The Intrinsic Flat Convergence of Riemannian Manifolds

C. Sormani and S. Wenger

Abstract:

We define a new distance between oriented Riemannian manifolds that we call the *intrinsic flat distance* based upon Ambrosio-Kirchheim's extension of Federer-Fleming's theory of integral currents on metric spaces. Limits of sequences of manifolds, M_j^k, with $vol(M_j) \leq V_0$, $vol(\partial M_j) \leq V_1$ and $diam(M_j) \leq D$ are countably H^k rectifiable metric spaces with an orientation and multiplicity that we call *integral current spaces*.

In general the Gromov-Hausdorff and intrinsic flat limits do not agree. However, we show that they do agree when the sequence of manifolds has nonnegative Ricci curvature and a uniform lower bound on volume and also when the sequence of manifolds has a uniform linear local geometric contractibility function. These results can be proven using work of Greene-Petersen, Gromov, Cheeger-Colding and Perelman.

This is joint work with S. Wenger. See http://comet.lehman.cuny.edu/sormani/intrinsicflat.html

Convergence of Riemannian Manifolds

A brief history...

Convergence of Riemannian Manifolds

A brief history...

$C^{1,\alpha}$ Convergence :

Limits are $C^{1,\alpha}$ manifolds diffeomorphic to the sequence

Convergence of Riemannian Manifolds

A brief history...

$C^{1,\alpha}$ Convergence :

Limits are $C^{1,\alpha}$ manifolds diffeomorphic to the sequence

Cheeger Compactness: $|R| \leq \Lambda, \ vol \geq V_0, \ diam \leq D$

Convergence of Riemannian Manifolds

A brief history...

$C^{1,\alpha}$ Convergence :

Limits are $C^{1,\alpha}$ manifolds diffeomorphic to the sequence

Cheeger Compactness: $|R| \leq \Lambda, \; vol \geq V_0, \; diam \leq D$

Anderson Compactness: $|Ric| \leq \Lambda, \; inj \geq i_0, \; \; diam \leq D$

Convergence of Riemannian Manifolds

A brief history...

$C^{1,\alpha}$ Convergence :

Limits are $C^{1,\alpha}$ manifolds diffeomorphic to the sequence

Cheeger Compactness: $|R| \leq \Lambda, \; vol \geq V_0, \; diam \leq D$

Anderson Compactness: $|Ric| \leq \Lambda, \; inj \geq i_0, \; \; diam \leq D$

Gromov-Hausdorff Convergence :

Limits are compact geodesic metric spaces

Topology and dimension can change in the limit.

Convergence of Riemannian Manifolds

A brief history...

$C^{1,\alpha}$ Convergence :

Limits are $C^{1,\alpha}$ manifolds diffeomorphic to the sequence

Cheeger Compactness: $|R| \leq \Lambda,\ vol \geq V_0,\ diam \leq D$

Anderson Compactness: $|Ric| \leq \Lambda,\ inj \geq i_0,\ \ diam \leq D$

Gromov-Hausdorff Convergence :

Limits are compact geodesic metric spaces

Topology and dimension can change in the limit.

Gromov Compactness: $Ric \geq -\Lambda,\ diam \leq D$.

Convergence of Riemannian Manifolds

A brief history...

$C^{1,\alpha}$ Convergence :

Limits are $C^{1,\alpha}$ manifolds diffeomorphic to the sequence

Cheeger Compactness: $|R| \le \Lambda,\ vol \ge V_0,\ diam \le D$

Anderson Compactness: $|Ric| \le \Lambda,\ inj \ge i_0,\ \ diam \le D$

Gromov-Hausdorff Convergence :

Limits are compact geodesic metric spaces

Topology and dimension can change in the limit.

Gromov Compactness: $Ric \ge -\Lambda,\ diam \le D$.

Introducing the Intrinsic Flat Convergence:

Limits are weighted oriented countably $\mathcal{H}^m$ rectifiable spaces

Convergence of Riemannian Manifolds

A brief history...

$C^{1,\alpha}$ Convergence :

Limits are $C^{1,\alpha}$ manifolds diffeomorphic to the sequence

Cheeger Compactness: $|R| \leq \Lambda, \ vol \geq V_0, \ diam \leq D$

Anderson Compactness: $|Ric| \leq \Lambda, \ inj \geq i_0, \ \ diam \leq D$

Gromov-Hausdorff Convergence :

Limits are compact geodesic metric spaces

Topology and dimension can change in the limit.

Gromov Compactness: $Ric \geq -\Lambda, \ diam \leq D$.

Introducing the Intrinsic Flat Convergence:

Limits are weighted oriented countably $\mathcal{H}^m$ rectifiable spaces

Wenger Compactness: $diam \leq D, \ vol \leq V$

Background

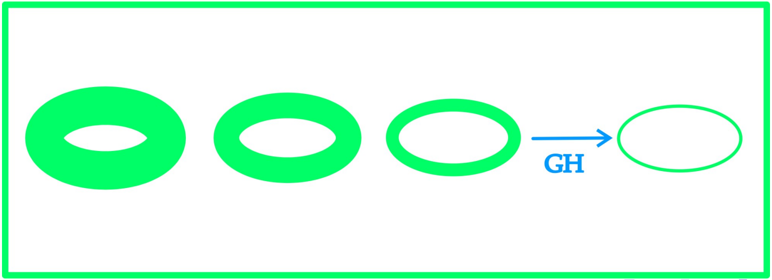
Defn: The Hausdorff distance between subsets $A, B \subset Z$ is

$$d_H^Z(A,B) = \inf\{r : \; A \subset T_r(B) \text{ and } B \subset T_r(A)\}.$$

Defn: The Gromov-Hausdorff distance between metric spaces, X, Y is

$$d_{GH}(X,Y) = \inf d_H^Z(\varphi(X), \psi(Y))$$

where the infimum is taken over all metric spaces Z and isometries $\varphi : X \to Z$ and $\psi : Y \to Z$.



Background

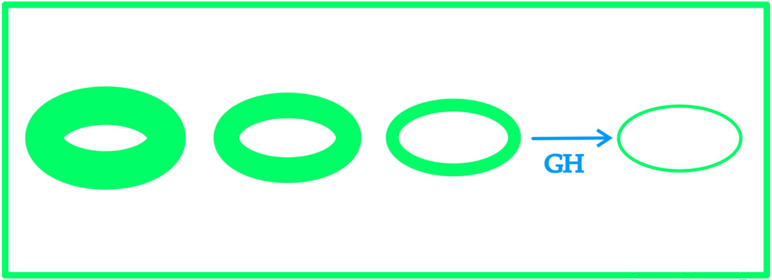
Defn: The Hausdorff distance between subsets $A, B \subset Z$ is

$$d_H^Z(A,B) = \inf\{r : \; A \subset T_r(B) \text{ and } B \subset T_r(A)\}.$$

Defn: The Gromov-Hausdorff distance between metric spaces, X, Y is

$$d_{GH}(X,Y) = \inf d_H^Z(\varphi(X), \psi(Y))$$

where the infimum is taken over all metric spaces Z and isometries $\varphi : X \to Z$ and $\psi : Y \to Z$.



Gromov's Compactness Theorem

Gromov's Compactness Theorem: If metric spaces X_j have $\mathrm{diam}(X_j) \le D$ and a uniform upper bound, $N(r)$, on the number of disjoint balls of radius r, then a subsequence converges: $X_{j_i} \xrightarrow{\mathrm{GH}} Y$.

Gromov's Compactness Theorem

Gromov's Compactness Theorem: If metric spaces X_j have $\operatorname{diam}(X_j) \le D$ and a uniform upper bound, $N(r)$, on the number of disjoint balls of radius r, then a subsequence converges: $X_{j_i} \xrightarrow{\mathrm{GH}} Y$.

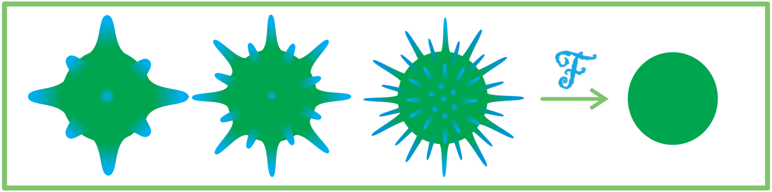
The converse also holds: if Y is compact and $X_j \xrightarrow{\mathrm{GH}} Y$, then there is a uniform upper bound on diameter $\operatorname{diam}(X_j)$ and on $N(r)$.

Gromov's Compactness Theorem

Gromov's Compactness Theorem: If metric spaces X_j have $\mathrm{diam}(X_j) \le D$ and a uniform upper bound, $N(r)$, on the number of disjoint balls of radius r, then a subsequence converges: $X_{j_i} \xrightarrow{\mathrm{GH}} Y$.

The converse also holds: if Y is compact and $X_j \xrightarrow{\mathrm{GH}} Y$, then there is a uniform upper bound on diameter $\mathrm{diam}(X_j)$ and on $N(r)$.

The following sequence has no Gromov-Hausdorff limit:

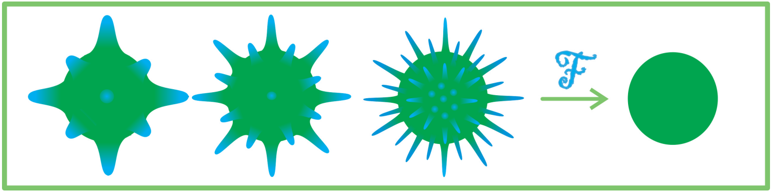


Gromov's Compactness Theorem

Gromov's Compactness Theorem: If metric spaces X_j have $\operatorname{diam}(X_j) \le D$ and a uniform upper bound, $N(r)$, on the number of disjoint balls of radius r, then a subsequence converges: $X_{j_i} \xrightarrow{\mathrm{GH}} Y$.

The converse also holds: if Y is compact and $X_j \xrightarrow{\mathrm{GH}} Y$, then there is a uniform upper bound on diameter $\operatorname{diam}(X_j)$ and on $N(r)$.

The following sequence has no Gromov-Hausdorff limit:



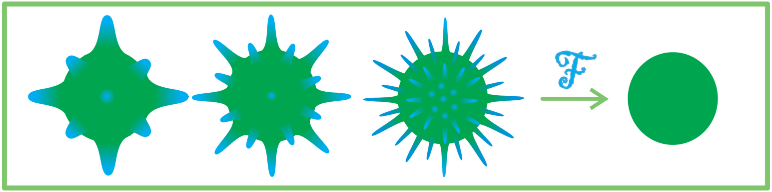
Ilmanen: can one define a weak convergence where this sequence of M_j^3 with positive scalar curvature converges to the three sphere?

Gromov's Compactness Theorem

Gromov's Compactness Theorem: If metric spaces X_j have $\mathrm{diam}(X_j) \le D$ and a uniform upper bound, $N(r)$, on the number of disjoint balls of radius r, then a subsequence converges: $X_{j_i} \xrightarrow{\mathrm{GH}} Y$.

The converse also holds: if Y is compact and $X_j \xrightarrow{\mathrm{GH}} Y$, then there is a uniform upper bound on diameter $\mathrm{diam}(X_j)$ and on $N(r)$.

The following sequence has no Gromov-Hausdorff limit:



Ilmanen: can one define a weak convergence where this sequence of M_j^3 with positive scalar curvature converges to the three sphere?

Background

Recall that by the Bishop-Gromov Volume Comparison Theorem, a sequence M_j^n with $Ricci(M_j) \geq 0$ and $diam(M_j) \leq D$ has a uniform upper bound on $N(r)$ and so a subsequence converges in the Gromov-Hausdorff sense.

Background

Recall that by the Bishop-Gromov Volume Comparison Theorem, a sequence M_j^n with $Ricci(M_j) \geq 0$ and $diam(M_j) \leq D$ has a uniform upper bound on $N(r)$ and so a subsequence converges in the Gromov-Hausdorff sense.

Cheeger-Colding Noncollapsing Theorem:
If M_j^m have $\mathrm{Ricci} \geq 0$ and $\mathrm{vol}(M_j^m) \geq V_0 > 0$ then the limit spaces are countably $\mathcal{H}^m$ rectifiable spaces.

Background

Recall that by the Bishop-Gromov Volume Comparison Theorem, a sequence M_j^n with $Ricci(M_j) \geq 0$ and $diam(M_j) \leq D$ has a uniform upper bound on $N(r)$ and so a subsequence converges in the Gromov-Hausdorff sense.

Cheeger-Colding Noncollapsing Theorem:
If M_j^m have $\mathrm{Ricci} \geq 0$ and $\mathrm{vol}(M_j^m) \geq V_0 > 0$ then the limit spaces are countably $\mathcal{H}^m$ rectifiable spaces.

This is intriguing, because ordinarily a Gromov-Hausdorff limit could have higher or lower dimension than the sequence and no rectifiability.

Background

Recall that by the Bishop-Gromov Volume Comparison Theorem, a sequence M_j^n with $Ricci(M_j) \geq 0$ and $diam(M_j) \leq D$ has a uniform upper bound on $N(r)$ and so a subsequence converges in the Gromov-Hausdorff sense.

Cheeger-Colding Noncollapsing Theorem:
If M_j^m have $\mathrm{Ricci} \geq 0$ and $\mathrm{vol}(M_j^m) \geq V_0 > 0$ then the limit spaces are countably $\mathcal{H}^m$ rectifiable spaces.

This is intriguing, because ordinarily a Gromov-Hausdorff limit could have higher or lower dimension than the sequence and no rectifiability.

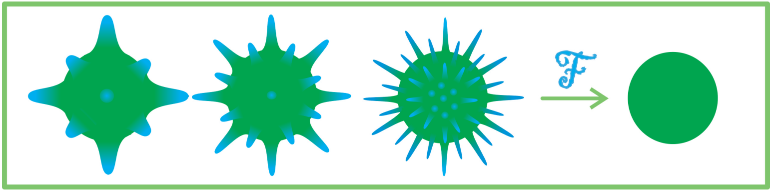
A new convergence?

Can one find a new kind of convergence with a nice compactness theorem usable in the setting of manifolds with positive scalar curvature?

