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The Covering Spectrum

and the Cut-off Covering Spectra

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The **Covering Spectrum**, $CovSpec(M)$, measures the size of holes in a complete length space or Riemannian manifold.

Defn: M is a complete length space iff
$d_M(p,q) = \inf\{L(\sigma) : \sigma(0) = p,\, \sigma(1) = q\}$
and the infimum is achived for all $p, q \in M$.

Defn: A geodesic, $\gamma : S^1 \to M$ or $\gamma : [0, L] \to M$, is a locally length minimizing curve.

S-Wei 2004: On a **compact** length space K

$$CovSpec(K) \subset (1/2)Length(K)$$

where $Length(K)$ is the set of lengths $\{L(\gamma)\}$ of closed geodesics $\gamma : S^1 \to K$.

S-Wei 2004: If compact spaces K_i Gromov-Hausdorff converge to K then

$$CovSpec(K_i) \to CovSpec(K) \cup \{0\}.$$

DeSmit-Gornet-Sutton 2007:
Isospectral examples with different CovSpec

S-Wei 2007: complete noncompact spaces
and cut off covering spectra...

In this talk M is a complete noncompact length space and K is a compact length space.

I will define and discuss:

* The Covering Spectrum of M
* The Length Spectrum of M
* The R Cut-off Spectrum of M
* The Cut-off Spectrum of M
* Gromov-Hausdorff Convergence
* Delta homotopies
* Applications

Towards defining $CovSpec(M)$ when the universal cover, $\tilde{M}$, exists:

Defn: $L(g) = \inf_{x \in M} d_{\tilde{M}}(g\tilde{x}, \tilde{x})$.
On a compact space, $L(g)$ is achieved by a closed geodesic.

Defn: The δ cover of M is

$$\tilde{M}^\delta = \tilde{M}/\pi_1(M, \delta)$$

where $\pi_1(M, \delta) = < g \, : L(g) < 2\delta >$.

Example 1: A $2\pi \times 4\pi$ flat torus T
$\delta \in (0, \pi]$ implies $\tilde{T}^\delta = \tilde{T}$.
$\delta \in (\pi, 2\pi]$ implies $\tilde{T}^\delta = S^1 \times R$
$\delta \in (2\pi, \infty)$ implies $\tilde{T}^\delta = T$.

Defn: $\delta \in CovSpec(M)$ iff
$\forall \, \delta' > \delta \ \tilde{M}^{\delta'} \neq \tilde{M}^\delta$

Example 1: $CovSpec(T) = \{\pi, 2\pi\}$

Example 2: $M = (-\infty, \infty) \times_f S^1$
where $f(r) = e^{-r}$ has a cusp.

$L(g) = \inf_{x \in M} d_{\tilde{M}}(g\tilde{x}, \tilde{x}) = 0.$

$\pi_1(M, \delta) =< g : L(g) < 2\delta >= \pi_1(M).$

Thus all the δ covers of M are trivial:

$$\tilde{M}^\delta = \tilde{M}/\pi_1(M, \delta) = M$$

$CovSpec(M) = \emptyset$

Example 3: $M = (-\infty, \infty) \times_f S^1$
where $f(r) = e^{-r} + 1$.

For g generating $\pi_1(M)$:

$$L(g) = \inf_{x \in M} d_{\tilde{M}}(g\tilde{x}, \tilde{x}) = 2\pi.$$

So $\tilde{M}^\delta = M$ when $\delta \geq \pi$
$\tilde{M}^\delta = \tilde{M}$ when $\delta < \pi$

$CovSpec(M) = \{\pi\}$

S-Wei 2004: Compact K has

$CovSpec(K) \subset (1/2)Length(K)$.

That is, $\delta \in CovSpec(K) \Longrightarrow 2\delta \in Length(K)$.

Not true for **noncompact** M because holes need not have geodesics.

Extend $Length(M)$?

Defn: The Shift Spectrum:

$Shift(M) = \{L(g) : g \in \pi_1(M)\}$

where $L(g) = \inf_{x \in M} d_{\tilde{M}}(g\tilde{x}, \tilde{x})$.

Note $Shift(K) \subset Length(K)$.

Recall Example 2: with cusp

$Shift(M) = \{0\}$.

