The Codimension One Homology of a Complete Manifold with Nonnegative Ricci Curvature

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Abstract

In this paper we prove that a complete noncompact manifold with nonnegative Ricci curvature has a trivial codimension one homology unless it is a flat normal bundle over a compact totally geodesic submanifold. In particular, we prove the conjecture that a complete noncompact manifold with positive Ricci curvature has a trivial codimension one integer homology. We also have a corollary stating when the codimension two integer homology of such a manifold is torsion free.

1 Introduction

In this paper we prove that a complete noncompact manifold with nonnegative Ricci curvature has a trivial codimension one homology unless it is a split or flat normal bundle over a compact totally geodesic submanifold (Thms 1.1 and 1.2). In particular, we prove the conjecture that a complete noncompact manifold with positive Ricci curvature has a trivial codimension one integer homology (Cor 1.1). We also have a corollary stating when the codimension two integer homology of such a manifold is torsion free (Cor 1.2).

In 1976, Shing Tung Yau proved that a complete noncompact manifold with positive Ricci curvature has a trivial codimension one real homology [Yau]. It was then conjectured that the integer homology might be trivial as well. In 1993, the first author proved that if the manifold is also proper, then the codimension one integer homology is trivial [Sh1]. Recall that a manifold is proper if the manifold's Busemann function has compact level sets. It is unknown whether all complete noncompact manifolds with positive Ricci curvature are proper. However, the authors have proven that such a manifold is proper when it has Euclidean volume growth [Sh2] or linear volume growth [So1].

Recently Itokawa and Kobayashi have proven that if a complete noncompact manifold with nonnegative Ricci curvature has more than linear volume growth then the codimension one integer homology is trivial [ItKo, Thm 1]. Since a manifold with nonnegative Ricci curvature must have at least linear volume growth [Yau], the three papers [ItKo], [So1] and [Sh1] prove the conjecture regarding manifolds with positive Ricci curvature. In fact they demonstrate that codimension one integer homology is trivial unless the manifold has linear volume growth and nonnegative Ricci curvature. Note that split or flat normal bundles over a compact manifolds with nonnegative Ricci curvature are examples of such exceptional manifolds. It is not hard to see that these examples can have codimension one integer homology equal to $\mathbf{Z}$.

Itokawa and Kobayashi have also proven that if a manifold with nonnegative Ricci curvature has bounded diameter growth then the integer codimension one homology is either $\mathbf{Z}$, $\mathbf{Z}_2$ or 0 [ItKo Thm 2]. Note that the split or flat normal bundles over compact manifolds have bounded diameter growth, but there also exist examples of manifolds with nonnegative Ricci curvature, linear volume growth and unbounded diameter growth [So1]. Itokawa and Kobayashi used minimizing currents to prove their results, and the first author used an adaption of Morse theory to prove his result regarding proper manifolds.

We reprove these results and complete the classification of the codimension one integer homology using Poincare Duality, the Cheeger Gromoll Splitting Theorem [ChGl] and properties of noncontractible loops developed by the second author [So2].

Theorem 1.1 *Let M^n be an orientable complete noncompact manifold with nonnegative Ricci curvature and G is an abelian group, then one of the following holds:*

$$\begin{array}{rrcl} (i) & H_{n-1}(M,G) & = & 0, \\ (ii) & H_{n-1}(M,G) & = & G, \\ (iii) & H_{n-1}(M,G) & = & \ker(G \stackrel{\times 2}{\to} G). \end{array}$$

Case (ii) can only occur when M^n is an an isometrically split manifold over a compact totally geodesic orientable submanifold.

Case (iii) can only occur if M^n is a one-ended flat normal bundle over a compact totally geodesic unorientable submanifold.

Recall that $\ker(\mathbf{Z} \stackrel{\times 2}{\to} \mathbf{Z}) = 0$ and $\ker(\mathbf{Z}_k \stackrel{\times 2}{\to} \mathbf{Z}_k) = 0$ if k is odd and is $\mathbf{Z}_2$ if k is even.

