such that $\lim_{i\to\infty} \varphi_i(p_i) = \varphi_\infty(p)$ and such that for $k = 0$ we have

$$\mathbf{M}(S(p_i, r)) \geq \mathbf{SF}_k(p_i, r) > C_{p,r} > 0 \text{ for all } r < r_p. \tag{433}$$

By (431) we have

$$\mathbf{M}(S(q_i, r)) \geq \mathbf{SF}_k(q_i, r)) \geq C \cdot \mathbf{SF}_k(p, \eta r) > C_{p,\eta r} > 0 \text{ for all } r < \min\{r_0, \eta r_p\}. \tag{434}$$

We may then continue exactly as in the proof of Theorem 5.4 after (to-get-limit-F) up to the construction of $F_\infty : P \to \bar{X}'_\infty$ satisfying (417) for all $p \in P$.

Since $\mathrm{Lip}(F_i) \leq \lambda$, by (417) we have $\mathrm{Lip}(F) \leq \lambda$ and so we may extend F continuously to $F : \bar{X}_\infty \to \bar{X}'_\infty$.

We need to show $F_\infty(X_\infty) \subset X'_\infty$. Suppose $p \in X_\infty$, then by Proposition 4.26, there exists $r_p > 0$, $C_p > 0$ and $p_i \in X'_i$ such that p_i converges to p and, for $k = 0$,

$$\mathbf{M}(S(p_i, r)) \geq \mathbf{SF}_k(p_i, r) > C_{p,r} > 0 \text{ for all } r < r_p. \tag{435}$$

Since F_i are $C, \eta r_0$ uniformly open filling functions and $\mathbf{SF}_0(p_i, r) = \mathrm{FillVol}(\partial S(p_i, r))$ we have

$$\mathbf{SF}_0(F_i(p_i), r) > CC_{p,\eta r} \text{ for all } r < \min\{r_p, r_0/\eta\}. \tag{436}$$

Applying Theorem 4.20 and the fact that $F_i(p_i) \to F_\infty(p)$ we have

$$\mathbf{M}(S(F_\infty(p), r) \geq \mathbf{SF}_0(F_\infty(p), r) \geq CC_{p,\eta r} \text{ for all} r < \min\{r_p, r_0/\eta\}. \tag{437}$$

So $F_\infty(p) \in \bar{X}'_\infty$. Thus $F_\infty : \bar{X}_\infty \to \bar{X}'_\infty$ and it satisfies (417). So it is λ Lipschitz as well. □

Remark 5.17. *With extra work it may be possible to show that F_∞ is a uniformly open sliced filling function. Such a proof would be rather technical and it is unclear whether it would have any applications.*

Remark 5.18. *It would be nice to extend Theorem 5.16 to apply to uniformly open k-sliced filling functions as in (431) for $k \in \{1, 2, ...(m-1)\}$. We would need to assume in addition that the M'_i converge in the Gromov-Hausdorff sense to some limit space. This extra hypothesis is required to apply Theorem 4.31 (and may not even be necessary if that theorem is improves as described in Remark 4.32). With this extra hypothesis, the proofs Theorems 5.16 and 5.4 (which were written with general k) would suffice except for one additional problem: we would also need to strengthen (375) of Proposition 4.26 to include $k \neq 0$. Improving that proposition would require a serious technical analysis of slices and fillings, which the author has begun but it is rather long and not worthwhile unless an application is found which requires this.*

5.4. **Additional Arzela-Ascoli Theorems as needed.**

A newly announced subsection in v2: More Arzela-Ascoli Theorems have been proven building upon Theorem 5.16. They may be included here in the future or placed in papers which apply them. Please let the author know if there is an Arzela-Ascoli Theorem which might be useful and should be proven here.

6. Compactness Theorems

This used to be Subsection 4.9 in v1 on the arxiv, otherwise unchanged except for the introduction.

In this section we complete the proofs of our main two compactness theorems: Theorem 6.2 and Theorem 6.1]. Both of these theorems prove that certain sequences of spaces have subsequences which converge in both the intrinsic flat and the Gromov-Hausdorff sense to the same space. This the Gromov-Hausdorff limits are countable $\mathcal{H}^m$ rectifiable and the Intrinsic Flat limits are not zero. Theorem 6.2 is the Tetrahedral Compactness Theorem concerning sequences of Riemannian manifolds satisfying the tetrahedral property. It was partially stated in the introduction. It is a consequence of Theorem 6.1 which applies to integral current spaces which have uniform lower bounds on the sliced filling volumes of the form $\mathbf{SF}_k(p, r) \ge C_{SF} r^m$.

