The Covering Spectrum
and the Cut-off Covering Spectra

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The **Covering Spectrum**, $CovSpec(M)$, measures the size of holes in a complete length space or Riemannian manifold.

**Defn:** $M$ is a complete length space iff

$$d_M(p, q) = \inf\{L(\sigma) : \sigma(0) = p, \sigma(1) = q\}$$

and the infimum is achieved for all $p, q \in M$.

**Defn:** A geodesic, $\gamma : S^1 \to M$ or $\gamma : [0, L] \to M$, is a locally length minimizing curve.

**S-Wei 2004:** On a **compact** length space $K$

$$CovSpec(K) \subset (1/2)\text{Length}(K)$$

where $\text{Length}(K)$ is the set of lengths $\{L(\gamma)\}$ of closed geodesics $\gamma : S^1 \to K$.

**S-Wei 2004:** If compact spaces $K_i$ Gromov-Hausdorff converge to $K$ then

$$CovSpec(K_i) \to CovSpec(K) \cup \{0\}.$$

**DeSmit-Gornet-Sutton 2007:**

Isospectral examples with different CovSpec

**S-Wei 2007:** complete noncompact spaces and cut off covering spectra...
In this talk $M$ is a complete noncompact length space and $K$ is a compact length space.

I will define and discuss:

* The Covering Spectrum of $M$
* The Length Spectrum of $M$
* The $R$ Cut-off Spectrum of $M$
* The Cut-off Spectrum of $M$
* Gromov-Hausdorff Convergence
* Delta homotopies
* Applications
Towards defining $CovSpec(M)$ when the universal cover, $\tilde{M}$, exists:

**Defn:** $L(g) = \inf_{x \in M} d_{\tilde{M}}(g\bar{x}, \bar{x})$.

On a compact space, $L(g)$ is achieved by a closed geodesic.

**Defn:** The $\delta$ cover of $M$ is

$$\tilde{M}^\delta = \tilde{M}/\pi_1(M, \delta)$$

where $\pi_1(M, \delta) = \langle g : L(g) < 2\delta \rangle$.

**Example 1:** A $2\pi \times 4\pi$ flat torus $T$

- $\delta \in (0, \pi]$ implies $T^\delta = \tilde{T}$.
- $\delta \in (\pi, 2\pi]$ implies $T^\delta = S^1 \times R$
- $\delta \in (2\pi, \infty)$ implies $T^\delta = T$.

**Defn:** $\delta \in CovSpec(M)$ iff

$$\forall \delta' > \delta \text{ } \tilde{M}^{\delta'} \neq \tilde{M}^\delta$$

**Example 1:** $CovSpec(T) = \{\pi, 2\pi\}$
Example 2: $M = (-\infty, \infty) \times f S^1$
where $f(r) = e^{-r}$ has a cusp.

$L(g) = \inf_{x \in M} d_{\tilde{M}}(g\tilde{x}, \tilde{x}) = 0.$

$\pi_1(M, \delta) = \varnothing \quad \text{if} \quad L(g) < 2\delta \quad \Rightarrow \quad \pi_1(M).$

Thus all the $\delta$ covers of $M$ are trivial:

$\tilde{M}^\delta = \tilde{M} / \pi_1(M, \delta) = M$

$\text{CovSpec}(M) = \emptyset$

Example 3: $M = (-\infty, \infty) \times f S^1$
where $f(r) = e^{-r} + 1.$

For $g$ generating $\pi_1(M):$

$L(g) = \inf_{x \in M} d_{\tilde{M}}(g\tilde{x}, \tilde{x}) = 2\pi.$

So $\tilde{M}^\delta = M$ when $\delta \geq \pi$
$\tilde{M}^\delta = \tilde{M}$ when $\delta < \pi$

$\text{CovSpec}(M) = \{\pi\}$
S-Wei 2004: Compact $K$ has

\[ \text{CovSpec}(K) \subset \left( \frac{1}{2} \right) \text{Length}(K). \]

That is, $\delta \in \text{CovSpec}(K) \implies 2\delta \in \text{Length}(K)$.

**Not true** for noncompact $M$

because holes need not have geodesics.

Extend $\text{Length}(M)$?

**Defn:** The Shift Spectrum:

\[ \text{Shift}(M) = \{ L(g) : g \in \pi_1(M) \} \]

where $L(g) = \inf_{x \in M} d_M(g\tilde{x}, \tilde{x})$.

Note $\text{Shift}(K) \subset \text{Length}(K)$.

**Recall Example 2:** with cusp

$\text{Shift}(M) = \{0\}$.