Recall Ex 3: asymptotic to a cylinder

$Shift(M) = \{2\pi k : k = 1, 2, 3...\}$

Is $CovSpec(M) \subset (1/2)Shift(M)$?

Is $CovSpec(M) \subset (1/2)Shift(M)$? No!

Example 4: M a line with circles of circumference, L_i, attached at the integers, where L_i decrease to $L > 0$.

Then $\tilde{M}^\delta$ opens all the handles with length $L_i \geq 2\delta$. Thus,

$$CovSpec(M) =$$
$$= \{\delta : \forall \delta' > \delta \, \tilde{M}^{\delta'} \neq \tilde{M}^\delta\}$$
$$= \{L_1/2, L_2/2, L_3/2, ...\} \cup \{L/2\}.$$

S-Wei 2007:

$$CovSpec(M) \subset Cl_{lower}((1/2)Shift(M))$$

where $Cl_{lower}(A)$ includes all the limits of decreasing sequences in A.

$CovSpec(M) \subset Cl_{lower}((1/2)Shift(M))$

Example 5: The line with circles where L_i run through the rationals has $Shift(M)$ equal to the rationals and $CovSpec(M) = (0, \infty)$.

S-Wei 2004: If K is compact with a universal cover, $\tilde{K}$, then K has a finite spectrum.

But what is the definition of $CovSpec$ when there is no universal cover?

Could we to adapt $CovSpec(M)$ instead of $Length(M)$? Yes.

Next we give the definition of $CovSpec(M)$ without universal covers which helps define $CovSpec_{cut}(M)$.

Recall Defn: The δ cover of M is

$$\tilde{M}^\delta \;=\; \tilde{M}/\pi_1(M,\delta)$$

where $\pi_1(M,\delta) \;=< g \,:\, L(g) < 2\delta \;>$.

In general $\tilde{M}^\delta$ is the *Spanier cover* defined using the covering of M by $B_q(\delta)$ so that curves $\alpha\beta\alpha^{-1}$ where
the loop $\beta \subset B_q(\delta)$
lift as closed curves to $\tilde{M}^\delta$.

Lemma (S-Wei): When $\tilde{M}$ exists, then $g = [\alpha\beta\alpha^{-1}]$ where $\beta \subset B_q(\delta)$ are generated by g with $L(g) < 2\delta$.

When $\tilde{K}$ does not exist we still have
$CovSpec(K) \subset (1/2)L(K)$.

Towards the cut-off spectra:

Defn: The R *cut-off* δ *cover* of M, $\tilde{M}^{\delta,R}_{cut}$, is the Spanier cover of M defined so that loops $\alpha\beta\alpha^{-1}$ where

$\beta \subset B_q(\delta)$ or $\beta \subset M \setminus \bar{B}_p(\delta)$

lift to loops and all else lifts to paths.

Example 6: If M is a line with circles of circumference L_i attached at the integers then

$\tilde{M}^{\delta,R}_{cut}$ is the cover which

opens all the circles except those with $L_i \geq 2\delta$ and those lying outside $\bar{B}_p(R)$.

Example 7: If M a cylinder, then all the $\tilde{M}^{\delta,R}_{cut}$ are just M

because all loops are freely homotopic to a loop lying outside $\bar{B}_p(R)$.

Recall Defn:
The R *cut-off delta cover* of M, $\tilde{M}^{\delta,R}_{cut}$,
is the Spanier cover of M defined so that
loops $\alpha\beta\alpha^{-1}$ where

$\beta \subset B_q(\delta)$ or $\beta \subset M \setminus \bar{B}_p(\delta)$

lift as closed curves.

Defn:
The R *cut-off covering spectrum* of M,

$CovSpec^R_{cut}(M) =$

$$= \{\delta : \forall \delta' > \delta \, \tilde{M}^{\delta',R}_{cut} \neq \tilde{M}^{\delta,R}_{cut}\}$$

Recall Example 6: M the line with circles.
If k of the circles $L_1, L_2, ...L_k$
are lying inside $\bar{B}_p(R)$ then
$CovSpec^R_{cut}(M) = \{L_1/2, L_2/2, ...L_k/2\}$

Recall Example 7:
M is a cylinder then all the $\tilde{M}^{\delta,R}_{cut} = M$!
because all loops are freely homotopic
to a loop lying outside $\bar{B}_p(R)$. So

$$CovSpec^R_{cut}(M) = \emptyset.$$

Defn: The *cut-off delta cover* of M,

$\tilde{M}^{\delta}_{cut} = \lim_{R\to\infty} \tilde{M}^{\delta,R}_{cut}$.