Theorem 1.2 *Let M^n be a complete noncompact unorientable manifold with nonnegative Ricci curvature and $G = \mathbf{Z}_2$ or $\mathbf{Z}$. Then one of the following holds:*

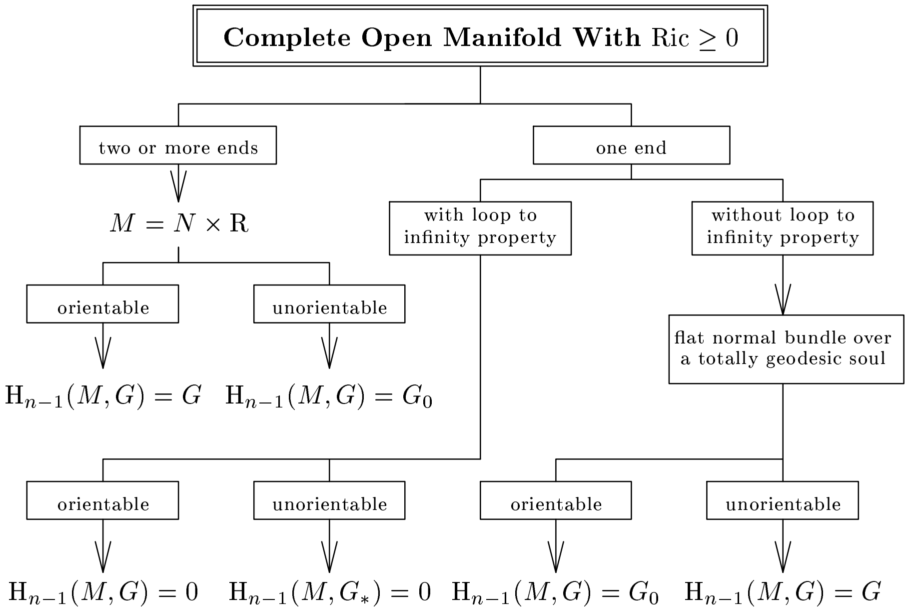
$$\begin{array}{rrcl} (i) & H_{n-1}(M,G) & = & 0, \\ (ii) & H_{n-1}(M,G) & = & G, \\ (iii) & H_{n-1}(M,G) & = & \ker(G \stackrel{\times 2}{\to} G). \end{array}$$

Case (ii) can only occur when M is a one-ended flat normal bundle over a compact totally geodesic orientable submanifold.

Case (iii) can only occur when M is an isometrically split manifold over a compact unorientable submanifold.

Remark: These two cases are opposite to the two cases in Theorem 1.1.

For an outline of our proof of Theorems 1.1 and 1.2 and the various cases involved, see the following diagram:



$$G_0 := \ker(G \overset{\times 2}{\to} G), \quad G_* := \mathbf{Z}_2 \text{ or } \mathbf{Z}$$

Corollary 1.1 *Let M^n be a complete noncompact manifold with nonnegative Ricci curvature. If there is a point $p \in M$ such that $Ricci_p(v,v) > 0$ then $H_{n-1}(M, \mathbf{Z})$ is trivial.*

Note that we have eliminated the possibility that $H_{n-1}(M, \mathbf{Z}) = \mathbf{Z}_2$. This was indicated as a possibility in [ItKo, Thm 2] if M had a cover of degree one or two isometric to $\bar{M} \times [0, \infty) \cup K$ where K and $\bar{M}$ are compact. Their second theorem also indicated that $\mathbf{Z}$ was a possibility only if M had a cover of degree one or two which split isometrically exactly as we have.

The paper is divided into two sections. In the first section, we define a topological property called the *loops to infinity property* [Defn 2.2]. We then prove that orientable manifolds with this property and only one end have trivial codimension one G homology where G is any abelian group [Prop 2.1]. This section of the paper is purely topological and uses Poincare Lefschetz Duality and the Universal Coefficient Theorem (c.f. [Mun] and [Mas]).

In the second section of the paper, we eliminate the topological conditions in Propositions 2.1 using the properties of manifolds with nonnegative Ricci curvature [Props 3.1- 3.3]. In particular, we use the Splitting Theorem [ChGl] and a theorem in [So2].