**Recall Ex 3:** asymptotic to a cylinder

$\text{Shift}(M) = \{ 2\pi k : k = 1, 2, 3 \ldots \}$

Is $\text{CovSpec}(M) \subset \left( \frac{1}{2} \right) \text{Shift}(M)$?
Is $\text{CovSpec}(M) \subset (1/2)\text{Shift}(M)$? No!

**Example 4:** $M$ a line with circles of circumference, $L_i$, attached at the integers, where $L_i$ decrease to $L > 0$.

Then $M^\delta$ opens all the handles with length $L_i \geq 2\delta$. Thus,

$\text{CovSpec}(M) = \{\delta : \forall \delta' > \delta \ M^{\delta'} \neq \tilde{M}^{\delta}\}$

$= \{L_1/2, L_2/2, L_3/2, \ldots\} \cup \{L/2\}$.

**S-Wei 2007:**

$\text{CovSpec}(M) \subset \text{Cl}_{\text{lower}}((1/2)\text{Shift}(M))$

where $\text{Cl}_{\text{lower}}(A)$ includes all the limits of decreasing sequences in $A$. 
$\text{CovSpec}(M) \subset Cl_{\text{lower}}((1/2)\text{Shift}(M))$

**Example 5:** The line with circles where $L_i$ run through the rationals has $\text{Shift}(M)$ equal to the rationals and $\text{CovSpec}(M) = (0, \infty)$.

**S-Wei 2004:** If $K$ is compact with a universal cover, $\tilde{K}$, then $K$ has a finite spectrum.

But what is the definition of $\text{CovSpec}$ when there is no universal cover?

Could we adapt $\text{CovSpec}(M)$ instead of $\text{Length}(M)$? Yes.

*Next we give the definition of $\text{CovSpec}(M)$ without universal covers which helps define $\text{CovSpec}_{\text{cut}}(M)$.*
Recall Defn: The $\delta$ cover of $M$ is 

$$\tilde{M}^\delta = \tilde{M}/\pi_1(M, \delta)$$

where $\pi_1(M, \delta) = < g : L(g) < 2\delta >$.

In general $\tilde{M}^\delta$ is the Spanier cover defined using the covering of $M$ by $B_q(\delta)$ so that curves $\alpha \beta \alpha^{-1}$ where the loop $\beta \subset B_q(\delta)$ lift as closed curves to $\tilde{M}^\delta$.

Lemma (S-Wei): When $\tilde{M}$ exists, then $g = [\alpha \beta \alpha^{-1}]$ where $\beta \subset B_q(\delta)$ are generated by $g$ with $L(g) < 2\delta$.

When $\tilde{K}$ does not exist we still have $CovSpec(K) \subset (1/2)L(K)$.
Towards the cut-off spectra:

**Defn:** The $R$ cut-off $\delta$ cover of $M$, $\tilde{M}^{\delta,R}_{\text{cut}}$, is the Spanier cover of $M$ defined so that loops $\alpha \beta \alpha^{-1}$ where

- $\beta \subset B_q(\delta)$ or $\beta \subset M \setminus \overline{B_p}(\delta)$

lift to loops and all else lifts to paths.

**Example 6:** If $M$ is a line with circles of circumference $L_i$ attached at the integers then

$\tilde{M}^{\delta,R}_{\text{cut}}$ is the cover which opens all the circles except those with $L_i \geq 2\delta$ and those lying outside $\overline{B_p}(R)$.

**Example 7:** If $M$ a cylinder, then all the $\tilde{M}^{\delta,R}_{\text{cut}}$ are just $M$

because all loops are freely homotopic to a loop lying outside $B_p(R)$. 

Recall Defn:
The $R$ cut-off delta cover of $M$, $\tilde{M}^{\delta,R}_{\text{cut}}$, is the Spanier cover of $M$ defined so that loops $\alpha\beta\alpha^{-1}$ where

$\beta \subset B_q(\delta)$ or $\beta \subset M \setminus \bar{B}_p(\delta)$

lift as closed curves.

Defn:
The $R$ cut-off covering spectrum of $M$,

$\text{CovSpec}^R_{\text{cut}}(M) = \{ \delta : \forall \delta' > \delta \, \tilde{M}^{\delta',R}_{\text{cut}} \neq \tilde{M}^{\delta,R}_{\text{cut}} \}$

Recall Example 6: $M$ the line with circles. If $k$ of the circles $L_1, L_2, \ldots L_k$ are lying inside $\bar{B}_p(R)$ then

$\text{CovSpec}^R_{\text{cut}}(M) = \{ L_1/2, L_2/2, \ldots L_k/2 \}$

Recall Example 7: $M$ is a cylinder then all the $\tilde{M}^{\delta,R}_{\text{cut}} = M$ because all loops are freely homotopic to a loop lying outside $\bar{B}_p(R)$. So

$\text{CovSpec}^R_{\text{cut}}(M) = \emptyset$. 
Defn: The cut-off delta cover of $M$, 
\[ \tilde{M}^\delta_{\text{cut}} = \lim_{R \to \infty} \tilde{M}^\delta_{\text{cut}}. \]

Thm (S-Wei): It exists and is unique.