A new convergence?

Can one find a new kind of convergence with a nice compactness theorem usable in the setting of manifolds with positive scalar curvature?

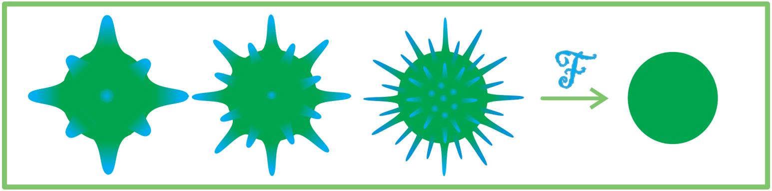
What kind of notion has sequences like this converge?



A new convergence?

Can one find a new kind of convergence with a nice compactness theorem usable in the setting of manifolds with positive scalar curvature?

What kind of notion has sequences like this converge?

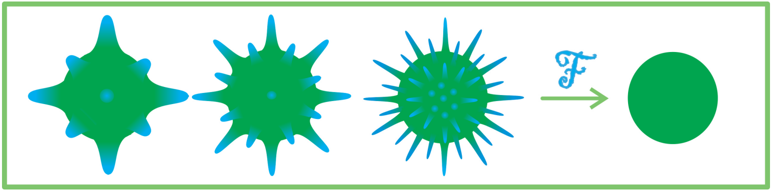


What about having countably $\mathcal{H}^m$ rectifiable limits just like sequences of manifolds with nonnegative Ricci curvature?

A new convergence?

Can one find a new kind of convergence with a nice compactness theorem usable in the setting of manifolds with positive scalar curvature?

What kind of notion has sequences like this converge?



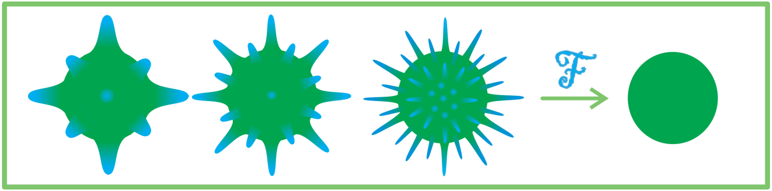
What about having countably $\mathcal{H}^m$ rectifiable limits just like sequences of manifolds with nonnegative Ricci curvature?

Maybe some sort of intrinsic flat convergence?

A new convergence?

Can one find a new kind of convergence with a nice compactness theorem usable in the setting of manifolds with positive scalar curvature?

What kind of notion has sequences like this converge?



What about having countably $\mathcal{H}^m$ rectifiable limits just like sequences of manifolds with nonnegative Ricci curvature?

Maybe some sort of intrinsic flat convergence?

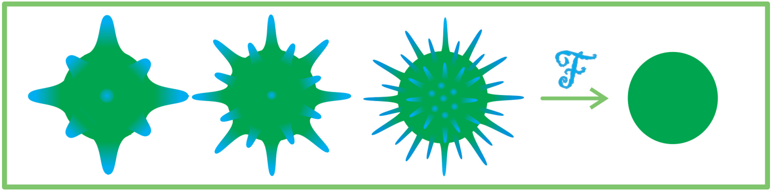
Reviewing Flat Convergence of Integral Currents

Recall Federer-Fleming's notion of an integral current.
By viewing a k dim submanifold of Euclidean space as an integral current and studying its behavior as a means of integrating k forms, Federer-Fleming defined a new notion of convergence of submanifolds: weak convergence as integral currents.

Reviewing Flat Convergence of Integral Currents

Recall Federer-Fleming's notion of an integral current.
By viewing a k dim submanifold of Euclidean space as an integral current and studying its behavior as a means of integrating k forms, Federer-Fleming defined a new notion of convergence of submanifolds: weak convergence as integral currents.

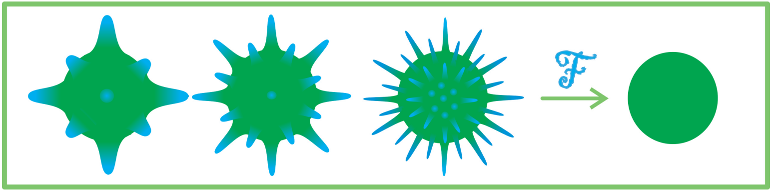
This theory was used to study Plateau's problems and other problems in which one is concerned with infimum of area:



Reviewing Flat Convergence of Integral Currents

Recall Federer-Fleming's notion of an integral current.
By viewing a k dim submanifold of Euclidean space as an integral current and studying its behavior as a means of integrating k forms, Federer-Fleming defined a new notion of convergence of submanifolds: weak convergence as integral currents.

This theory was used to study Plateau's problems and other problems in which one is concerned with infimum of area:

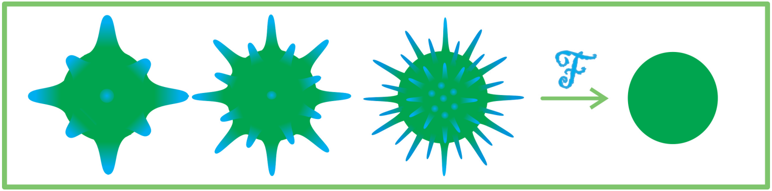


The limits are also integral currents: integrals over k dimensional countably rectifiable subsets of Euclidean space.

Reviewing Flat Convergence of Integral Currents

Recall Federer-Fleming's notion of an integral current.
By viewing a k dim submanifold of Euclidean space as an integral current and studying its behavior as a means of integrating k forms, Federer-Fleming defined a new notion of convergence of submanifolds: weak convergence as integral currents.

This theory was used to study Plateau's problems and other problems in which one is concerned with infimum of area:



The limits are also integral currents: integrals over k dimensional countably rectifiable subsets of Euclidean space.

Reviewing Flat convergence of Integral Currents

Recall Federer-Fleming's notion of an integral current.
By viewing a k dim submanifold of Euclidean space as an integral current and studying its behavior as a means of integrating k forms, Federer-Fleming defined a new notion of convergence of submanifolds: weak convergence as integral currents.

Reviewing Flat convergence of Integral Currents

Recall Federer-Fleming's notion of an integral current.
By viewing a k dim submanifold of Euclidean space as an integral current and studying its behavior as a means of integrating k forms, Federer-Fleming defined a new notion of convergence of submanifolds: weak convergence as integral currents.

Federer-Fleming defined the **flat distance**. When the integral currents have uniformly bounded total mass, they converge with respect to the flat norm iff they converge in the weak sense.

Reviewing Flat convergence of Integral Currents

Recall Federer-Fleming's notion of an integral current.
By viewing a k dim submanifold of Euclidean space as an integral current and studying its behavior as a means of integrating k forms, Federer-Fleming defined a new notion of convergence of submanifolds: weak convergence as integral currents.

Federer-Fleming defined the **flat distance**. When the integral currents have uniformly bounded total mass, they converge with respect to the flat norm iff they converge in the weak sense.
The key concept needed to understand the flat distance is the mass of the current and the boundary. For a submanifold, mass is just the volume and the boundary is as in a manifold. In general, boundaries are defined via Stoke's Theorem.

Reviewing Flat convergence of Integral Currents

Recall Federer-Fleming's notion of an integral current.
By viewing a k dim submanifold of Euclidean space as an integral current and studying its behavior as a means of integrating k forms, Federer-Fleming defined a new notion of convergence of submanifolds: weak convergence as integral currents.

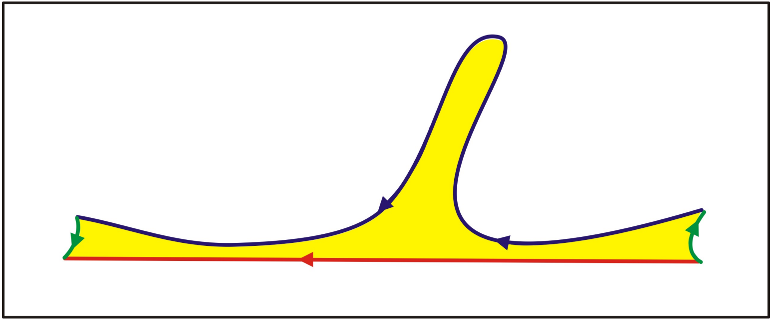
Federer-Fleming defined the **flat distance**. When the integral currents have uniformly bounded total mass, they converge with respect to the flat norm iff they converge in the weak sense.
The key concept needed to understand the flat distance is the mass of the current and the boundary. For a submanifold, mass is just the volume and the boundary is as in a manifold. In general, boundaries are defined via Stoke's Theorem.
Ambrosio-Kirchheim extended the notion of an integral current to an arbitrary complete metric space. **Wenger** extended the notion of the flat distance to this setting.

The flat distance between integral currents

The flat distance between two integral currents, $T, S \in \mathbf{I}_k(Z)$ is

$$d_F^Z(T, S) = \inf\{\mathbf{M}(A) + \mathbf{M}(B) \,:\, A + \partial B = S - T\},$$

where the infimum is taken over all $A \in \mathbf{I}_k(Z)$, $B \in \mathbf{I}_{k+1}(Z)$.



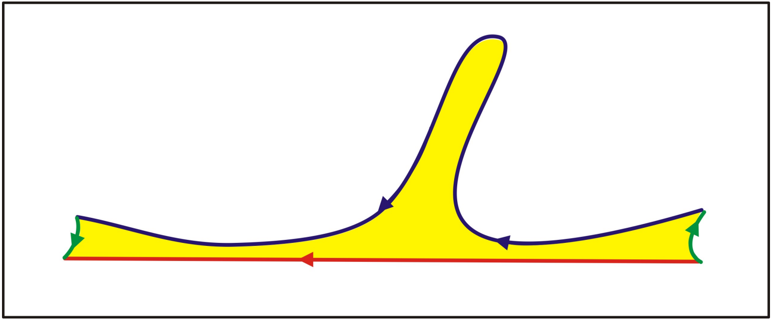
Wenger has proven that this infimum is achieved by a pair of integral currents, $A \in \mathbf{I}_k(Z)$, $B \in \mathbf{I}_{k+1}(Z)$ [W-GAFA].

The flat distance between integral currents

The flat distance between two integral currents, $T, S \in \mathbf{I}_k(Z)$ is

$$d_F^Z(T, S) = \inf\{\mathbf{M}(A) + \mathbf{M}(B) : A + \partial B = S - T\},$$

where the infimum is taken over all $A \in \mathbf{I}_k(Z)$, $B \in \mathbf{I}_{k+1}(Z)$.



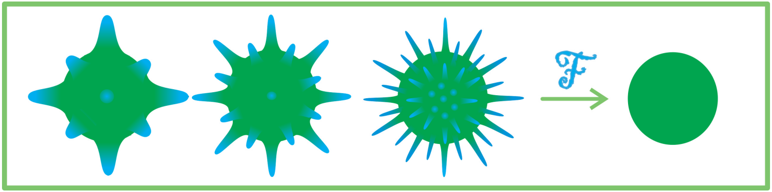
Wenger has proven that this infimum is achieved by a pair of integral currents, $A \in \mathbf{I}_k(Z)$, $B \in \mathbf{I}_{k+1}(Z)$ [W-GAFA].

The flat distance between integral currents

The flat distance between two integral currents, $T_j, S \in \mathbf{I}_k(Z)$ is

$$d_F^Z(T_j, S) = \inf \{\mathbf{M}(A_j) + \mathbf{M}(B_j) \,:\, A_j + \partial B_j = S - T_j\},$$

where the infimum is taken over all $A_j \in \mathbf{I}_k(Z), B_j \in \mathbf{I}_{k+1}(Z)$.



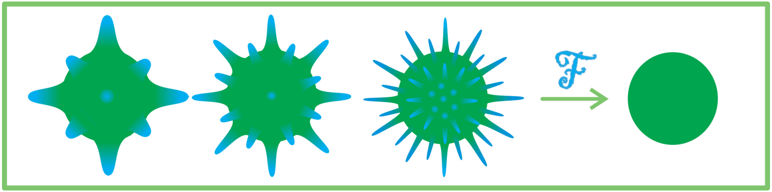
Here, the limit is the sphere S, and if one thinks of the bumpy spheres as T_j lying directly on top of S then we can take $A_j = 0$ and B_j to be the region between the sphere and the bumpy sphere.

The flat distance between integral currents

The flat distance between two integral currents, $T_j, S \in \mathbf{I}_k(Z)$ is

$$d_F^Z(T_j, S) = \inf \{\mathbf{M}(A_j) + \mathbf{M}(B_j) \,:\, A_j + \partial B_j = S - T_j\},$$

where the infimum is taken over all $A_j \in \mathbf{I}_k(Z), B_j \in \mathbf{I}_{k+1}(Z)$.



Here, the limit is the sphere S, and if one thinks of the bumpy spheres as T_j lying directly on top of S then we can take $A_j = 0$ and B_j to be the region between the sphere and the bumpy sphere.

To make an intrinsic notion, we cannot allow any dependence on the embedding of the manifolds. So we imitate Gromov's intrinsic Hausdorff distance and take an infimum over all possible isometric embeddings into all possible complete metric spaces Z...

The Definition of the Intrinsic Flat Distance

Given oriented Riemannian manifolds with boundary, M, N
the intrinsic flat distance is the infimum of the flat distances:

$$d_{\mathcal{F}}(M, N) := \inf d_F^Z(\varphi_\#[M], \psi_\#[N]),$$

where the infimum is taken over all complete metric spaces, Z, and over all isometric embeddings,

$$\varphi : M \to Z \text{ and } \psi : N \to Z$$

and $\phi_\#$ denotes the push forward of the manifolds viewed as integral currents into the metric space Z.

The Definition of the Intrinsic Flat Distance

Given oriented Riemannian manifolds with boundary, M, N the intrinsic flat distance is the infimum of the flat distances:

$$d_{\mathcal{F}}(M, N) := \inf d_F^Z(\varphi_\#[M], \psi_\#[N]),$$

where the infimum is taken over all complete metric spaces, Z, and over all isometric embeddings,

$$\varphi : M \to Z \text{ and } \psi : N \to Z$$

and $\phi_\#$ denotes the push forward of the manifolds viewed as integral currents into the metric space Z.