In addition to proving Theorems 1.1 and 1.2, we prove the following corollary in Section 2.

Corollary 1.2 *Let M^n be an orientable complete noncompact manifold with nonnegative Ricci curvature, then M is either an isometrically split manifold or a flat normal bundle over a compact totally geodesic submanifold or, for any abelian group G,*

$$H_{n-2}(M, \mathbf{Z}) * G = 0,$$

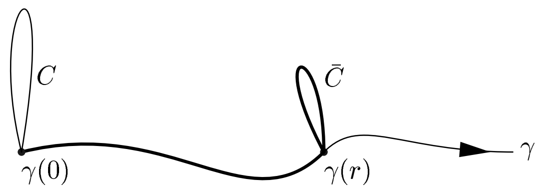
in which case $H_{n-2}(M, \mathbf{Z})$ has no elements of finite order.

The authors would like to thank Dan Christensen for helpful discussions regarding homology theory.

2 Topology and $H_{n-1}(M, \mathbf{Z})$

Before we prove Theorems 1.1 and 1.2 we introduce some topological concepts from [So2] concerning noncontractible loops in complete noncompact manifolds.

Definition 2.1 Given a ray γ and a loop $C : [0, L] \to M$ based at $\gamma(0)$, we say that a loop $\bar{C} : [0, L] \to M$ is *homotopic to C* along γ if there exists $r > 0$ with $\bar{C}(0) = \bar{C}(L) = \gamma(r)$ and the loop, constructed by joining γ from 0 to r with $\bar{C}$ from 0 to L and then with γ from r to 0, is homotopic to C (see diagram below).



Definition 2.2 A complete noncompact manifold M^n has the *loops to infinity property* if there is a ray γ, such that for any element, $g \in \pi_1(M, \gamma(0))$, and any compact set, $K \subset M$, there exists a closed loop, $\bar{C}$, contained in $M \setminus K$ which is homotopic along γ to a representative loop, C, such that $g = [C]$.

When a ray, γ, is specified, we say M^n has the loops to infinity property along γ.

In [So2] it is proven that a manifold with nonnegative Ricci curvature either has the loops to infinity property along some ray or the manifold is a flat normal bundle over a compact totally geodesic soul and has a double cover which splits isometrically.

Recall that a manifold is said to have *one end* if given any compact set $K \subset M$, there is only one unbounded connected component of $M \setminus K$. Note that if a complete manifold has two or more ends then it has a length minimizing geodesic defined for all values of $\mathbf{R}$, called a *line*. Recall also that by the Cheeger-Gromoll Splitting Theorem [ChGl], if a complete noncompact manifold with nonnegative Ricci curvature contains a line, then it splits isometrically. So a manifold with nonnegative Ricci curvature either splits isometrically or has only one end.

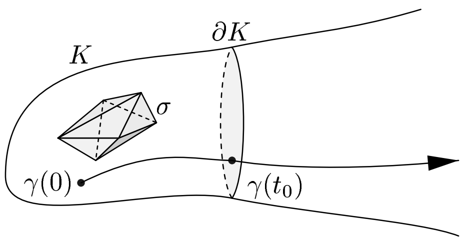
Thus, in some sense, most manifolds with nonnegative Ricci curvature have the loops to infinity property and only one end. We now prove the following simplified version of Theorem 1.1 relating the loops to infinity property to the codimension one integer homology of a manifold.

Proposition 2.1 *Let M^n be a complete noncompact manifold with one end and the loops to infinity property. Then $H_{n-1}(M, \mathbf{Z}_2)$ is trivial.*

If M^n is also orientable, then $H_{n-1}(M, G)$ is trivial where G is an abelian group.

Proof: Suppose $H_{n-1}(M, G)$ is not trivial. Then there is a chain, σ, which has no boundary but which is not a boundary . See the diagram below. The images of the simplices of this chain must be contained in some compact set, $K_0 = Cl(U_0)$, containing $\gamma(0)$. Since M^n has only one end, there is a compact set, K, consisting of K_0 and the bounded connected components of $M \setminus K_0$. Furthermore, K can be chosen with smooth boundary. Thus K is an n dimensional manifold with boundary such that ∂K is connected and

$$H_{n-1}(K, G) \neq 0. \tag{1}$$



We now proceed to show that (1) cannot hold under the appropriate conditions on M.