Defn: The cut-off covering spectrum 
\[ \text{CovSpec}_{\text{cut}}(M) = \{ \delta : \forall \delta' > \delta \tilde{M}^\delta' \neq \tilde{M}^\delta \} \]

Recall Example 6: $M$ a line with circles attached of circumference, $L_i$, then $\tilde{M}^\delta_{\text{cut}}$ unravels all circles with 
$L_i \geq 2\delta$, so $\tilde{M}^\delta_{\text{cut}} = \tilde{M}^\delta$
and $\text{CovSpec}_{\text{cut}}(M) = \{L_1/2, L_2/2, \ldots\}$.

Recall Example 7:
$M$ is a cylinder. All the $\tilde{M}^{\delta,R} = M$.
So $\tilde{M}^\delta_{\text{cut}} = M$ and $\text{CovSpec}_{\text{cut}}(M) = \emptyset$.

Thm: If $M^n$ has the loops to infinity property then $\text{CovSpec}_{\text{cut}}(M) = \emptyset$. 
S-Wei 2007: For any complete $M$

$$CovSpec^R_{cut}(M) \subset CovSpec_{cut}(M)$$

S-Wei 2007: For any complete $M$

$$CovSpec_{cut}(M) =$$

$$= Cl_{lower}( \bigcup_{R>0} CovSpec^R_{cut}(M))$$

S-Wei 2007: Locally compact $M$

$$CovSpec^R_{cut}(M) \subset Length(M).$$

$$CovSpec_{cut}(M) = Cl_{lower}(Length(M))$$

The cut-off covering spectrum does not detect holes that:

* disappear into cusps [Ex 2] or

* are asymptotic to a cylinders [Ex 3].

It detects geodesics so it is a natural object of study for spectral theorists.
Gromov-Hausdorff Convergence:

$K_i$ compact converge to $K$ iff

$\epsilon_i \to 0$ and $f_i : K_i \to K$ which are

almost onto: $K \subset T_{\epsilon_i}(f_i(K_i))$

and almost distance preserving:

$|d_K(f_i(x), f_i(y)) - d_{K_i}(x, y)| < \epsilon_i$.

**Theorem [S-Wei 2004]:**

If $K_i$ simply connected and $K_i \to K$
then $\text{CovSpec}(K) = \emptyset$.

**Noncompact setting?**

$M_i$ a sequence of stretching spheres
converging to $M$ a cylinder
then $\text{CovSpec}(M) \neq \emptyset$.

**We prove:** If $M_i$ are locally compact
and simply connected and $M_i \to M$
then $\text{CovSpec}_{\text{cut}}(M) = \emptyset$.

**details and definitions...**
Gromov-Hausdorff Convergence:

**Thm '04:** If $K_j \to K$ in the GH sense then
a) $\delta_j \in \text{CovSpec}(K_j)$ and $\delta_j \to \delta$
   implies $\delta \in \text{CovSpec}(K) \cup \{0\}$.
b) $\forall \delta \in \text{CovSpec}(K)$ there exists
   $\delta_j \in \text{CovSpec}(K_j)$ such that $\delta_j \to \delta$.

Locally Compact $(M_j, p_j)$:

**Defn:** $(M_j, p_j)$ coverge to $(M, p)$ in the **pointed Gromov Hausdorff** sense if
$\forall R$, the balls $B(p_j, R)$ GH converge to $B(p, R)$.

**Example 8:** (a) fails for $M_j$ with a handle of fixed size sliding out to infinity.

**Example 9:** (b) fails for cusped $M_j$

\[ M_j = (-\infty, \infty) \times f_j S^1 \]

with $f_j(r) = e^{-jr}$ converging to a cylinder.

* $\text{CovSpec}(M_j) = \emptyset \neq \text{CovSpec}(M) = \{\pi\}$.

**Example 10:** (b) fails for $M_k$ that are the same capped cylinder with $p_k \to \infty$.

So $M$ is a cylinder and * holds again.
Thm '07: If locally compact $M_i \to M$ in the pointed Gromov Hausdorff sense then
\begin{itemize}
  \item[a)] $\delta_i \in CovSpec_{cut}^R(M_i)$ and $\delta_i \to \delta$
  \hspace{1em} implies $\delta \in CovSpec_{cut}^R(M) \cup \{0\}$.
  \item[b)] $\forall \delta \in CovSpec^R_{cut}(M)$ and $R' > R$,
  \hspace{1em} $\exists \delta_i \in CovSpec^R_{cut}(M_i)$ with $\delta_i \to \delta$.
\end{itemize}

Recall Example 8: (a) holds
Handle is eventually outside $B(p_i, R)$ so $CovSpec_{cut}^R(M_i) = \emptyset = CovSpec_{cut}^R(M)$.