As in Gromov-Hausdorff distance, an isometric embedding, $\phi : X \to Y$ satisfies $d_Y(\phi(x_1)\phi(x_2)) = d_X(x_1, x_2) \qquad \forall x_1, x_2 \in X$.

The Definition of the Intrinsic Flat Distance

Given oriented Riemannian manifolds with boundary, M, N the intrinsic flat distance is the infimum of the flat distances:

$$d_{\mathcal{F}}(M, N) := \inf d_F^Z(\varphi_\#[M], \psi_\#[N]),$$

where the infimum is taken over all complete metric spaces, Z, and over all isometric embeddings,

$$\varphi : M \to Z \text{ and } \psi : N \to Z$$

and $\phi_\#$ denotes the push forward of the manifolds viewed as integral currents into the metric space Z.

As in Gromov-Hausdorff distance, an isometric embedding, $\phi : X \to Y$ satisfies $d_Y(\phi(x_1)\phi(x_2)) = d_X(x_1, x_2) \qquad \forall x_1, x_2 \in X$.

Theorem [S-W]: If M, N are compact and $d_{\mathcal{F}}(M, N) = 0$ then there is an orientation preserving isometry between M and N.

The Definition of the Intrinsic Flat Distance

Given oriented Riemannian manifolds with boundary, M, N the intrinsic flat distance is the infimum of the flat distances:

$$d_{\mathcal{F}}(M,N) := \inf d_F^Z(\varphi_\#[M], \psi_\#[N]),$$

where the infimum is taken over all complete metric spaces, Z, and over all isometric embeddings,

$$\varphi : M \to Z \text{ and } \psi : N \to Z$$

and $\phi_\#$ denotes the push forward of the manifolds viewed as integral currents into the metric space Z.

As in Gromov-Hausdorff distance, an isometric embedding, $\phi : X \to Y$ satisfies $d_Y(\phi(x_1)\phi(x_2)) = d_X(x_1, x_2) \qquad \forall x_1, x_2 \in X$.

Theorem [S-W]: If M, N are compact and $d_{\mathcal{F}}(M,N) = 0$ then there is an orientation preserving isometry between M and N.

Examples

Defn: The intrinsic flat distance between oriented Riemannian manifolds, M^k, N^k

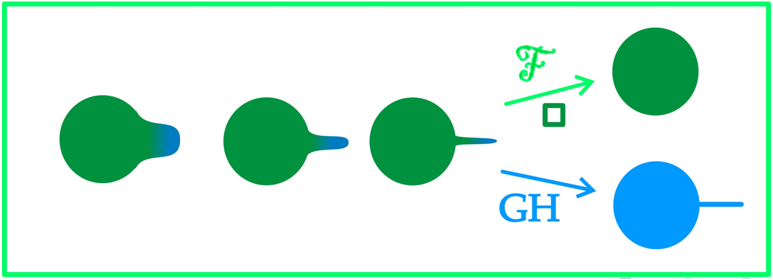
$$d_{\mathcal{F}}(M,N) := \inf d_F^Z(\varphi_\#[M], \psi_\#[N])$$

where $d_F^Z(T,S) = \inf\{\mathbf{M}(A) + \mathbf{M}(B) \, : \, A + \partial B = T - S\}$.

Defn: The Gromov-Hausdorff distance

$$d_{\mathrm{GH}}(M,N) := \inf d_H^Z(\varphi(M), \psi(N))$$

where $d_H^Z(A,B) = \inf\{r : A \subset T_r(B), B \subset T_r(A)\}$.



Examples

Defn: The intrinsic flat distance between oriented Riemannian manifolds, M^k, N^k

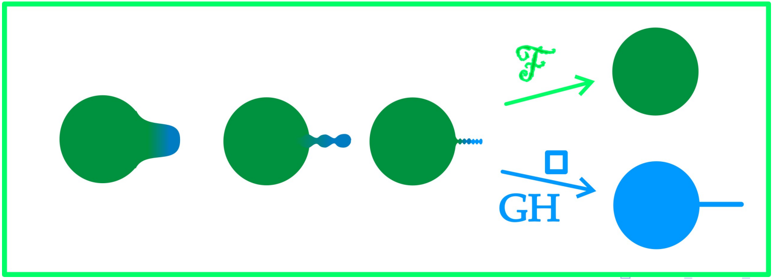
$$d_{\mathcal{F}}(M,N) := \inf d_F^Z(\varphi_\#[M], \psi_\#[N])$$

where $d_F^Z(T,S) = \inf\{\mathbf{M}(A) + \mathbf{M}(B) \,:\, A + \partial B = T - S\}$.

Defn: The Gromov-Hausdorff distance

$$d_{\mathrm{GH}}(M,N) := \inf d_H^Z(\varphi(M), \psi(N))$$

where $d_H^Z(A,B) = \inf\{r : A \subset T_r(B), B \subset T_r(A)\}$.



Examples

Defn: The intrinsic flat distance between oriented Riemannian manifolds, M^k, N^k

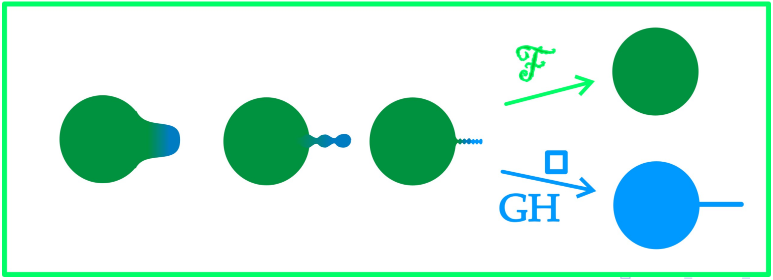
$$d_{\mathcal{F}}(M,N) := \inf d_F^Z(\varphi_\#[M], \psi_\#[N])$$

where $d_F^Z(T,S) = \inf\{\mathbf{M}(A) + \mathbf{M}(B) \,:\, A + \partial B = T - S\}$.

Defn: The Gromov-Hausdorff distance

$$d_{\mathrm{GH}}(M,N) := \inf d_H^Z(\varphi(M), \psi(N))$$

where $d_H^Z(A,B) = \inf\{r : A \subset T_r(B), B \subset T_r(A)\}$.



Examples

Defn: The intrinsic flat distance between oriented Riemannian manifolds, M^k, N^k

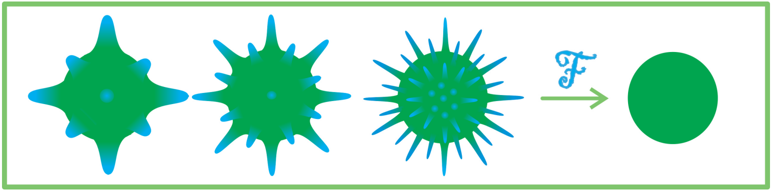
$$d_{\mathcal{F}}(M,N) := \inf d_F^Z(\varphi_\#[M], \psi_\#[N])$$

where $d_F^Z(T,S) = \inf\{\mathbf{M}(A) + \mathbf{M}(B) \,:\, A + \partial B = T - S\}$.

Defn: The Gromov-Hausdorff distance

$$d_{\mathrm{GH}}(M,N) := \inf d_H^Z(\varphi(M), \psi(N))$$

where $d_H^Z(A,B) = \inf\{r : A \subset T_r(B), B \subset T_r(A)\}$.



Examples

Defn: The intrinsic flat distance between oriented Riemannian manifolds, M^k, N^k

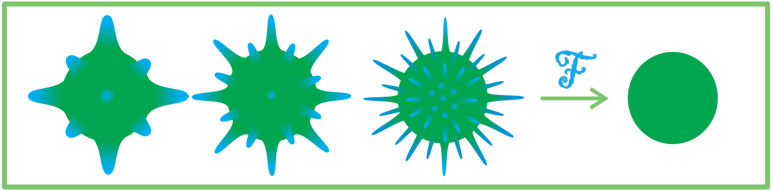
$$d_{\mathcal{F}}(M,N) := \inf d_F^Z(\varphi_\#[M], \psi_\#[N])$$

where $d_F^Z(T,S) = \inf\{\mathbf{M}(A) + \mathbf{M}(B) \ : \ A + \partial B = T - S\}$.

Defn: The Gromov-Hausdorff distance

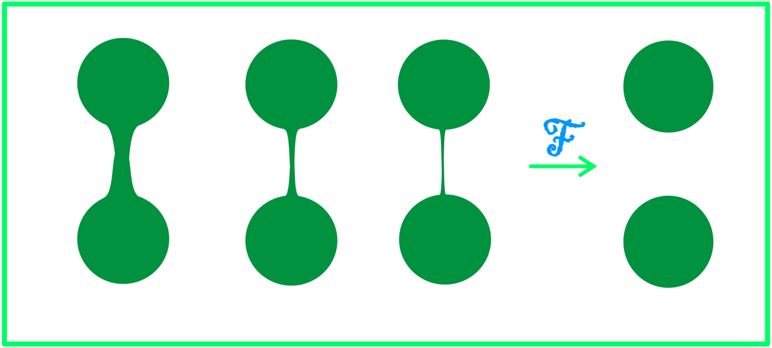
$$d_{\mathrm{GH}}(M,N) := \inf d_H^Z(\varphi(M), \psi(N))$$

where $d_H^Z(A,B) = \inf\{r : A \subset T_r(B), B \subset T_r(A)\}$.



Example of a Cauchy sequence:

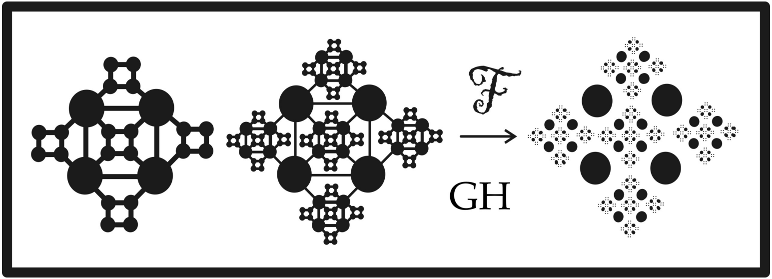
This example is Cauchy with respect to the intrinsic flat distance:



The Gromov-Hausdorff limit in this setting is the pair of spheres joined by a line segment. Intuitively it seems the intrinsic flat limit should be the disjoint pair of spheres without the line segment.

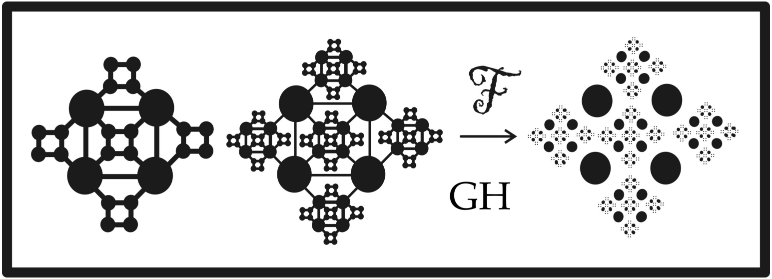
Another Cauchy sequence:

In this example we join spheres by tubes, so that we get an intrinsic flat Cauchy sequence with a uniform upper bound on volume. The Gromov-Hausdorff limit is the metric space on the right connected by line segments while the intrinsic flat limit should just be the space on the right.



Another Cauchy sequence:

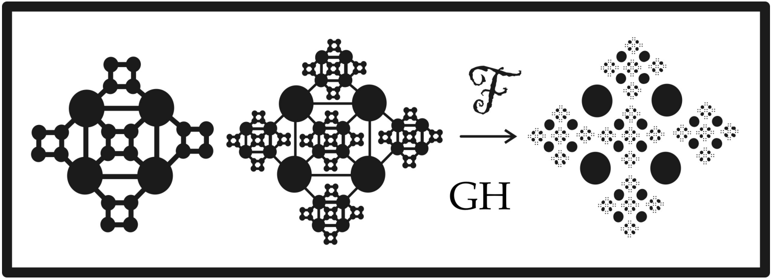
In this example we join spheres by tubes, so that we get an intrinsic flat Cauchy sequence with a uniform upper bound on volume. The Gromov-Hausdorff limit is the metric space on the right connected by line segments while the intrinsic flat limit should just be the space on the right.



In fact one could construct a sequence of manifolds of bounded volume, whose Gromov-Hausdorff limit has dimension $k+1$.

Another Cauchy sequence:

In this example we join spheres by tubes, so that we get an intrinsic flat Cauchy sequence with a uniform upper bound on volume. The Gromov-Hausdorff limit is the metric space on the right connected by line segments while the intrinsic flat limit should just be the space on the right.



In fact one could construct a sequence of manifolds of bounded volume, whose Gromov-Hausdorff limit has dimension $k+1$.
We next define intrinsic flat limits: integral current spaces.

Integral Current Spaces

A countably $\mathcal{H}^k$ rectifiable metric space is a metric space, X with countably many Lipschitz maps ϕ_i from Borel measurable sets $A_i \subset \mathbf{E}^k$ to X such that the Hausdorff measure

$$\mathcal{H}^k(X \setminus \bigcup \phi_i(A_i)) = 0.$$

Integral Current Spaces

A countably $\mathcal{H}^k$ rectifiable metric space is a metric space, X with countably many Lipschitz maps ϕ_i from Borel measurable sets $A_i \subset \mathbf{E}^k$ to X such that the Hausdorff measure

$$\mathcal{H}^k(X \setminus \bigcup \phi_i(A_i)) = 0.$$

ϕ_i can be chosen to be biLipschitz with $\phi_i(A_i) \cap \phi_j(A_j) = \emptyset$.

Integral Current Spaces

A countably $\mathcal{H}^k$ rectifiable metric space is a metric space, X with countably many Lipschitz maps ϕ_i from Borel measurable sets $A_i \subset \mathbf{E}^k$ to X such that the Hausdorff measure

$$\mathcal{H}^k(X \setminus \bigcup \phi_i(A_i)) = 0.$$

ϕ_i can be chosen to be biLipschitz with $\phi_i(A_i) \cap \phi_j(A_j) = \emptyset$.
We can define *orientation* by choosing a preferred atlas of charts.