Let γ be the ray along which M^n has the loops to infinity property. Since γ is a ray, it must leave K, so let $t_0 > 0$ be defined such that $\gamma(t_0) \in \partial K$. In [So2], it is proven that if M^n has the loops to infinity property along γ and ∂K is smooth and connected, then

$$i_* : \pi_1(\partial K, \gamma(t_0)) \to \pi_1(K, \gamma(t_0)) \tag{2}$$

is onto. Here i_* is the map induced by the inclusion map $i : \partial K \to K$.

Now $H_1(N, \mathbf{Z})$ is just the abelianization of $\pi_1(N, p)$ (c.f. [Mas, Ch. VIII, Thm 7.1]). Thus we know

$$i_* : H_1(\partial K, \mathbf{Z}) \to H_1(K, \mathbf{Z}) \tag{3}$$

is onto. This is part of a long exact sequence involving the relative homology and the reduced homology,

$$H_1(\partial K, \mathbf{Z}) \xrightarrow{i_*} H_1(K, \mathbf{Z}) \xrightarrow{\pi_*} H_1(K, \partial K, \mathbf{Z}) \xrightarrow{\partial_*} \tilde{H}_0(\partial K, \mathbf{Z}), \tag{4}$$

(c.f. [Mun, Thm 23.3]). So $Ker(\pi_*) = Im(i_*) = H_1(K, \mathbf{Z})$. Thus $\pi_* = 0$ and ∂_* is one-to-one. Now, ∂K is connected so the reduced homology, $\tilde{H}_0(\partial K, \mathbf{Z})$, is trivial. Thus the relative homology, $H_1(K, \partial K, \mathbf{Z})$, is trivial as well.

We now claim that the cohomology $H^1(K, \partial K, G)$ must be trivial as well. To prove this we will apply the Universal Coefficient Theorem for Cohomology (c.f. [Mun, Thm 53.1]), which provides the following short exact sequence:

$$0 \to Ext(H_0(K, \partial K; \mathbf{Z}), G) \to H^1(K, \partial K; G) \to Hom(H_1(K, \partial K; \mathbf{Z}), G) \to 0.$$

Since $H_1(K, \partial K; \mathbf{Z})$ is trivial, the homomorphisms from it to G must be trivial as well. Thus

$$Ext(H_0(K, \partial K; \mathbf{Z}), G) = H^1(K, \partial K; G). \tag{5}$$

On the other hand, since K and ∂K are connected, the long exact relative homology sequence (c.f. [Mun, Thm 23.3]),

$$\ldots \to H_0(\partial K, \mathbf{Z}) \xrightarrow{i_*} H_0(K, \mathbf{Z}) \xrightarrow{\pi_*} H_0(K, \partial K; \mathbf{Z}) \to 0 \tag{6}$$

is

$$\ldots \to \mathbf{Z} \xrightarrow{i_*} \mathbf{Z} \xrightarrow{\pi_*} H_0(K, \partial K; \mathbf{Z}) \to 0, \tag{7}$$

where i_* is an isomorphism. Thus $ker(\pi_*) = Im(i_*) = \mathbf{Z}$ and π_* is onto, so $H_0(K, \partial K; \mathbf{Z}) = 0$. Thus $Ext(H_0(K, \partial K; \mathbf{Z}), G) = 0$ and, by (5), $H^1(K, \partial K, G)$ must be trivial as well.

Now by Poincare Lefschetz Duality [Mun, Thm 70.7], we know that

$$H^1(K, \partial K, \mathbf{Z}_2) \to H_{n-1}(K, \mathbf{Z}_2) \tag{8}$$

is an isomorphism. Since $H^1(K, \partial K, \mathbf{Z}_2)$ is trivial and, by (1), $H_{n-1}(K, \mathbf{Z}_2)$ is not trivial, we have a contradiction. Thus $H_{n-1}(M, \mathbf{Z}_2)$ must be trivial.