Recall Example 9: (b) holds
Cusped $M_k$ have
** $CovSpec(M_k) = CovSpec_{cut}^R(M_k) = \emptyset$.
Limit $M$ is cylinder so $CovSpec_{cut}^R(M_k) = \emptyset$.

Recall Example 10: (b) holds for $M_k$ which the same capped cylinder with $p_k \to \infty$
So $M$ is a cylinder and ** holds again.
Thm ’07: Locally compact $M_i$ and $M$
If $M_i \rightarrow M$ in the pointed GH sense then
b) $\forall \delta \in CovSpec_{cut}(M)$ there exists,
   $\delta_i \in CovSpec_{cut}(M_i)$ with $\delta_i \rightarrow \delta$.

Recall Example 8: (a) fails because a handle can slide off to infinity.

Proof Idea:

1) Key step in the compact setting:
   show the $R$ cut-off $\delta$ covers converge.
2) Define $\delta$ homotopies.
3) Localize long homotopies and loops to $\infty$.
4) Use local compactness to control lengths of generating $g$ by covering balls with nets and applying the pigeon-hole principle.
Summarizing our paper:

$K$ compact, $M$ locally compact:

$CovSpec(K) \subset (1/2)\text{Length}(K)$

$CovSpec^R_{cut}(M) \subset (1/2)\text{Length}(M)$

$CovSpec_{cut}(M) \subset \text{Cl}_{\text{lower}}((1/2)\text{Length}(M))$

**Gromov-Hausdorff Convergence:**

If $K_i$ simply connected and $K_i \rightarrow K$  
then $CovSpec(K) = \emptyset$.

Note $M_i$ a sequence of stretching spheres 
converging to $M$ a cylinder  
then $CovSpec(M) \neq \emptyset$.

If $M_i$ simply connected and $M_i \rightarrow M$  
then $CovSpec_{cut}(M) = \emptyset$. 
Soul Thm (Cheeger-Gromoll): If $M$ has $\text{sect} \geq 0$ then $M$ has a compact totally geodesic submanifold, $S$, and $M$ is a normal bundle over $S$.

Thm (Sharafutdinov): There is a distance nonincreasing deformation retraction from $M$ to the soul $S$.

Cor (S-Wei): $\text{CovSpec}(M) \subset L(M)$ and is determined by the MLS.

Thm (S): If $N$ has $\text{Ricci} \geq 0$ then $N$ either has the loops to infinity property or it is a flat normal bundle over a compact totally geodesic soul, $S^k$.

Cor (S-Wei): $\text{CovSpec}_{\text{cut}}(N^n) = \emptyset$ or it is a flat normal bundle...
The Rescaled Spectra:

Recall Defn: The covering spectrum was defined using lengths of \( g \in \pi_1(M) \):

\[
L(g) = \inf_{x \in M} d_M(g\tilde{x}, \tilde{x})
\]

and groups \( \pi_1(\delta) = \{ g : L(g) < 2\delta \} \).

So \( CovSpec(M) = \{ \delta : \pi_1(\delta') \neq \pi_1(\delta) \ \forall \delta' > \delta \} \)

Defn: The rescaled covering spectrum is defined the same way using the rescaled length

\[
L_{rs}(g) = \inf_{x \in M} \frac{d_M(g\tilde{x}, \tilde{x})}{d(x,p)}
\]

\( CovSpec_{rs}(M) \) is scale invariant and always \( \leq 1 \).

Defn: The infinite rescaled covering spectrum is defined using the infinite rescaled length

\[
L_{rs}^{\infty}(g) = \lim_{R \to \infty} \inf_{x \in M \setminus B_p(R)} \frac{d_M(g\tilde{x}, \tilde{x})}{d(x,p)}
\]

\( CovSpec_{rs}^{\infty}(M) \in (0, 1] \) is scale and basepoint invariant. \( 1 \in CovSpec_{rs}^{\infty}(M) \) if \( M \) has a handle. Other elements detect linearly opening holes.
Preprints soon to appear (S-Wei):

The Cut-off Covering Spectrum
Various Covering Spectra of Complete Spaces

Already published (S-Wei):

The covering spectrum of a compact length space
JDG 67 2004 (erratum JDG 74 2006, Ex 10.3)

All available on my webpage:
http://comet.lehman.cuny.edu/sormani

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