Integral Current Spaces

A countably $\mathcal{H}^k$ rectifiable metric space is a metric space, X with countably many Lipschitz maps ϕ_i from Borel measurable sets $A_i \subset \mathbf{E}^k$ to X such that the Hausdorff measure

$$\mathcal{H}^k(X \setminus \bigcup \phi_i(A_i)) = 0.$$

ϕ_i can be chosen to be biLipschitz with $\phi_i(A_i) \cap \phi_j(A_j) = \emptyset$.
We can define *orientation* by choosing a preferred atlas of charts.
We can define *multiplicity* (weight) by choosing positive integer valued Borel measurable function, θ, on X.

Integral Current Spaces

A countably $\mathcal{H}^k$ rectifiable metric space is a metric space, X with countably many Lipschitz maps ϕ_i from Borel measurable sets $A_i \subset \mathbf{E}^k$ to X such that the Hausdorff measure

$$\mathcal{H}^k(X \setminus \bigcup \phi_i(A_i)) = 0.$$

ϕ_i can be chosen to be biLipschitz with $\phi_i(A_i) \cap \phi_j(A_j) = \emptyset$.
We can define *orientation* by choosing a preferred atlas of charts.
We can define *multiplicity* (weight) by choosing positive integer valued Borel measurable function, θ, on X.
Defn: a k dimensional integral current space, (X, d, T), is a countably $\mathcal{H}^k$ rectifiable metric space (X, d) with a current structure, $T \in \mathbf{I}_k(\bar{X})$, such that $\mathrm{set}(T) = X$.

The integral current T is the same dimension as X and is determined by the multiplicity/weight and orientation on the metric space X.

Integral Current Spaces

A countably $\mathcal{H}^k$ rectifiable metric space is a metric space, X with countably many Lipschitz maps ϕ_i from Borel measurable sets $A_i \subset \mathbf{E}^k$ to X such that the Hausdorff measure

$$\mathcal{H}^k(X \setminus \bigcup \phi_i(A_i)) = 0.$$

ϕ_i can be chosen to be biLipschitz with $\phi_i(A_i) \cap \phi_j(A_j) = \emptyset$.
We can define *orientation* by choosing a preferred atlas of charts.
We can define *multiplicity* (weight) by choosing positive integer valued Borel measurable function, θ, on X.
Defn: a k dimensional integral current space, (X, d, T), is a countably $\mathcal{H}^k$ rectifiable metric space (X, d) with a current structure, $T \in \mathbf{I}_k(\bar{X})$, such that $\mathrm{set}(T) = X$.

The integral current T is the same dimension as X and is determined by the multiplicity/weight and orientation on the metric space X.

Integral Current Spaces: (X, d, T) such that $X = \mathrm{set}(T)$:

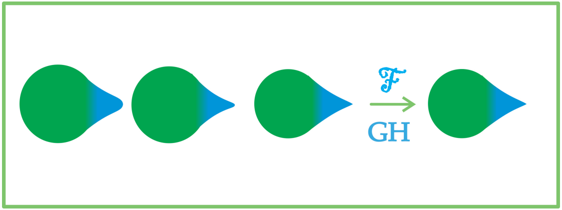
The set of positive lower density, $\mathrm{set}(T)$, is the set of $y \in \bar{X}$ with

$$\liminf_{r \to 0} \frac{\mathbf{M}(T \llcorner B(y,r))}{r^k} > 0.$$

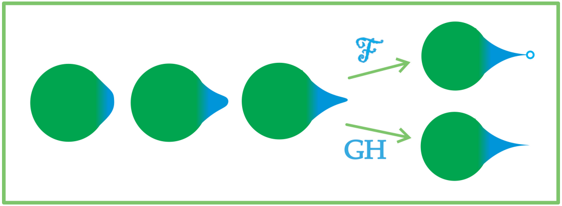
When the integral current space is a manifold with singularities,

$$\mathbf{M}(T \llcorner B(y,r)) = \mathrm{vol}(B(y,r)).$$

So conical singularities have positive lower density:



While cusp singularities do not:



Integral Current Spaces: (X, d, T) such that $X = \mathrm{set}(T)$:

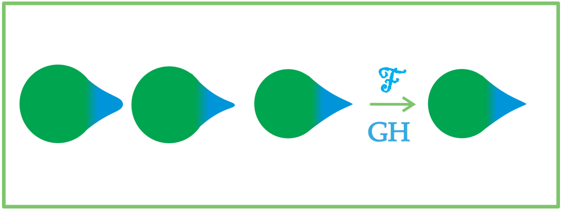
The set of positive lower density, $\mathrm{set}(T)$, is the set of $y \in \bar{X}$ with

$$\liminf_{r \to 0} \frac{\mathbf{M}(T \llcorner B(y,r))}{r^k} > 0.$$

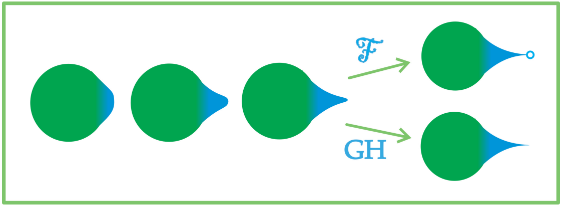
When the integral current space is a manifold with singularities,

$$\mathbf{M}(T \llcorner B(y,r)) = \mathrm{vol}(B(y,r)).$$

So conical singularities have positive lower density:



While cusp singularities do not:



Integral Current Spaces

Defn: a k dimensional integral current space $M = (X, d, T)$ is a countably $\mathcal{H}^k$ rectifiable metric spaces (X, d) with a current structure, $T \in \mathbf{I}_k(\bar{X})$, such that $set(T) = X$. M may be $(0, 0, 0)$.

Defn: The intrinsic flat distance between integral current spaces $M_1 = (X_1, d_1, T_1)$ and $M_2 = (X_2, d_2, T_2)$ is defined by taking the infimum of the flat distance

$$d_{\mathcal{F}}(M_1, M_2) := \inf d_F^Z(\varphi_{1\#} T_1, \varphi_{2\#} T_2).$$

over all isometric embeddings $\varphi_1 : X_1 \to Z$ and $\varphi_2 : X_2 \to Z$.

Theorem [S-W]: If $d_{\mathcal{F}}(M_1, M_2) = 0$ then there is a current preserving isometry $f : X_1 \to X_2$ such that $f_\# T_1 = T_2$. So when M_1 and M_2 are oriented Riemannian manifolds with boundary, they have an orientation preserving isometry between them.

It is crucial in the proof of this theorem that $X = set(T)$ is the points of positive lower density and not the support...

Integral Current Spaces

Defn: a k dimensional integral current space $M = (X, d, T)$ is a countably $\mathcal{H}^k$ rectifiable metric spaces (X, d) with a current structure, $T \in \mathbf{I}_k(\bar{X})$, such that $set(T) = X$. M may be $(0, 0, 0)$.

Defn: The intrinsic flat distance between integral current spaces $M_1 = (X_1, d_1, T_1)$ and $M_2 = (X_2, d_2, T_2)$ is defined by taking the infimum of the flat distance

$$d_{\mathcal{F}}(M_1, M_2) := \inf d_F^Z(\varphi_{1\#} T_1, \varphi_{2\#} T_2).$$

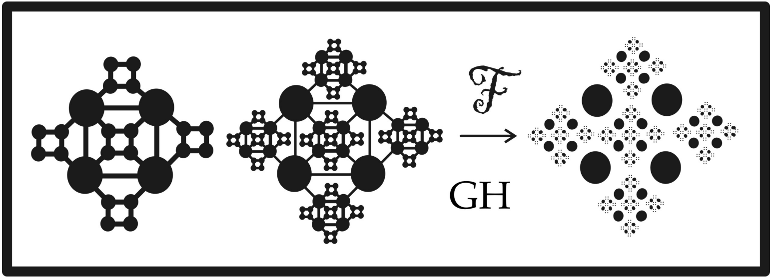
over all isometric embeddings $\varphi_1 : X_1 \to Z$ and $\varphi_2 : X_2 \to Z$.

Theorem [S-W]: If $d_{\mathcal{F}}(M_1, M_2) = 0$ then there is a current preserving isometry $f : X_1 \to X_2$ such that $f_\# T_1 = T_2$. So when M_1 and M_2 are oriented Riemannian manifolds with boundary, they have an orientation preserving isometry between them.

It is crucial in the proof of this theorem that $X = set(T)$ is the points of positive lower density and not the support...

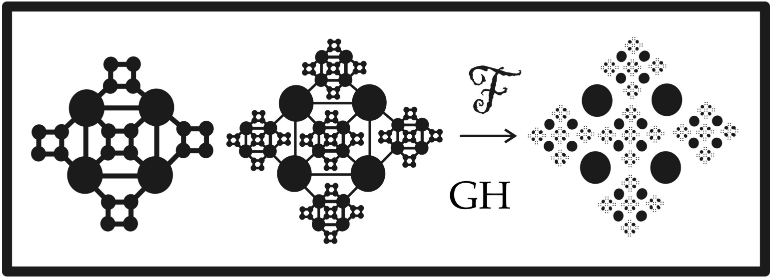
Examples:

Here we see the intrinsic flat limit is a countably $\mathcal{H}^2$ rectifiable space without boundary, while the Gromov-Hausdorff limit has lower dimensional line segments joining the spheres:



Examples:

Here we see the intrinsic flat limit is a countably $\mathcal{H}^2$ rectifiable space without boundary, while the Gromov-Hausdorff limit has lower dimensional line segments joining the spheres:

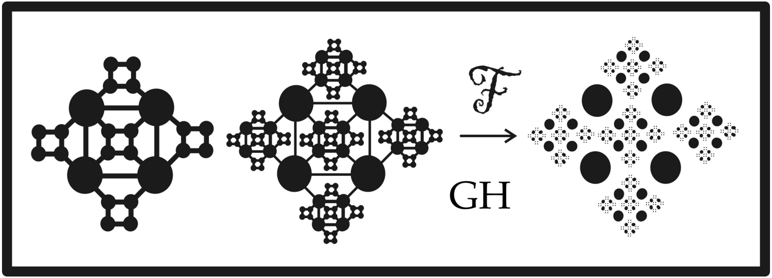


It is possible to construct sequences of Riemannian manifolds with a uniform upper bound on volume which converge to an integral current space whose completion has higher dimension.

By requiring $X = set(T)$, the set of positive lower density, we are guaranteed that X is countably $\mathcal{H}^k$ rectifiable whenever T is a k dim integral current by Ambrosio-Kircheim. We do not take the closure which could be a set of higher dimension.

Examples:

Here we see the intrinsic flat limit is a countably $\mathcal{H}^2$ rectifiable space without boundary, while the Gromov-Hausdorff limit has lower dimensional line segments joining the spheres:



It is possible to construct sequences of Riemannian manifolds with a uniform upper bound on volume which converge to an integral current space whose completion has higher dimension.

By requiring $X = set(T)$, the set of positive lower density, we are guaranteed that X is countably $\mathcal{H}^k$ rectifiable whenever T is a k dim integral current by Ambrosio-Kircheim. We do not take the closure which could be a set of higher dimension.

Gromov-Hausdorff to Intrinsic Flat Convergence

Theorem [S-W]: If M_j^k are oriented Riemannian manifolds with

$$\mathrm{vol}(M_j) \le V_1 \text{ and } \mathrm{vol}(\partial M_j) \le V_2$$

such that $M_j^k \xrightarrow{\mathrm{GH}} Y$, where Y is compact, then a subsequence of the M_j^k converge in the intrinsic flat sense to an integral current space $M^k = (X, d_X, T)$ such that $X \subset Y$ and $d_X = d_Y$.

Gromov-Hausdorff to Intrinsic Flat Convergence

Theorem [S-W]: If M_j^k are oriented Riemannian manifolds with

$$\mathrm{vol}(M_j) \le V_1 \text{ and } \mathrm{vol}(\partial M_j) \le V_2$$

such that $M_j^k \xrightarrow{\mathrm{GH}} Y$, where Y is compact, then a subsequence of the M_j^k converge in the intrinsic flat sense to an integral current space $M^k = (X, d_X, T)$ such that $X \subset Y$ and $d_X = d_Y$.

Proof: by Ambrosio-Kirchheim and Gromov Compactness Thms

Gromov-Hausdorff to Intrinsic Flat Convergence

Theorem [S-W]: If M_j^k are oriented Riemannian manifolds with

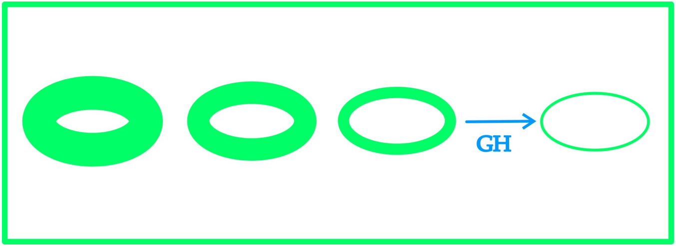
$$\mathrm{vol}(M_j) \le V_1 \text{ and } \mathrm{vol}(\partial M_j) \le V_2$$

such that $M_j^k \xrightarrow{\mathrm{GH}} Y$, where Y is compact, then a subsequence of the M_j^k converge in the intrinsic flat sense to an integral current space $M^k = (X, d_X, T)$ such that $X \subset Y$ and $d_X = d_Y$.

Proof: by Ambrosio-Kirchheim and Gromov Compactness Thms

Cor [S-W]: When $M_j^k \xrightarrow{\mathrm{GH}} Y$ and the dimension of Y is $< k$, then the intrinsic flat limit must be the zero integral current space, 0.

This is called *collapsing*.