In prior work of the author with Wenger [22] a another pair of compactness theorems was proven providing subsequences of manifolds which converge both in the intrinsic flat and Gromov Hausdorff sense to the same limit. One theorem concerned noncollapsing sequences of Riemannian manifolds with nonnegative Ricci curvature (extending Gromov's Ricci Compactness Theorem [11]). The other concerned sequences of Riemannian manifolds with a uniform linear contractibility function and a uniform upper bound on volume (extending Greene-Petersen's Compactness Theorem [8]). The techniques used in the proof of the Contractibility Function Compactness Theorem in [22] involve the continuity of the filling volumes of balls. Here we use the continuity of sliced filling volumes in a similar way.

6.1. **Sliced Filling Compactness Theorem.**

Theorem 6.1. *Given a sequence of m dimensional integral current spaces $M_i = (X_i, d_i, T_i)$ with $\mathbf{M}(M_i) \le V_0$, $\mathbf{M}(\partial M_i) \le A_0$, $\operatorname{Diam}(M_i) \le D_0$ and a uniform constant $C_{SF} > 0$ such that for some $k \in 0..(m-1)$ we have*

$$\mathbf{SF}_k(p, r) \ge C_{SF} r^m \tag{438}$$

then a subsequence of the M_i converge in the Gromov-Hausdorff sense and the Intrinsic Flat sense to a nonzero integral current space M_∞.

Proof. By Theorem 3.23, we know a subsequence (X_i, d_i) has a Gromov-Hausdorff limit (Y, d_Y). Thus by Gromov, there exists a common compact metric space Z and isometric embeddings $\varphi_i : X_i \to Z$, $\varphi : Y \to Z$, such that $d_H^Z(\varphi(X_i), \varphi(Y)) \to 0$. By Ambrosio-Kirchheim Compactness Theorem, a subsequence of $\varphi_{1\#} T_i$ converges to $T_\infty \in \mathbf{I}_m(Z)$. Let $M_\infty = (\mathrm{set}(\mathrm{T}_\infty), \mathrm{d}_Z, \mathrm{T}_\infty)$.

We need only show $\varphi(Y) = \mathrm{set}(\mathrm{T}_\infty)$. Let $z_\infty \in Y$, and let $p_i \in X_i$ such that $z_i = \varphi_i(p_i) \to z$. By Theorem 4.27, we see that $z_\infty = \varphi(p_\infty)$. □

6.2. **Tetrahedral Compactness Theorem.**

Theorem 6.2. *Given $r_0 > 0, \beta \in (0,1), C > 0, V_0 > 0, A_0 > 0$. If a sequence of compact Riemannian manifolds, M^m, has $\operatorname{Vol}(M^m) \le V_0$, $\operatorname{Diam}(M^m) \le D_0$ and the C,β (integral) tetrahedral property for all balls of radius $\le r_0$, then a subsequence converges in the Gromov-Hausdorff and Intrinsic Flat sense to a nonzero integral current space.*

Here our manifolds do not have boundary.

Proof. The C,β (integral) tetrahedral property implies that there exists $C_{SF} > 0$ such that

$$\mathbf{SF}_{m-1}(p,r) \geq C_{SF} r^m. \tag{439}$$

Theorem 3.41 implies there exists a uniform upper bound on diameter. So we apply Theorem 6.1. □

Remark 6.3. *As a consequence of this theorem, we see that there is no uniform tetrahedral property on manifolds with positive scalar curvature even when the volume of the balls are uniformly bounded below by that of Euclidean balls. In fact there exist a sequence of such manifolds, M_j^3, whose intrinsic flat limit is* 0 *described in* [22].

6.3. **Sliced Interval Compactness Theorem.** New announced subsection for v2 which may appear in v3

There should be a natural compactness theorem involving the sliced interval filling volumes. If there is a natural application of such a compactness theorem, we might include it here in this subsection in the future.

7. Consequences of Continuity

In v1 this section was announced as Subsection 4.11. Here in v2 we see it is still in progress. It is possible this material may be put in v3 or postponed to a future paper.

Here we will itemize the consequence of the continuity of $\mathbf{SF}_k$ and $\mathrm{FillVol}(\partial S(p, r))$ on classes of spaces where intrinsic flat and Gromov-Hausdorff limits of spaces agree including the classes in our compactness theorems given above and manifolds with nonnegative Ricci curvature and uniform lower bounds on volume.