Thm (S-Wei): It exists and is unique.

Defn: The *cut-off covering spectrum* $CovSpec_{cut}(M) =$

$= \{\delta : \forall \delta' > \delta \, \tilde{M}^{\delta'} \neq \tilde{M}^{\delta}\}$

Recall Example 6: M a line with circles attached of circumference, L_i, then $\tilde{M}^{\delta}_{cut}$ unravels all circles with

$L_i \geq 2\delta$, so $\tilde{M}^{\delta}_{cut} = \tilde{M}^{\delta}$

and $CovSpec_{cut}(M) = \{L_1/2, L_2/2, ...\}$.

Recall Example 7:
M is a cylinder. All the $\tilde{M}^{\delta,R} = M$.
So $\tilde{M}^{\delta}_{cut} = M$ and $CovSpec_{cut}(M) = \emptyset$.

Thm: If M^n has the loops to infinity property then $CovSpec_{cut}(M) = \emptyset$.

S-Wei 2007: For any complete M

$$CovSpec^R_{cut}(M) \subset CovSpec_{cut}(M)$$

S-Wei 2007: For any complete M

$$CovSpec_{cut}(M) =$$

$$= Cl_{lower}(\bigcup_{R>0} CovSpec^R_{cut}(M))$$

S-Wei 2007: Locally compact M

$$CovSpec^R_{cut}(M) \subset Length(M).$$

$$CovSpec_{cut}(M) = Cl_{lower}(Length(M))$$

The cut-off covering spectrum does not detect holes that:
** disappear into cusps* [Ex 2] *or*
** are asymptotic to a cylinders* [Ex 3].
It detects geodesics so it is a natural object of study for spectral theorists.

Gromov-Hausdorff Convergence:

K_i compact converge to K iff

$\epsilon_i \to 0$ and $f_i : K_i \to K$ which are

almost onto: $K \subset T_{\epsilon_i}(f_i(K_i))$

and almost distance preserving:

$|d_K(f_i(x), f_i(y)) - d_{K_i}(x, y)| < \epsilon_i$.

Theorem [S-Wei 2004]:

If K_i simply connected and $K_i \to K$
then $CovSpec(K) = \emptyset$.

Noncompact setting?

M_i a sequence of stretching spheres
converging to M a cylinder
then $CovSpec(M) \neq \emptyset$.

We prove: If M_i are locally compact
and simply connected and $M_i \to M$
then $CovSpec_{cut}(M) = \emptyset$.

details and definitions...

Gromov-Hausdorff Convergence:

Thm '04: If $K_j \to K$ in the GH sense then

a) $\delta_j \in CovSpec(K_j)$ and $\delta_j \to \delta$
implies $\delta \in CovSpec(K) \cup \{0\}$.

b) $\forall \delta \in CovSpec(K)$ there exists
$\delta_j \in CovSpec(K_j)$ such that $\delta_j \to \delta$.

Locally Compact (M_j, p_j):

Defn: (M_j, p_j) coverge to (M, p) in the **pointed Gromov Hausdorff** sense if $\forall R$, the balls $B(p_j, R)$ GH converge to $B(p, R)$.

Example 8: (a) fails for M_j with a handle of fixed size sliding out to infinity.

Example 9: (b) fails for cusped M_j

$M_j = (-\infty, \infty) \times_{f_j} S^1$

with $f_j(r) = e^{-jr}$ converging to a cylinder.

$*$ $CovSpec(M_j) = \emptyset \neq CovSpec(M) = \{\pi\}$.

Example 10: (b) fails for M_k that are the same capped cylinder with $p_k \to \infty$.

So M is a cylinder and * holds again.

Thm '07: If locally compact $M_i \to M$
in the pointed Gromov Hausdorff sense then
a) $\delta_i \in CovSpec^R_{cut}(M_i)$ and $\delta_i \to \delta$
implies $\delta \in CovSpec^R_{cut}(M) \cup \{0\}$.
b) $\forall \delta \in CovSpec^R_{cut}(M)$ and $R' > R$,
$\exists \delta_i \in CovSpec^{R'}_{cut}(M_i)$ with $\delta_i \to \delta$.