Gromov-Hausdorff to Intrinsic Flat Convergence

Theorem [S-W]: If M_j^k are oriented Riemannian manifolds with

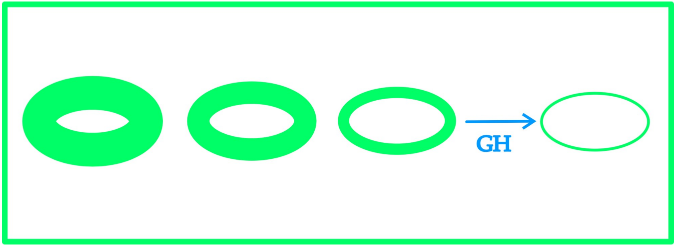
$$\mathrm{vol}(M_j) \le V_1 \text{ and } \mathrm{vol}(\partial M_j) \le V_2$$

such that $M_j^k \xrightarrow{\text{GH}} Y$, where Y is compact, then a subsequence of the M_j^k converge in the intrinsic flat sense to an integral current space $M^k = (X, d_X, T)$ such that $X \subset Y$ and $d_X = d_Y$.

Proof: by Ambrosio-Kirchheim and Gromov Compactness Thms

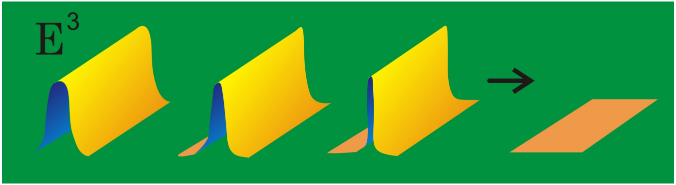
Cor [S-W]: When $M_j^k \xrightarrow{\text{GH}} Y$ and the dimension of Y is $< k$, then the intrinsic flat limit must be the zero integral current space, 0.

This is called *collapsing*.



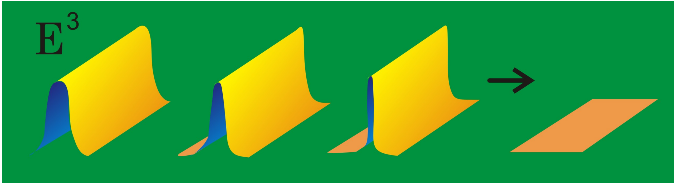
Cancellation

Integral currents in Euclidean space may also disappear due to cancellation:



Cancellation

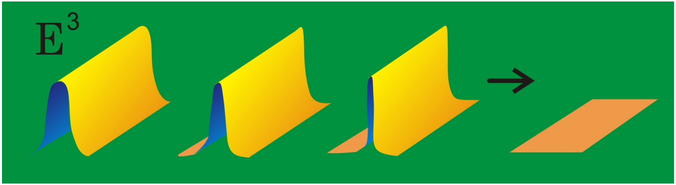
Integral currents in Euclidean space may also disappear due to cancellation:



Note in this $\mathbb{E}^3$ example the embedding is not isometric.

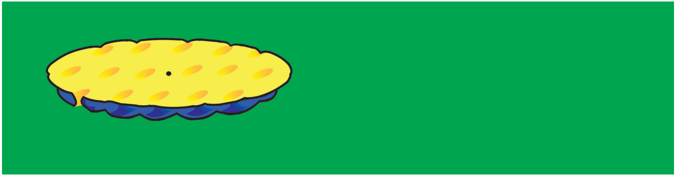
Cancellation

Integral currents in Euclidean space may also disappear due to cancellation:



Note in this $\mathbb{E}^3$ example the embedding is not isometric.

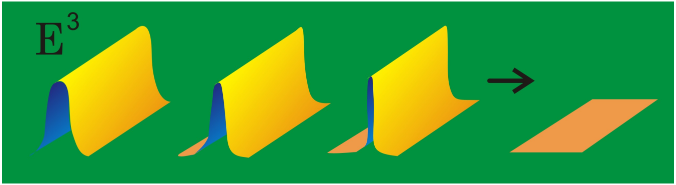
Cancellation of integral current spaces can be seen in examples with many small tunnels between two sheets:



Notice the increasing topology in these examples...

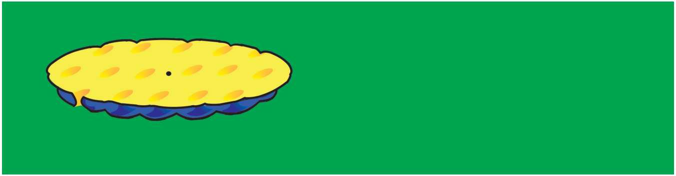
Cancellation

Integral currents in Euclidean space may also disappear due to cancellation:



Note in this $\mathbb{E}^3$ example the embedding is not isometric.

Cancellation of integral current spaces can be seen in examples with many small tunnels between two sheets:



Notice the increasing topology in these examples...

Avoiding Cancellation

Avoiding Cancellation

Gromov, Greene-Petersen: When M_j^k have a uniform linear geometric contractibility function and $Vol(M_j^k) \leq V_1$ then a subsequence converges in the Gromov-Hausdorff sense to Y.

Avoiding Cancellation

Gromov, Greene-Petersen: When M_j^k have a uniform linear geometric contractibility function and $Vol(M_j^k) \leq V_1$ then a subsequence converges in the Gromov-Hausdorff sense to Y.

Theorem [S-W]: In this setting the Gromov-Hausdorff and Intrinsic Flat limits agree: $M_j^k \xrightarrow{\mathrm{GH}} Y$ and $M_j^k \xrightarrow{\mathcal{F}} (X, d, T)$ implies $Y = X$.

Avoiding Cancellation

Gromov, Greene-Petersen: When M_j^k have a uniform linear geometric contractibility function and $Vol(M_j^k) \leq V_1$ then a subsequence converges in the Gromov-Hausdorff sense to Y.

Theorem [S-W]: In this setting the Gromov-Hausdorff and Intrinsic Flat limits agree: $M_j^k \xrightarrow{\mathrm{GH}} Y$ and $M_j^k \xrightarrow{\mathcal{F}} (X, d, T)$ implies $Y = X$. In particular the Gromov-Hausdorff limit is countably $\mathcal{H}^k$ rectifiable.

Avoiding Cancellation

Gromov, Greene-Petersen: When M_j^k have a uniform linear geometric contractibility function and $Vol(M_j^k) \leq V_1$ then a subsequence converges in the Gromov-Hausdorff sense to Y.

Theorem [S-W]: In this setting the Gromov-Hausdorff and Intrinsic Flat limits agree: $M_j^k \xrightarrow{\mathrm{GH}} Y$ and $M_j^k \xrightarrow{\mathcal{F}} (X, d, T)$ implies $Y = X$. In particular the Gromov-Hausdorff limit is countably $\mathcal{H}^k$ rectifiable.

Gromov, Cheeger-Colding: When a sequence of manifolds, M_j^k, have $\mathrm{Ricci}(M_j^k) \geq 0$, vol$(M_j^k) \geq V_0 > 0$ and diam$(M_j^k) \leq D$ then a subsequence converges in the Gromov-Hausdorff sense to a countably $\mathcal{H}^k$ rectifiable metric space, Y.

Avoiding Cancellation

Gromov, Greene-Petersen: When M_j^k have a uniform linear geometric contractibility function and $Vol(M_j^k) \leq V_1$ then a subsequence converges in the Gromov-Hausdorff sense to Y.

Theorem [S-W]: In this setting the Gromov-Hausdorff and Intrinsic Flat limits agree: $M_j^k \xrightarrow{\mathrm{GH}} Y$ and $M_j^k \xrightarrow{\mathcal{F}} (X, d, T)$ implies $Y = X$. In particular the Gromov-Hausdorff limit is countably $\mathcal{H}^k$ rectifiable.

Gromov, Cheeger-Colding: When a sequence of manifolds, M_j^k, have $\mathrm{Ricci}(M_j^k) \geq 0$, $\mathrm{vol}(M_j^k) \geq V_0 > 0$ and $\mathrm{diam}(M_j^k) \leq D$ then a subsequence converges in the Gromov-Hausdorff sense to a countably $\mathcal{H}^k$ rectifiable metric space, Y.

Theorem [S-W]: In this setting the Gromov-Hausdorff and Intrinsic Flat limits agree: $M_j^k \xrightarrow{\mathrm{GH}} Y$ and $M_j^k \xrightarrow{\mathcal{F}} (X, d, T)$ implies $Y = X$. This is a new perspective on Cheeger-Colding.

Avoiding Cancellation

Gromov, Greene-Petersen: When M_j^k have a uniform linear geometric contractibility function and $Vol(M_j^k) \leq V_1$ then a subsequence converges in the Gromov-Hausdorff sense to Y.

Theorem [S-W]: In this setting the Gromov-Hausdorff and Intrinsic Flat limits agree: $M_j^k \xrightarrow{\text{GH}} Y$ and $M_j^k \xrightarrow{\mathcal{F}} (X, d, T)$ implies $Y = X$. In particular the Gromov-Hausdorff limit is countably $\mathcal{H}^k$ rectifiable.

Gromov, Cheeger-Colding: When a sequence of manifolds, M_j^k, have $\mathrm{Ricci}(M_j^k) \geq 0$, $\mathrm{vol}(M_j^k) \geq V_0 > 0$ and $\mathrm{diam}(M_j^k) \leq D$ then a subsequence converges in the Gromov-Hausdorff sense to a countably $\mathcal{H}^k$ rectifiable metric space, Y.

Theorem [S-W]: In this setting the Gromov-Hausdorff and Intrinsic Flat limits agree: $M_j^k \xrightarrow{\text{GH}} Y$ and $M_j^k \xrightarrow{\mathcal{F}} (X, d, T)$ implies $Y = X$. This is a new perspective on Cheeger-Colding.

Menguy: no uniform geometric contractibility in this setting.

Avoiding Cancellation

Gromov, Greene-Petersen: When M_j^k have a uniform linear geometric contractibility function and $Vol(M_j^k) \leq V_1$ then a subsequence converges in the Gromov-Hausdorff sense to Y.

Theorem [S-W]: In this setting the Gromov-Hausdorff and Intrinsic Flat limits agree: $M_j^k \xrightarrow{\text{GH}} Y$ and $M_j^k \xrightarrow{\mathcal{F}} (X, d, T)$ implies $Y = X$. In particular the Gromov-Hausdorff limit is countably $\mathcal{H}^k$ rectifiable.

Gromov, Cheeger-Colding: When a sequence of manifolds, M_j^k, have $\mathrm{Ricci}(M_j^k) \geq 0$, $\mathrm{vol}(M_j^k) \geq V_0 > 0$ and $\mathrm{diam}(M_j^k) \leq D$ then a subsequence converges in the Gromov-Hausdorff sense to a countably $\mathcal{H}^k$ rectifiable metric space, Y.

Theorem [S-W]: In this setting the Gromov-Hausdorff and Intrinsic Flat limits agree: $M_j^k \xrightarrow{\text{GH}} Y$ and $M_j^k \xrightarrow{\mathcal{F}} (X, d, T)$ implies $Y = X$. This is a new perspective on Cheeger-Colding.

Menguy: no uniform geometric contractibility in this setting.

Examples with positive scalar curvature

Example: We construct M_j^3 with positive scalar curvature and $vol(M_j^3) \geq V > 0$ such that $M_j^3 \xrightarrow{\mathrm{GH}} S^3$ but $M_j^3 \xrightarrow{\mathcal{F}} \mathbf{0}$.
Each M_j^3 is a pair of standard spheres with Gromov-Lawson tunnels running between them. As j increases we have more and more tinier tunnels evenly placed about the spheres.

Examples with positive scalar curvature

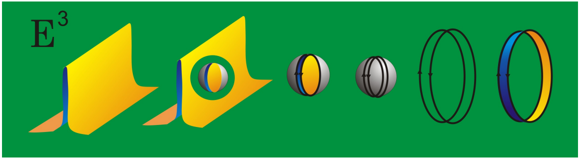
Example: We construct M_j^3 with positive scalar curvature and $vol(M_j^3) \geq V > 0$ such that $M_j^3 \xrightarrow{\text{GH}} S^3$ but $M_j^3 \xrightarrow{\mathcal{F}} \mathbf{0}$.
Each M_j^3 is a pair of standard spheres with Gromov-Lawson tunnels running between them. As j increases we have more and more tinier tunnels evenly placed about the spheres.
Example: We construct M_j^3 with positive scalar curvature and $vol(M_j^3) \geq V > 0$ such that $M_j^3 \xrightarrow{\text{GH}} S^3$ and $M^3 \xrightarrow{\mathcal{F}} (S^3, d, 2T)$ with multiplicity 2.
This construction is as above except that we glue the tunnels with a twist so that the two copies of S^3 have the same orientation as they come together.

Examples with positive scalar curvature

Example: We construct M_j^3 with positive scalar curvature and $vol(M_j^3) \geq V > 0$ such that $M_j^3 \xrightarrow{\text{GH}} S^3$ but $M_j^3 \xrightarrow{\mathcal{F}} \mathbf{0}$.
Each M_j^3 is a pair of standard spheres with Gromov-Lawson tunnels running between them. As j increases we have more and more tinier tunnels evenly placed about the spheres.
Example: We construct M_j^3 with positive scalar curvature and $vol(M_j^3) \geq V > 0$ such that $M_j^3 \xrightarrow{\text{GH}} S^3$ and $M^3 \xrightarrow{\mathcal{F}} (S^3, d, 2T)$ with multiplicity 2.
This construction is as above except that we glue the tunnels with a twist so that the two copies of S^3 have the same orientation as they come together.
Conjecture: We believe this cancellation cannot occur if we take sequences of manifolds with positive scalar curvature and no interior minimal surfaces. Regions may disappear as in the hairy sphere due to collapse but not cancellation.