There are also some nice consequences of knowing a sequence converges to a ball in Euclidean space or to a sphere.

8. Slicing to Specific Limits

In v1 this section was announced as Subsection 4.11. Here in v2 it is still in progress. It is possible this material may be put in v3 or postponed to a future paper.

In order to prove conjectures like those stated in [15] and [16], one needs to prove a sequence of Riemannian manifolds has a specific intrinsic flat limit. In this section we will prove theorems which allow one to prove results about slices in order to achieve a specific limit.

References

[1] Luigi Ambrosio and Bernd Kirchheim. Currents in metric spaces. *Acta Math.*, 185(1):1–80, 2000.
[2] D. Burago and S. Ivanov. On asymptotic volume of tori. *Geom. Funct. Anal.*, 5(5):800–808, 1995.
[3] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.
[4] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. I. *J. Differential Geom.*, 46(3):406–480, 1997.
[5] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. III. *J. Differential Geom.*, 54(1):37–74, 2000.
[6] Tobias H. Colding. Ricci curvature and volume convergence. *Ann. of Math. (2)*, 145(3):477–501, 1997.
[7] Herbert Federer and Wendell H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960.
[8] Robert E. Greene and Peter Petersen V. Little topology, big volume. *Duke Math. J.*, 67(2):273–290, 1992.
[9] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.
[10] Mikhael Gromov. Filling Riemannian manifolds. *J. Differential Geom.*, 18(1):1–147, 1983.
[11] Misha Gromov. *Metric structures for Riemannian and non-Riemannian spaces*, volume 152 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1999. Based on the 1981 French original [MR0682063 (85e:53051)], With appendices by M. Katz, P. Pansu and S. Semmes, Translated from the French by Sean Michael Bates.
[12] Misha Gromov. Hilbert volume in metric spaces. Part 1. *Cent. Eur. J. Math.*, 10(2):371–400, 2012.
[13] Karsten Grove and Peter Petersen. Manifolds near the boundary of existence. *J. Differential Geom.*, 33(2):379–394, 1991.
[14] Sajjad Lakzian and Christina Sormani. Smooth convergence away from singular sets. *to appear in Communications in Analysis and Geometry*, 2012.
[15] Dan A. Lee and Christina Sormani. Stability of the positive mass theorem for rotationally symmetric riemannian manifolds. *to appear in Crelle's Journal*, 2011.
[16] Dan A. Lee and Christina Sormani. Almost inequality in the Penrose equality for rotationally symmetric manifolds. *Annales Henri Poincare*, to appear.
[17] Fanghua Lin and Xiaoping Yang. *Geometric measure theory—an introduction*, volume 1 of *Advanced Mathematics (Beijing/Boston)*. Science Press, Beijing, 2002.
[18] Frank Morgan. *Geometric measure theory*. Elsevier/Academic Press, Amsterdam, fourth edition, 2009. A beginner's guide.
[19] Christina Sormani. The Tetrahedral Property and a new Gromov-Hausdorff Compactness Theorem. *preprint on arxiv*.
[20] Christina Sormani and Guofang Wei. Hausdorff convergence and universal covers. *Trans. Amer. Math. Soc.*, 353(9):3585–3602 (electronic), 2001.
[21] Christina Sormani and Guofang Wei. Universal covers for Hausdorff limits of noncompact spaces. *Trans. Amer. Math. Soc.*, 356(3):1233–1270 (electronic), 2004.
[22] Christina Sormani and Stefan Wenger. Weak convergence and cancellation, appendix by Raanan Schul and Stefan Wenger. *Calculus of Variations and Partial Differential Equations*, 38(1-2), 2010.
[23] Christina Sormani and Stefan Wenger. Intrinsic flat convergence of manifolds and other integral current spaces. *Journal of Differential Geometry*, 87, 2011.
[24] Stefan Wenger. Flat convergence for integral currents in metric spaces. *Calc. Var. Partial Differential Equations*, 28(2):139–160, 2007.
[25] Stefan Wenger. Compactness for manifolds and integral currents with bounded diameter and volume. *Calculus of Variations and Partial Differential Equations*, 40(3-4):423–448, 2011.
[26] Hassler Whitney. *Geometric integration theory*. Princeton University Press, Princeton, N. J., 1957.

CUNY Graduate Center and Lehman College
E-mail address: sormanic@member.ams.org