If $(K, \partial K)$ is orientable, then Poincare Lefschetz Duality implies that

$$H^1(K, \partial K, G) \to H_{n-1}(K, G) \tag{9}$$

is an isomorphism. Thus, when M is orientable, we contradict (1), so $H_{n-1}(M, G)$ must be trivial. Q.E.D.

3 Ricci Curvature and $H_{n-1}(M, \mathbf{Z})$

To prove the main theorems, we must deal with complete noncompact manifolds with nonnegative Ricci curvature that either have more than one end, are not orientable or fail to satisfy the loops to infinity property. We deal with these three cases seperately. Refer to the diagram at the end of the paper.

Proposition 3.1 *Let M^n be a complete noncompact with nonnegative Ricci curvature and two or more ends. Let G be an abelian group. Then $M = N^{n-1} \times \mathbf{R}$ and*

$$H_{n-1}(M, G) = \begin{cases} G, & \text{if } M \text{ is orientable} \\ \ker(G \overset{\times 2}{\to} G), & \text{if } M \text{ is not orientable} \end{cases}.$$

Proposition 3.2 *Let M^n be a one-ended complete noncompact manifold with nonnegative Ricci curvature satisfying the loops to infinity property. Then, regardless of the orientability of M, $H_{n-1}(M, \mathbf{Z}) = 0$.*

Proposition 3.3 *Let M^n be a complete noncompact manifold with nonnegative Ricci curvature, that has one end and doesn't have a ray with the loops to infinity property. Let G be an abelian group. Then M is a flat normal bundle over a totally geodesic soul and*

$$H_{n-1}(M, G) = \begin{cases} G, & \text{if } M \text{ is unorientable} \\ \ker(G \overset{\times 2}{\to} G), & \text{if } M \text{ is orientable} \end{cases}.$$

Note that Propositions 2.1, 3.1 and 3.3 imply Theorem 1.1, and Propositions 2.1- 3.3 imply Theorem 1.2. The proof of Corollary 1.1 will appear at the end of the paper.

Proof of Proposition 3.1: By the Cheeger-Gromoll Splitting Theorem, a complete noncompact manifold with nonnegative Ricci curvature and two or more ends must split isometrically, $M^n = N^{n-1} \times \mathbf{R}$, where N^{n-1} is compact. Thus $H_{n-1}(M^n, G) = H_{n-1}(N^{n-1}, G)$, which is

$$\ker(G \overset{\times 2}{\to} G)$$

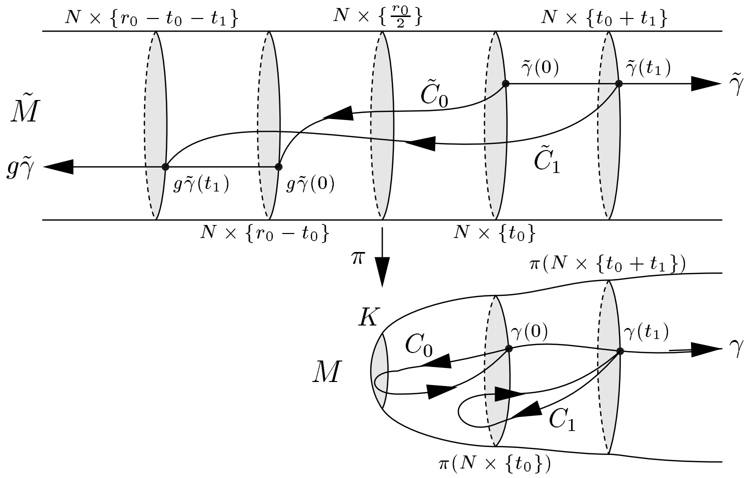
if N^{n-1} is not orientable and is G if N^{n-1} is orientable (c.f. [Mun Cor 65.5]). The proposition then follows from the fact that N^{n-1} is orientable iff M^n is orientable. Q.E.D.