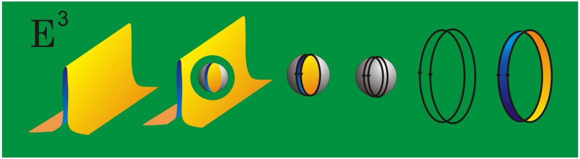
Key observation:

When T_j are flat converging sequences in Euclidean space, the spheres of cancelling balls are the boundaries of currents of small mass:

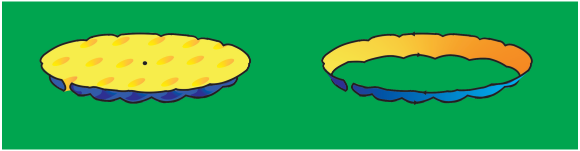


Key observation:

When T_j are flat converging sequences in Euclidean space, the spheres of cancelling balls are the boundaries of currents of small mass:

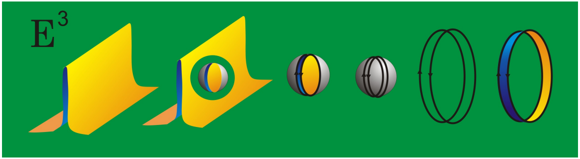


When M_j are intrinsic flat converging sequences of manifolds, the spheres of cancelling balls have small filling volumes:

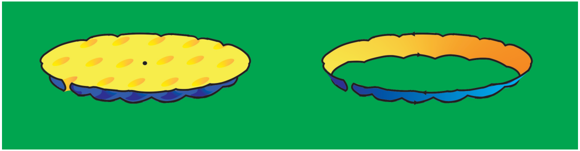


Key observation:

When T_j are flat converging sequences in Euclidean space, the spheres of cancelling balls are the boundaries of currents of small mass:



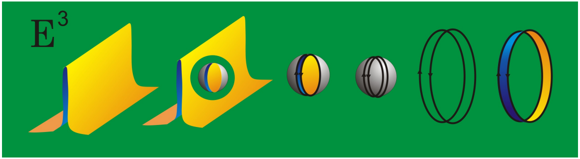
When M_j are intrinsic flat converging sequences of manifolds, the spheres of cancelling balls have small filling volumes:



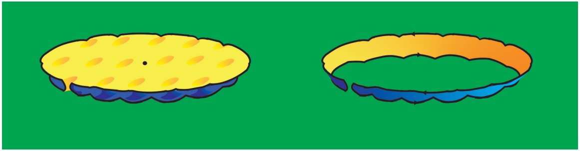
This is made rigorous using Ambrosio-Kirchheim's Slicing Theorem.

Key observation:

When T_j are flat converging sequences in Euclidean space, the spheres of cancelling balls are the boundaries of currents of small mass:



When M_j are intrinsic flat converging sequences of manifolds, the spheres of cancelling balls have small filling volumes:



This is made rigorous using Ambrosio-Kirchheim's Slicing Theorem.

Avoiding Cancellation with Linear Contractibility

Greene-Petersen: If M_j^k have $\mathrm{vol}(M_j^k) \leq V$ and a *uniform geometric contractibility function*, ρ, such that

$$\textit{any ball } B_p(r) \subset M_j^k \textit{ is contractible in } B_p(\rho(r)) \subset M_j^k,$$

then $\mathrm{vol}(B_p(r)) \geq V(r) > 0$. So a subsequence $M_j^k \xrightarrow{\mathrm{GH}} Y$:

Avoiding Cancellation with Linear Contractibility

Greene-Petersen: If M_j^k have $\mathrm{vol}(M_j^k) \le V$ and a *uniform geometric contractibility function*, ρ, such that

$$\textit{any ball } B_p(r) \subset M_j^k \textit{ is contractible in } B_p(\rho(r)) \subset M_j^k,$$

then $\mathrm{vol}(B_p(r)) \ge V(r) > 0$. So a subsequence $M_j^k \xrightarrow{\mathrm{GH}} Y$:

Theorem [S-W]: If ρ is linear, then a subsequence $M_{j_i}^k \xrightarrow{\mathcal{F}} M^k = (X, d, T)$ where $X = Y$. In particular the Gromov-Hausdorff limit is countably $\mathcal{H}^k$ rectifiable.

Avoiding Cancellation with Linear Contractibility

Greene-Petersen: If M_j^k have $\mathrm{vol}(M_j^k) \leq V$ and a *uniform geometric contractibility function*, ρ, such that

$$\textit{any ball } B_p(r) \subset M_j^k \textit{ is contractible in } B_p(\rho(r)) \subset M_j^k,$$

then $\mathrm{vol}(B_p(r)) \geq V(r) > 0$. So a subsequence $M_j^k \xrightarrow{\mathrm{GH}} Y$:

Theorem [S-W]: If ρ is linear, then a subsequence $M_{j_i}^k \xrightarrow{\mathcal{F}} M^k = (X, d, T)$ where $X = Y$. In particular the Gromov-Hausdorff limit is countably $\mathcal{H}^k$ rectifiable.

Example [Schul-Wenger]: Without the assumption of linearity of the contractibility function, the limit space need not be countably $\mathcal{H}^k$ rectifiable.

Avoiding Cancellation with Linear Contractibility

Greene-Petersen: If M_j^k have $\mathrm{vol}(M_j^k) \le V$ and a *uniform geometric contractibility function*, ρ, such that

$$\textit{any ball } B_p(r) \subset M_j^k \textit{ is contractible in } B_p(\rho(r)) \subset M_j^k,$$

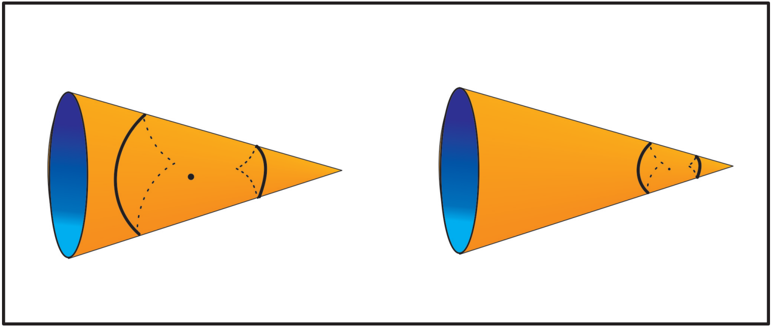
then $\mathrm{vol}(B_p(r)) \ge V(r) > 0$. So a subsequence $M_j^k \xrightarrow{\mathrm{GH}} Y$:

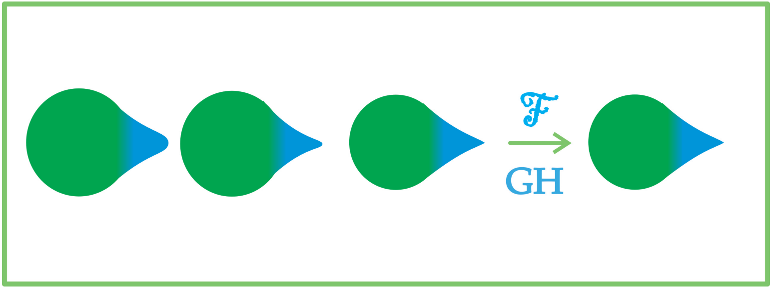
Theorem [S-W]: If ρ is linear, then a subsequence $M_{j_i}^k \xrightarrow{\mathcal{F}} M^k = (X, d, T)$ where $X = Y$. In particular the Gromov-Hausdorff limit is countably $\mathcal{H}^k$ rectifiable.

Example [Schul-Wenger]: Without the assumption of linearity of the contractibility function, the limit space need not be countably $\mathcal{H}^k$ rectifiable.

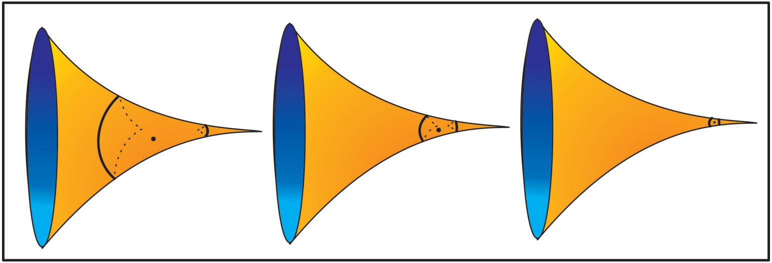
Before proving this contractibility theorem, a few remarks...

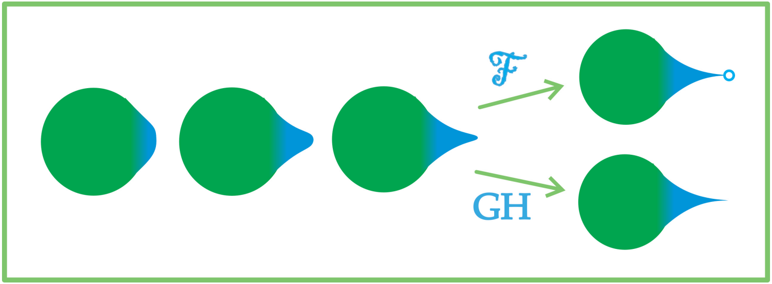
Linear Contractibility and GH vs Flat Limits





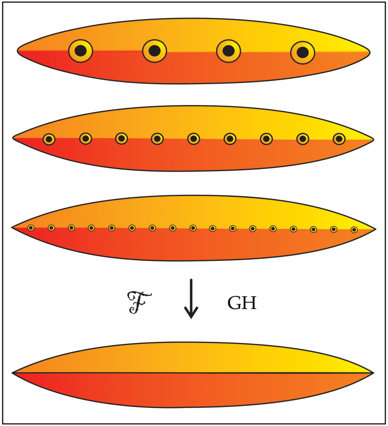
No Linear Contractibility and GH vs Flat Limits





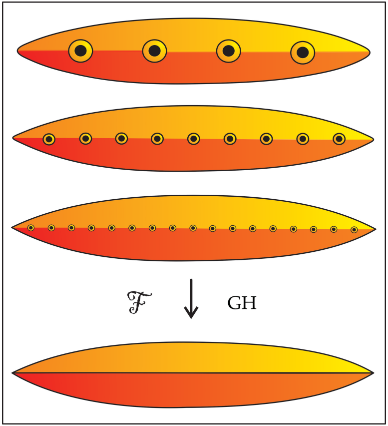
Sequences with $\mathrm{Ricci} \geq 0$ are Not Uniformly Contractible

Perelman constructed noncollapsing sequences M_j^4 with nonnegative Ricci curvature with increasing topology:



Sequences with $\mathrm{Ricci} \geq 0$ are Not Uniformly Contractible

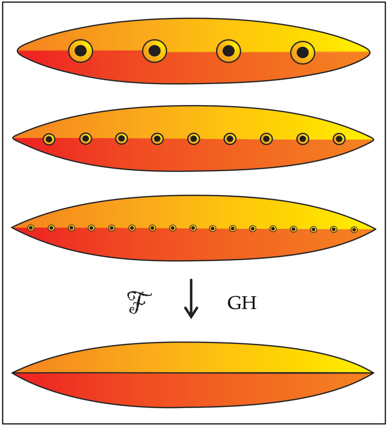
Perelman constructed noncollapsing sequences M^4_j with nonnegative Ricci curvature with increasing topology:



Theorem [S-W]: When M^k_j have $\mathrm{Ricci}(M^k_j) \geq 0$, $\mathrm{vol}(M^k_j) \geq V_0$ and $\mathrm{diam}(M^k_j) \leq D$ then the Gromov-Hausdorff and Intrinsic Flat limits agree: $M^k_j \xrightarrow{\mathrm{GH}} Y$ and $M^k_j \xrightarrow{\mathcal{F}} (X, d, T)$ implies $Y = X$.

Sequences with $\mathrm{Ricci} \geq 0$ are Not Uniformly Contractible

Perelman constructed noncollapsing sequences M_j^4 with nonnegative Ricci curvature with increasing topology:



Theorem [S-W]: When M_j^k have $\mathrm{Ricci}(M_j^k) \geq 0$, $\mathrm{vol}(M_j^k) \geq V_0$ and $\mathrm{diam}(M_j^k) \leq D$ then the Gromov-Hausdorff and Intrinsic Flat limits agree: $M_j^k \xrightarrow{\mathrm{GH}} Y$ and $M_j^k \xrightarrow{\mathcal{F}} (X, d, T)$ implies $Y = X$.

Towards Proofs of the Contractibility and Ricci Theorems

Towards Proofs of the Contractibility and Ricci Theorems

In both settings:

1) we have a sequence $M_j^k \xrightarrow{\mathrm{GH}} Y$ and $\mathrm{vol}(M_j) \le V$

and we want to show $M_{j_i}^k \xrightarrow{\mathcal{F}} M = (X, d, T)$ where $X = Y$.

Towards Proofs of the Contractibility and Ricci Theorems

In both settings:

1) we have a sequence $M_j^k \xrightarrow{\mathrm{GH}} Y$ and $\mathrm{vol}(M_j) \leq V$

and we want to show $M_{j_i}^k \xrightarrow{\mathcal{F}} M = (X, d, T)$ where $X = Y$.

2) By Gromov, there is a compact metric space, Z, and isometric embeddings: $\phi_j : M_j^k \to Z$ and $\phi : Y \to Z$ where $\phi_j(M_j^k)$ Hausdorff converge to Y in Z.

Towards Proofs of the Contractibility and Ricci Theorems

In both settings:

1) we have a sequence $M_j^k \xrightarrow{\mathrm{GH}} Y$ and $\mathrm{vol}(M_j) \le V$
and we want to show $M_{j_i}^k \xrightarrow{\mathcal{F}} M = (X, d, T)$ where $X = Y$.
2) By Gromov, there is a compact metric space, Z, and
isometric embeddings: $\phi_j : M_j^k \to Z$ and $\phi : Y \to Z$
where $\phi_j(M_j^k)$ Hausdorff converge to Y in Z.
3) By Ambrosio-Kirchheim there is a flat converging subsequence:
$T_{j_i} = \phi_{j_i\#}[M_{j_i}]$ converging to $T \in \mathbf{I}_k(Z)$
with $X = \mathrm{set}(T) \subset \phi(Y) \subset Z$ and $d_F^Z(T_{j_i}, T) \to 0$.