Recall Example 8: (a) holds
Handle is eventually outside $B(p_i, R)$
so $CovSpec^R_{cut}(M_i) = \emptyset = CovSpec^R_{cut}(M)$.

Recall Example 9: (b) holds
Cusped M_k have
$** \ CovSpec(M_k) = CovSpec^R_{cut}(M_k) = \emptyset$.
Limit M is cylinder so $CovSpec^R_{cut}(M_k) = \emptyset$.

Recall Example 10: (b) holds for M_k which
the same capped cylinder with $p_k \to \infty$
So M is a cylinder and ** holds again.

Thm '07: Locally compact M_i and M
If $M_i \to M$ in the pointed GH sense then
b) $\forall \delta \in CovSpec_{cut}(M)$ there exists,
$\delta_i \in CovSpec_{cut}(M_i)$ with $\delta_i \to \delta$.

Recall Example 8: (a) fails
because a handle can slide off to infinity.

Proof Idea:

1) Key step in the compact setting:

show the R cut-off δ covers converge.

2) Define δ homotopies.

3) Localize long homotopies and loops to ∞.

4) Use local compactness to control lengths of

generating g by covering balls with nets and

applying the pigeon-hole principle.

Summarizing our paper:

K compact, M locally compact:

$CovSpec(K) \subset (1/2)Length(K)$

$CovSpec^R_{cut}(M) \subset (1/2)Length(M)$

$CovSpec_{cut}(M) \subset Cl_{lower}((1/2)Length(M))$

Gromov-Hausdorff Convergence:

If K_i simply connected and $K_i \to K$
then $CovSpec(K) = \emptyset$.

Note M_i a sequence of stretching spheres
converging to M a cylinder
then $CovSpec(M) \neq \emptyset$.

If M_i simply connected and $M_i \to M$
then $CovSpec_{cut}(M) = \emptyset$.

Soul Thm (Cheeger-Gromoll): If M has *sect* ≥ 0 then M has a compact totally geodesic submanifold, S, and M is a normal bundle over S.

Thm (Sharafutdinov): There is a distance nonincreasing deformation retraction from M to the soul S.

Cor (S-Wei): $CovSpec(M) \subset L(M)$ and is determined by the MLS.

Thm (S): If N has $Ricci \geq 0$ then N either has the *loops to infinity property* or it is a flat normal bundle over a compact totally geodesic soul, S^k.

Cor (S-Wei): $CovSpec_{cut}(N^n) = \emptyset$ or it is a flat normal bundle...

The Rescaled Spectra:

Recall Defn: The *covering spectrum* was defined using lengths of $g \in \pi_1(M)$:

$$L(g) = \inf_{x \in M} d_{\tilde{M}}(g\tilde{x}, \tilde{x})$$

and groups $\pi_1(\delta) = < g : L(g) < 2\delta >$.

So $CovSpec(M) = \{\delta : \pi_1(\delta') \neq \pi_1(\delta) \, \forall \delta' > \delta\}$

Defn: The *rescaled covering spectrum* is defined the same way using the *rescaled length*

$$L_{rs}(g) = \inf_{x \in M} \frac{d_{\tilde{M}}(g\tilde{x}, \tilde{x})}{d(x,p)}$$

$CovSpec_{rs}(M)$ is scale invariant and always ≤ 1.

Defn: The *infinite rescaled covering spectrum* is defined using the *infinite rescaled length*

$$L_{rs}^{\infty}(g) = \lim_{R \to \infty} \inf_{x \in M \setminus B_p(R)} \frac{d_{\tilde{M}}(g\tilde{x}, \tilde{x})}{d(x,p)}$$

$CovSpec_{rs}^{\infty}(M) \in (0, 1]$ is scale and basepoint invariant. $1 \in CovSpec_{rs}^{\infty}(M)$ if M has a handle. Other elements detect linearly opening holes.

Preprints soon to appear (S-Wei):

The Cut-off Covering Spectrum

Various Covering Spectra of Complete Spaces

Already published (S-Wei):

The covering spectrum of a compact length space
JDG 67 2004 (erratum JDG 74 2006, Ex 10.3)

All available on my webpage:

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