Proof of Proposition 3.2: This proposition holds if M is orientable by Proposition 2.1. If M is not orientable, then it has a double cover $\pi : \tilde{M} \to M$ such that $\tilde{M}$ is orientable. We first claim that $\tilde{M}$ has only one end.

Assume on the contrary that $\tilde{M}$ has two or more ends. By the Cheeger-Gromoll Splitting Theorem, it splits isometrically, $\tilde{M} = N^{n-1} \times \mathbf{R}$ where N is compact [ChGl]. Since $\tilde{M}$ is orientable, so is the totally geodesic submanifold, N^{n-1}. Let g be the deck transform acting on $\tilde{M}$. Then g takes lines to lines and acts as an isometry on each component, $g : \mathbf{R} \to \mathbf{R}$ and $g : N^{n-1} \to N^{n-1}$.

Now $g : \mathbf{R} \to \mathbf{R}$ is an isometry so $g(r) = \pm r + r_0$. If $g(r) = r + r_0$, then $\pi : \{p_0\} \times \mathbf{R} \to \pi(\{p_0\} \times \mathbf{R})$ would map a line to a line and M itself would split isometrically and have two ends. Thus $g(r) = -r + r_0$.

Now M has the loops to infinity property along a ray, γ, which lifts to a ray $\tilde{\gamma}$. Since N is compact, we can choose $\tilde{\gamma}(t) = (q_0, t + t_0)$. Let C_0 be a curve in $\tilde{M}$ joining (q_0, t_0) to $g(q_0, t_0) = (gq_0, -t_0 + r_0)$. Then for any compact set $K \subset M$, including $K = \pi(N^{n-1} \times \{r_0/2\})$, there exists a curve, $C_1 \in M \setminus K$, based at $\gamma(t_1)$ which is homotopic along γ to $\pi \circ C_0$ [Defn 2.2]. See the diagram below. By lifting the homotopy, we lift C_1 to a curve, $\tilde{C}_1$, which runs from $\tilde{\gamma}(t_1) = (q_0, t_1 + t_0)$ to $g\tilde{\gamma}(t_1) = (gq_0, -t_1 - t_0 + r_0)$ [Defn 2.1]. Thus $\tilde{C}_1$ passes through $N \times \{r_0/2\}$, and we have a contradiction. Any unorientable manifold with nonnegative Ricci curvature and the loops to infinity property, has an orientable double cover with only one end.



Now $\tilde{M}$ has only one end and, if we prove that $\tilde{M}$ has the loops to infinity property, then it satisfies all the conditions of Proposition 2.1.

The ray, γ, in M with the loops to infinity property can be lifted to a ray $\tilde{\gamma}$ in $\tilde{M}$. We claim that $\tilde{M}$ satisfies the loops to infinity property along $\tilde{\gamma}$ [Defn 2.2]. Given any element, $[C] \in \pi_1(\tilde{M}, \tilde{\gamma}(0))$, $\pi \circ C$ is

a loop in M based at $\gamma(0)$. Given any compact set $K \subset \tilde{M}$, $\pi(K)$ is compact in M, so by the loops to infinity property on M, there is a curve $\bar{C} \in M \setminus \pi(K)$ which is homotopic along γ to $\pi \circ C$. We can lift the homotopy map to the cover $\tilde{M}$, and it is easy to see that this gives us a curve, $\tilde{\bar{C}} \in \tilde{M} \setminus K$ which is homotopic along $\tilde{\gamma}$ to C.

Thus the orientable double cover, $\tilde{M}$, satisfies all the conditions of Proposition 2.1 and $H_{n-1}(\tilde{M}, G)$ is trivial. We claim that this implies that $H_{n-1}(M, Z)$ is trivial as well.

Given any simplicial map $\sigma : \Delta_k \to M$, there is a continuous lift of σ, $\tilde{\sigma} : \Delta_k \to \tilde{M}$, which is unique up to deck transform. Thus, there is a unique chain in $C_k(\tilde{M}, \mathbf{Z})$,

$$f(\sigma) = \tilde{\sigma} + g\tilde{\sigma}, \tag{10}$$

where g is the deck transform acting on the double cover. We can extend f to a well defined map, $f : C_k(M, \mathbf{Z}) \to C_k(\tilde{M}, \mathbf{Z})$. In fact, f commutes with the boundary operator, ∂, so $f_* : H_k(M, \mathbf{Z}) \to H_k(\tilde{M}, \mathbf{Z})$ is well defined. Note also, that $\pi_* \circ f_*([\sigma]) = 2[\sigma]$.