Towards Proofs of the Contractibility and Ricci Theorems

In both settings:

1) we have a sequence $M_j^k \xrightarrow{\mathrm{GH}} Y$ and $\mathrm{vol}(M_j) \le V$
and we want to show $M_{j_i}^k \xrightarrow{\mathcal{F}} M = (X, d, T)$ where $X = Y$.
2) By Gromov, there is a compact metric space, Z, and isometric embeddings: $\phi_j : M_j^k \to Z$ and $\phi : Y \to Z$
where $\phi_j(M_j^k)$ Hausdorff converge to Y in Z.
3) By Ambrosio-Kirchheim there is a flat converging subsequence:
$T_{j_i} = \phi_{j_i\#}[M_{j_i}]$ converging to $T \in \mathbf{I}_k(Z)$
with $X = \mathrm{set}(T) \subset \phi(Y) \subset Z$ and $d_F^Z(T_{j_i}, T) \to 0$.
4) We know $\exists y_{j_i} \in \phi_{j_i}(M_{j_i})$ converging to y.

Towards Proofs of the Contractibility and Ricci Theorems

In both settings:

1) we have a sequence $M_j^k \xrightarrow{\mathrm{GH}} Y$ and $\mathrm{vol}(M_j) \le V$
and we want to show $M_{j_i}^k \xrightarrow{\mathcal{F}} M = (X, d, T)$ where $X = Y$.
2) By Gromov, there is a compact metric space, Z, and isometric embeddings: $\phi_j : M_j^k \to Z$ and $\phi : Y \to Z$
where $\phi_j(M_j^k)$ Hausdorff converge to Y in Z.
3) By Ambrosio-Kirchheim there is a flat converging subsequence:
$T_{j_i} = \phi_{j_i\#}[M_{j_i}]$ converging to $T \in \mathbf{I}_k(Z)$
with $X = \mathrm{set}(T) \subset \phi(Y) \subset Z$ and $d_F^Z(T_{j_i}, T) \to 0$.
4) We know $\exists y_{j_i} \in \phi_{j_i}(M_{j_i})$ converging to y.
5) Note that the restrictions, $T_{j_i} \llcorner B(y_{j_i}, r)$ weakly converge to $T \llcorner B(y, r)$.

Towards Proofs of the Contractibility and Ricci Theorems

In both settings:

1) we have a sequence $M_j^k \xrightarrow{\mathrm{GH}} Y$ and $\mathrm{vol}(M_j) \le V$

and we want to show $M_{j_i}^k \xrightarrow{\mathcal{F}} M = (X, d, T)$ where $X = Y$.

2) By Gromov, there is a compact metric space, Z, and isometric embeddings: $\phi_j : M_j^k \to Z$ and $\phi : Y \to Z$

where $\phi_j(M_j^k)$ Hausdorff converge to Y in Z.

3) By Ambrosio-Kirchheim there is a flat converging subsequence:

$T_{j_i} = \phi_{j_i\#}[M_{j_i}]$ converging to $T \in \mathbf{I}_k(Z)$

with $X = \mathrm{set}(T) \subset \phi(Y) \subset Z$ and $d_F^Z(T_{j_i}, T) \to 0$.

4) We know $\exists y_{j_i} \in \phi_{j_i}(M_{j_i})$ converging to y.

5) Note that the restrictions, $T_{j_i} \llcorner B(y_{j_i}, r)$ weakly converge to $T \llcorner B(y, r)$.

We need only show $\forall y \in Y$, $\liminf_{r\to 0} \mathbf{M}(T \llcorner B(y, r))/r^k > 0$.

Towards Proofs of the Contractibility and Ricci Theorems

In both settings:

1) we have a sequence $M_j^k \xrightarrow{\mathrm{GH}} Y$ and $\mathrm{vol}(M_j) \le V$
and we want to show $M_{j_i}^k \xrightarrow{\mathcal{F}} M = (X, d, T)$ where $X = Y$.
2) By Gromov, there is a compact metric space, Z, and isometric embeddings: $\phi_j : M_j^k \to Z$ and $\phi : Y \to Z$
where $\phi_j(M_j^k)$ Hausdorff converge to Y in Z.
3) By Ambrosio-Kirchheim there is a flat converging subsequence:
$T_{j_i} = \phi_{j_i\#}[M_{j_i}]$ converging to $T \in \mathbf{I}_k(Z)$
with $X = \mathrm{set}(T) \subset \phi(Y) \subset Z$ and $d_F^Z(T_{j_i}, T) \to 0$.
4) We know $\exists y_{j_i} \in \phi_{j_i}(M_{j_i})$ converging to y.
5) Note that the restrictions, $T_{j_i} \llcorner B(y_{j_i}, r)$ weakly converge to $T \llcorner B(y, r)$.

We need only show $\forall y \in Y$, $\liminf_{r\to 0} \mathbf{M}(T \llcorner B(y, r))/r^k > 0$.
However, even if $\mathbf{M}(T \llcorner B(y_{j_i}, r)) \ge Cr^k$,
we can drop in the limit to $\mathbf{M}(T \llcorner B(y, r)) = 0$.

Towards Proofs of the Contractibility and Ricci Theorems

In both settings:

1) we have a sequence $M_j^k \xrightarrow{\text{GH}} Y$ and $\text{vol}(M_j) \le V$

and we want to show $M_{j_i}^k \xrightarrow{\mathcal{F}} M = (X, d, T)$ where $X = Y$.

2) By Gromov, there is a compact metric space, Z, and isometric embeddings: $\phi_j : M_j^k \to Z$ and $\phi : Y \to Z$ where $\phi_j(M_j^k)$ Hausdorff converge to Y in Z.

3) By Ambrosio-Kirchheim there is a flat converging subsequence: $T_{j_i} = \phi_{j_i\#}[M_{j_i}]$ converging to $T \in \mathbf{I}_k(Z)$ with $X = \mathrm{set}(T) \subset \phi(Y) \subset Z$ and $d_F^Z(T_{j_i}, T) \to 0$.

4) We know $\exists y_{j_i} \in \phi_{j_i}(M_{j_i})$ converging to y.

5) Note that the restrictions, $T_{j_i} \llcorner B(y_{j_i}, r)$ weakly converge to $T \llcorner B(y, r)$.

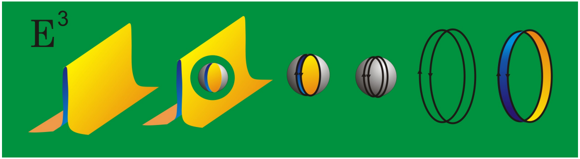
We need only show $\forall y \in Y$, $\liminf_{r\to 0} \mathbf{M}(T \llcorner B(y, r))/r^k > 0$.

However, even if $\mathbf{M}(T \llcorner B(y_{j_i}, r)) \ge Cr^k$,

we can drop in the limit to $\mathbf{M}(T \llcorner B(y, r)) = 0$.

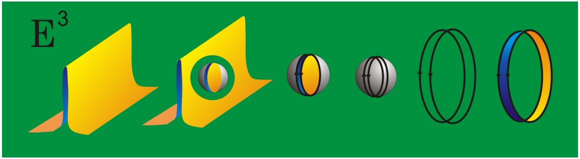
Examining balls in flat converging sequences:

When T_j are flat converging sequences in Euclidean space, the spheres of cancelling balls are the boundaries of currents of small mass:

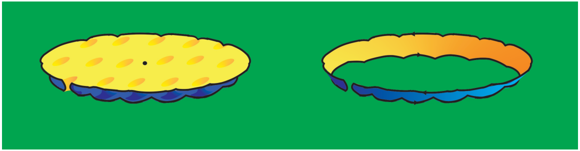


Examining balls in flat converging sequences:

When T_j are flat converging sequences in Euclidean space, the spheres of cancelling balls are the boundaries of currents of small mass:



When M_j are intrinsic flat converging sequences of manifolds, the spheres of cancelling balls have small filling volumes:



Contractibility and Filling Volumes

Contractibility and Filling Volumes

5) The restrictions, $T_{j_i} \llcorner B(y_{j_i}, r)$ weakly converge to $T \llcorner B(y, r)$.

Contractibility and Filling Volumes

5) The restrictions, $T_{j_i} \llcorner B(y_{j_i}, r)$ weakly converge to $T \llcorner B(y, r)$.
6) By the Ambrosio-Kirchheim Slicing Theorem, for a.e. r,
the spheres $\partial(T_{j_i} \llcorner B(y_{j_i}, r))$ weakly converge to $\partial(T \llcorner B(y, r))$.

Contractibility and Filling Volumes

5) The restrictions, $T_{j_i} \llcorner B(y_{j_i}, r)$ weakly converge to $T \llcorner B(y, r)$.
6) By the Ambrosio-Kirchheim Slicing Theorem, for a.e. r,
the spheres $\partial(T_{j_i} \llcorner B(y_{j_i}, r))$ weakly converge to $\partial(T \llcorner B(y, r))$.
7) By [Wenger-06], after isometrically embedding Z into a Banach space,

$$Fillvol_\infty(\partial T \llcorner B(y, r)) = \lim_{i\to\infty} Fillvol_\infty(\partial T_{j_i} \llcorner B(y_{j_i}, r))$$

Contractibility and Filling Volumes

5) The restrictions, $T_{j_i} \llcorner B(y_{j_i}, r)$ weakly converge to $T \llcorner B(y, r)$.
6) By the Ambrosio-Kirchheim Slicing Theorem, for a.e. r,
the spheres $\partial(T_{j_i} \llcorner B(y_{j_i}, r))$ weakly converge to $\partial(T \llcorner B(y, r))$.
7) By [Wenger-06], after isometrically embedding Z into a Banach space,

$$Fillvol_\infty(\partial T \llcorner B(y, r)) = \lim_{i\to\infty} Fillvol_\infty(\partial T_{j_i} \llcorner B(y_{j_i}, r))$$

8) By the definition, $\mathbf{M}(T \llcorner B(y, r)) \geq Fillvol_\infty(\partial T \llcorner B(y, r))$.

Contractibility and Filling Volumes

5) The restrictions, $T_{j_i} \llcorner B(y_{j_i}, r)$ weakly converge to $T \llcorner B(y, r)$.
6) By the Ambrosio-Kirchheim Slicing Theorem, for a.e. r,
the spheres $\partial(T_{j_i} \llcorner B(y_{j_i}, r))$ weakly converge to $\partial(T \llcorner B(y, r))$.
7) By [Wenger-06], after isometrically embedding Z into a Banach space,

$$Fillvol_\infty(\partial T \llcorner B(y, r)) = \lim_{i\to\infty} Fillvol_\infty(\partial T_{j_i} \llcorner B(y_{j_i}, r))$$

8) By the definition, $\mathbf{M}(T \llcorner B(y, r)) \geq Fillvol_\infty(\partial T \llcorner B(y, r))$.

So we need only bound the filling volumes of spheres:

$$Fillvol_\infty(\partial T_{j_i} \llcorner B(y_{j_i}, r)) \geq Cr^k.$$

Contractibility and Filling Volumes

5) The restrictions, $T_{j_i} \llcorner B(y_{j_i}, r)$ weakly converge to $T \llcorner B(y, r)$.
6) By the Ambrosio-Kirchheim Slicing Theorem, for a.e. r,
the spheres $\partial(T_{j_i} \llcorner B(y_{j_i}, r))$ weakly converge to $\partial(T \llcorner B(y, r))$.
7) By [Wenger-06], after isometrically embedding Z into a Banach space,

$$Fillvol_\infty(\partial T \llcorner B(y, r)) = \lim_{i\to\infty} Fillvol_\infty(\partial T_{j_i} \llcorner B(y_{j_i}, r))$$

8) By the definition, $\mathbf{M}(T \llcorner B(y, r)) \geq Fillvol_\infty(\partial T \llcorner B(y, r))$.

So we need only bound the filling volumes of spheres:

$$Fillvol_\infty(\partial T_{j_i} \llcorner B(y_{j_i}, r)) \geq Cr^k.$$

9) Greene-Petersen proved that if $B(y_{j_i}, r)$ has a linear geometric contractibility function, then $Fillvol_\infty(\partial T_{j_i} \llcorner B(y_{j_i}, r)) \geq Cr^k$ by applying Gromov's systole construction and filling inequality.

Contractibility and Filling Volumes

5) The restrictions, $T_{j_i} \llcorner B(y_{j_i}, r)$ weakly converge to $T \llcorner B(y, r)$.
6) By the Ambrosio-Kirchheim Slicing Theorem, for a.e. r,
the spheres $\partial(T_{j_i} \llcorner B(y_{j_i}, r))$ weakly converge to $\partial(T \llcorner B(y, r))$.
7) By [Wenger-06], after isometrically embedding Z into a Banach space,

$$Fillvol_\infty(\partial T \llcorner B(y, r)) = \lim_{i \to \infty} Fillvol_\infty(\partial T_{j_i} \llcorner B(y_{j_i}, r))$$

8) By the definition, $\mathbf{M}(T \llcorner B(y, r)) \geq Fillvol_\infty(\partial T \llcorner B(y, r))$.

So we need only bound the filling volumes of spheres:

$$Fillvol_\infty(\partial T_{j_i} \llcorner B(y_{j_i}, r)) \geq Cr^k.$$

9) Greene-Petersen proved that if $B(y_{j_i}, r)$ has a linear geometric contractibility function, then $Fillvol_\infty(\partial T_{j_i} \llcorner B(y_{j_i}, r)) \geq Cr^k$ by applying Gromov's systole construction and filling inequality.
We extend their result using work of Lang-Schlichenmaier.

Contractibility and Filling Volumes

5) The restrictions, $T_{j_i} \llcorner B(y_{j_i}, r)$ weakly converge to $T \llcorner B(y, r)$.
6) By the Ambrosio-Kirchheim Slicing Theorem, for a.e. r,
the spheres $\partial(T_{j_i} \llcorner B(y_{j_i}, r))$ weakly converge to $\partial(T \llcorner B(y, r))$.
7) By [Wenger-06], after isometrically embedding Z into a Banach space,

$$Fillvol_\infty(\partial T \llcorner B(y, r)) = \lim_{i\to\infty} Fillvol_\infty(\partial T_{j_i} \llcorner B(y_{j_i}, r))$$

8) By the definition, $\mathbf{M}(T \llcorner B(y, r)) \geq Fillvol_\infty(\partial T \llcorner B(y, r))$.

So we need only bound the filling volumes of spheres:

$$Fillvol_\infty(\partial T_{j_i} \llcorner B(y_{j_i}, r)) \geq Cr^k.$$

9) Greene-Petersen proved that if $B(y_{j_i}, r)$ has a linear geometric contractibility function, then $Fillvol_\infty(\partial T_{j_i} \llcorner B(y_{j_i}, r)) \geq Cr^k$ by applying Gromov's systole construction and filling inequality.
We extend their result using work of Lang-Schlichenmaier.

Thus we have proven:
The Gromov-Hausdorff and Intrinsic Flat limits agree when M_j have uniform linear geometric contractibility functions.

Completing the Ricci curvature Theorem

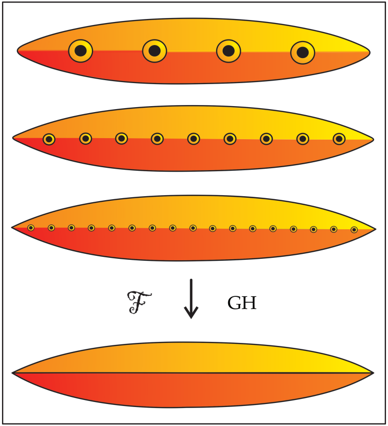
We know that if $y = \lim_{i\to\infty} y_{j_i}$ such that $B(y_{j_i}, r) \subset M_{j_i}$ have a linear geometric contractibility function, then y is in the intrinsic flat limit $X = \mathrm{set}(T)$.

Completing the Ricci curvature Theorem

We know that if $y = \lim_{i\to\infty} y_{j_i}$ such that $B(y_{j_i}, r) \subset M_{j_i}$ have a linear geometric contractibility function, then y is in the intrinsic flat limit $X = \mathrm{set}(T)$.

Now we have M_j with $\mathrm{Ricci} \geq 0$ and $\mathrm{vol}(M_j) \geq V_0$

Not all balls have linear geometric contractibility functions!

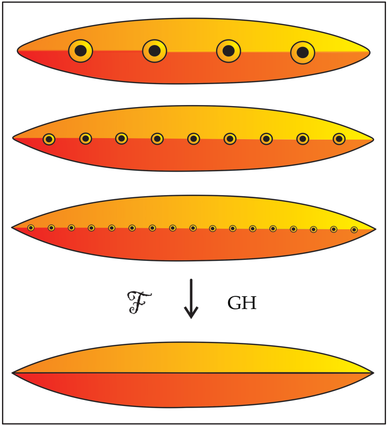


Completing the Ricci curvature Theorem

We know that if $y = \lim_{i\to\infty} y_{j_i}$ such that $B(y_{j_i}, r) \subset M_{j_i}$ have a linear geometric contractibility function, then y is in the intrinsic flat limit $X = \mathrm{set}(T)$.

Now we have M_j with $\mathrm{Ricci} \geq 0$ and $\mathrm{vol}(M_j) \geq V_0$

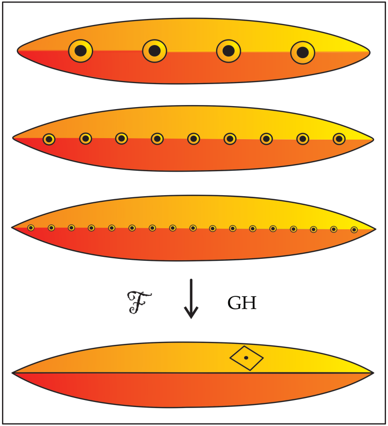
Not all balls have linear geometric contractibility functions!



Proving the Ricci Theorem

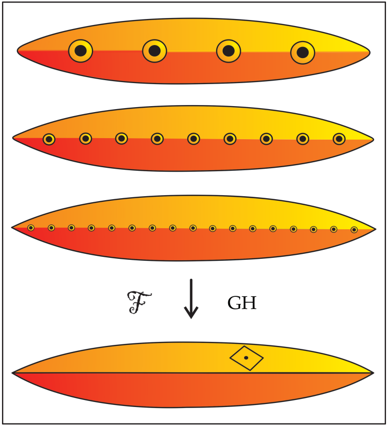
Cheeger-Colding: The set of regular points $\mathcal{R} \subset Y$,
$\mathcal{R} = \{y$ that have a Euclidean tangent cone $\mathbf{E}^k\}$, has full measure.

.
.
.
.
.
.
.
.
.
.
.
.
.
.



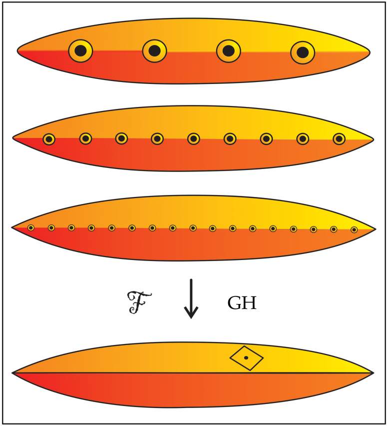
Proving the Ricci Theorem

Cheeger-Colding: The set of regular points $\mathcal{R} \subset Y$,
$\mathcal{R} = \{y$ that have a Euclidean tangent cone $\mathbf{E}^k\}$, has full measure.



Proving the Ricci Theorem

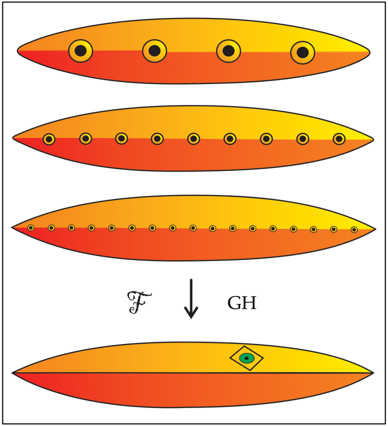
We will prove that $\mathcal{R} \subset X = \mathrm{set}(T)$ by finding $y_{j_i} \to y$ such that $B(y_{j_i}, r) \subset M_{j_i}$ have a linear geometric contractibility function.



Proving the Ricci Theorem

We will prove that $\mathcal{R} \subset X = \mathrm{set}(T)$ by finding $y_{j_i} \to y$ such that $B(y_{j_i}, r) \subset M_{j_i}$ have a linear geometric contractibility function.

Given $y \in \mathcal{R}$. For all $\alpha < \omega_k$ and r sufficiently small $r < r_{y,\alpha}$ $\mathrm{vol}(B(y,r)) \geq \alpha r^k$.



Proving the Ricci Theorem

We will prove that $\mathcal{R} \subset X = \mathrm{set}(T)$ by finding $y_{j_i} \to y$ such that $B(y_{j_i}, r) \subset M_{j_i}$ have a linear geometric contractibility function.

Given $y \in \mathcal{R}$. For all $\alpha < \omega_k$
and r sufficiently small $r < r_{y,\alpha}$
$\mathrm{vol}(B(y,r)) \geq \alpha r^k$.
Colding Volume Convergence:
$\exists y_{j_i} \in M^k_{j_i}$ such that
$\mathrm{vol}(B(y_{j_i}, r)) \geq \alpha r^k \;\; \forall i \geq N_y$.

.

.

.

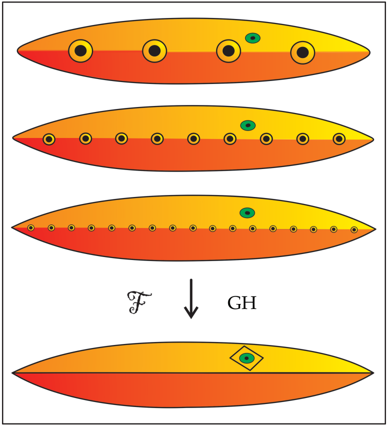
.

.

.

.

.



Proving the Ricci Theorem

We will prove that $\mathcal{R} \subset X = \mathrm{set}(T)$ by finding $y_{j_i} \to y$ such that $B(y_{j_i}, r) \subset M_{j_i}$ have a linear geometric contractibility function.

Given $y \in \mathcal{R}$. For all $\alpha < \omega_k$ and r sufficiently small $r < r_{y,\alpha}$ $\mathrm{vol}(B(y,r)) \geq \alpha r^k$.

Colding Volume Convergence:
$\exists y_{j_i} \in M^k_{j_i}$ such that $\mathrm{vol}(B(y_{j_i}, r)) \geq \alpha r^k \, \forall i \geq N_y$.

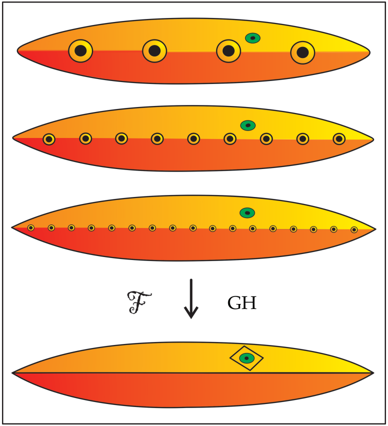
Perelman's Contractibility:
If $\alpha > \alpha_k$ and $\mathrm{Ricci} \geq 0$ then $B(y_{j_i}, r)$ is linearly contractible.

.

.

.

.



Proving the Ricci Theorem

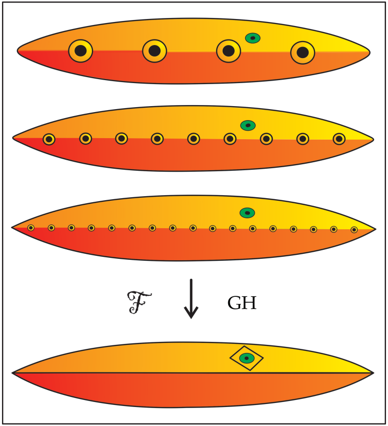
We will prove that $\mathcal{R} \subset X = \mathrm{set}(T)$ by finding $y_{j_i} \to y$ such that $B(y_{j_i}, r) \subset M_{j_i}$ have a linear geometric contractibility function.

Given $y \in \mathcal{R}$. For all $\alpha < \omega_k$ and r sufficiently small $r < r_{y,\alpha}$ $\mathrm{vol}(B(y,r)) \geq \alpha r^k$.

Colding Volume Convergence:
$\exists y_{j_i} \in M^k_{j_i}$ such that $\mathrm{vol}(B(y_{j_i}, r)) \geq \alpha r^k \, \forall i \geq N_y$.

Perelman's Contractibility:
If $\alpha > \alpha_k$ and $\mathrm{Ricci} \geq 0$ then $B(y_{j_i}, r)$ is linearly contractible.

So choosing $\alpha > \alpha_k$, we obtain a uniform contractibility function on $[0, r_{y,\alpha}]$. Thus $y \in \mathrm{set}(T) = X$, the flat limit.



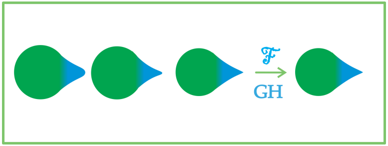
Completing the Ricci curvature Theorem

Now suppose $y \in Y \setminus \mathcal{R}$ is not a regular point.

Completing the Ricci curvature Theorem

Now suppose $y \in Y \setminus \mathcal{R}$ is not a regular point.
By Cheeger-Colding we still have Bishop-Gromov comparison on the limit space Y, so that

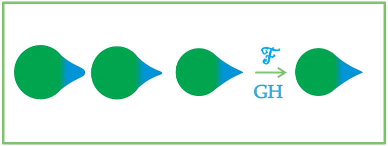
$$\mathcal{H}^k(B(y,r)) \geq \frac{r^k}{D^k}\mathcal{H}^k(Y) = \frac{r^k}{D^k}V_0.$$



Completing the Ricci curvature Theorem

Now suppose $y \in Y \setminus \mathcal{R}$ is not a regular point.
By Cheeger-Colding we still have Bishop-Gromov comparison on the limit space Y, so that

$$\mathcal{H}^k(B(y,r)) \geq \frac{r^k}{D^k}\mathcal{H}^k(Y) = \frac{r^k}{D^k}V_0.$$



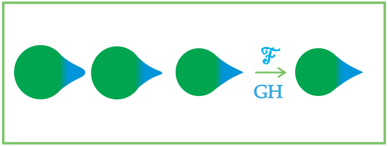
Since we already know $\mathrm{set}(T) = X$ has full measure in Y:

$$||T||(B(y,r)) \geq \frac{r^k}{D^k}\mathcal{H}^k(Y) = \frac{r^k}{D^k}V_0.$$

Completing the Ricci curvature Theorem

Now suppose $y \in Y \setminus \mathcal{R}$ is not a regular point.
By Cheeger-Colding we still have Bishop-Gromov comparison on the limit space Y, so that

$$\mathcal{H}^k(B(y,r)) \geq \frac{r^k}{D^k}\mathcal{H}^k(Y) = \frac{r^k}{D^k}V_0.$$



Since we already know $\mathrm{set}(T) = X$ has full measure in Y:

$$||T||(B(y,r)) \geq \frac{r^k}{D^k}\mathcal{H}^k(Y) = \frac{r^k}{D^k}V_0.$$

Thus $y \in \mathrm{set}(T) = X$ as well.

Closing Remarks

Papers related to this project are linked to from
http://comet.lehman.cuny.edu/sormani/intrinsicflat.html
or just google "Sormani Wenger Intrinsic Flat"

Open problems in progress:
Applications to manifolds with Positive Scalar Curvature (Tom Ilmanen and Stefan Wenger)
Applications to Isospectral Manifolds
Applications to Ricci Flow
Pointed Intrinsic Flat Convergence (Urs Lang and Stefan Wenger)

Other open problems will be posted on the webpage

Closing Remarks

Papers related to this project are linked to from
http://comet.lehman.cuny.edu/sormani/intrinsicflat.html
or just google "Sormani Wenger Intrinsic Flat"

Open problems in progress:
Applications to manifolds with Positive Scalar Curvature (Tom Ilmanen and Stefan Wenger)
Applications to Isospectral Manifolds
Applications to Ricci Flow
Pointed Intrinsic Flat Convergence (Urs Lang and Stefan Wenger)

Other open problems will be posted on the webpage

Thank you for the opportunity to speak.