Thus, given any $h \in H_{n-1}(M, \mathbf{Z})$, $f_*(h)$ is in the trivial group, $H_{n-1}(\tilde{M}, \mathbf{Z})$, so

$$2h = \pi_*(f_*(h)) = \pi_*(0) = 0. \tag{11}$$

Thus, $2H_{n-1}(M, \mathbf{Z}) = 0$ and

$$H_{n-1}(M, \mathbf{Z}) \otimes \mathbf{Z}_2 = \frac{H_{n-1}(M, \mathbf{Z})}{2H_{n-1}(M, \mathbf{Z})} = H_{n-1}(M, \mathbf{Z}). \tag{12}$$

However, by the Universal Coefficient Theorem (c.f. [Mun, Thm 55.1]), we have a short exact sequence,

$$0 \to H_{n-1}(M, \mathbf{Z}) \otimes \mathbf{Z}_2 \to H_{n-1}(M, \mathbf{Z}_2) \to H_{n-2}(M, \mathbf{Z}) * \mathbf{Z}_2 \to 0. \tag{13}$$

Now, by Proposition 2.1, $H_{n-1}(M, \mathbf{Z}_2)$ is trivial, so $H_{n-1}(M, \mathbf{Z}) \otimes \mathbf{Z}_2 = 0$. By (12), $H_{n-1}(M, \mathbf{Z}) = 0$ and the proposition follows.

Q.E.D.

Note that in the above proof of Proposition 3.2, we only use nonnegative Ricci curvature in the first step to eliminate the possibility of a two-ended double cover. This step could be proven without the nonnegative Ricci curvature condition, but it makes the proof unnecessarily long. Note also that it is in (13), when we apply the universal coefficient theorem, that we get our result for $\mathbf{Z}$. This step does not extend to a result with the more general $\mathbf{Z}_k$ or arbitrary abelian G.

Proof of Proposition 3.3 If M^n does not have a ray satisfying the loops to infinity property then by [So2], M^n is a flat normal bundle over a compact totally geodesic submanifold, N^{n-1}.

Since M^n has only one end, N^{n-1} is orientable if and only if M^n is unorientable. Since the base space of a normal bundle is homotopic to the bundle, $H_{n-1}(M, G) = H_{n-1}(N, G)$. Thus

$$H_{n-1}(M, G) = H_{n-1}(N, G) = \begin{cases} G, & \text{if M is unorientable} \\ \ker(G \overset{\times 2}{\to} G), & \text{if M is orientable} \end{cases}$$

(c.f. [Mun Cor 65.5]). Q.E.D.

Proof of Corollary 1.2: If M is neither a isometrically split manifold nor a flat normal bundle over a compact totally geodesic submanifold, then by

Theorem 1.1, we know $H_{n-1}(M,G) = 0$. The Universal Coefficient Theorem (c.f. [Mun, Thm 55.1]), states that there is a short exact sequence,

$$0 \to H_{n-1}(M,\mathbf{Z}) \otimes G \to H_{n-1}(M,G) \to H_{n-2}(M,\mathbf{Z}) * G \to 0. \qquad (14)$$

Substituting, $H_{n-1}(M,G) = 0$, we get

$$H_{n-2}(M,\mathbf{Z}) * G = 0. \qquad (15)$$

In particular, substituting $G = Z_k$, we have

$$0 = H_{n-2}(M,\mathbf{Z}) * \mathbf{Z}_k = \frac{H_{n-2}(M,\mathbf{Z})}{kH_{n-2}(M,\mathbf{Z})}. \qquad (16)$$

Thus for any finite number k, $H_{n-2}(M,\mathbf{Z})$ no elements of order k. Q.E.D.

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