Research Description

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My research is a combination of Classical Riemannian Geometry, Metric Geometry, Geometric Analysis, Topology and Mathematical Physics. One underlying theme is the convergence of Riemannian Manifolds. In this description, I begin with my early work and then organize the rest by topic.

1 Thesis and Postdoc Years:

I completed my dissertation with Jeff Cheeger at Courant in 1996 and then had a one year postdoc with Shing-Tung Yau at Harvard. At this time I focused on complete noncompact Riemannian manifolds with nonnegative Ricci curvature. Yau proved that such manifolds have at least linear volume growth [Yau1]:

$$\liminf_{R\to\infty} \frac{Vol(B_p(R))}{R} = V_0 > 0. \tag{1}$$

Bishop proved they have at most Euclidean volume growth [Bi1]. Cheeger and Colding had just begun working together on their notion of almost rigidity, in which they proved, among other things, that annular regions in Riemannian manifolds with nonnegative Ricci curvature and almost Euclidean volume growth were close in the Gromov-Hausdorff sense to cones over spheres [ChCo1]. Their technique involved studying the Laplacian of functions of the distance function and constructing explicit maps to the cones.

In my thesis, I applied their technique to Busemann functions which are defined using rays, γ, as follows:

$$b_\gamma(x) = \lim_{R\to\infty} R - d(x,\gamma(R)). \tag{2}$$

I proved that if a Riemannian manifold with nonnegative Ricci curvature has linear volume growth:

$$\limsup_{R\to\infty} \frac{Vol(B_p(R)))}{R} = V_1 < \infty, \tag{3}$$

then regions, $b_\gamma^{-1}(R, R+L)$, for R sufficiently large, are Gromov-Hausdorff close to isometric products: $b_\gamma^{-1}(R) \times (R, R+L)$. Suprisingly, I was able to prove that these level sets were compact, partially solving a conjecture posed by Yau in [Yau2]. A more general version of Yau's conjecture remains open to this day although another partial solutions had been found by Zhongmin Shen in the case of maximal volume growth. As this result was proven using geometric techniques unrelated to the work of Cheeger-Colding, it was published seperately in JDG [So1]. The rest of my thesis as well as a proof that the level sets of the Busemann function grow sublinearly was published in CAG [So2].

During the postdoc, I also proved a theorem concerning harmonic functions of polynomial growth in these manifolds which was published in Pacific Journal of Mathematics [So3]. This result has since been superceded by Colding-Minicozzi's work concerning such functions on all Riemannian manifolds with nonnegative Ricci curvature [CoMin].

2 The Topology of Riemannian Manifolds with $Ricci \geq 0$

When I completed my first postdoc, I took a postdoc at Johns Hopkins and began to work on a famous conjecture of Milnor: *Complete noncompact Riemannian manifolds with nonnegative Ricci curvature have a finitely generated fundamental group* [Mil1]. Intuitively, Milnor's conjecture states that there are finitely many one dimensional holes in such a manifold. Colding's student, Menguy, had constructed examples of such manifolds with infinite dimensional second homology by cutting an infinite sequence of holes in a manifold with positive Ricci curvature and pasting infinitely many Perelman necks, preserving the positive Ricci curvature [Men1]. Menguy's example even had maximal volume growth $\geq Cr^n$.

On the other hand Mike Anderson and Peter Li had seperately proven Milnor's conjecture for manifolds satisfying maximal volume growth using universal covers [And][Li]. Anderson measured balls in the universal cover and applied the Bishop-Gromov Comparison Theorem while Li used the heat kernal on the universal cover. I felt I could imitate one of their techniques to prove the corresponding results for manifolds with at most linear volume growth. The common idea in their work and in Milnor's was that the universal cover could be used to control the fundametal group even though it has no control on the rest of the topology. Since Ricci curvature bounds are local, the bound lifts to the universal cover and can be used to control the fundamental group.

Ultimately I did not apply their specific techniques but rather came up with a notion of a *uniform cut point*. Intuitively, I showed that whenever there is a one dimensional hole in the Riemannian manifold, there is not only a cut point, but all geodesics passing through a ball about that cut point are cut as well. The ball scales with the length of the geodesic. As a consequence I proved: *Any Riemannian manifold with nonnegative Ricci curvature and at most small linear diameter growth,*

$$\limsup_{R\to\infty} \frac{\mathrm{diam}(\partial B_p(R))}{R} \leq \frac{n}{n-1}\frac{1}{3^n}\left(\frac{n-2}{n-1}\right)^{n-1} \tag{4}$$

has a finitely generated fundamental group [So4]. This result applies to all manifolds with linear volume growth as well as those with sublinear diameter growth. It was published in JDG. The key step in the proof of the uniform cut lemma and the above theorem was the construction of a thin triangle in the universal cover whose height could be estimated using Abresch-Gromoll's Excess Theorem [AbGl].

After publishing this, I submitted an NSF grant proposal and a reviewer suggested that Milnor's conjecture was false. So I spent some time trying to construct an example by cutting out holes in Riemannian manifolds and attempting to paste in some pieces that included noncontactible loops. Instead I discovered this was impossible: *all noncontractible loops in complete noncompact spaces with positive Ricci curvature are homotopic to a sequence of loops extending out to infinity* [So5]. Investigating further, I discovered that, *if M^n has $Ricci \geq 0$ then either all noncontractible loops are homotopic to a sequence of loops to infinity or M^n has a double cover which is an isometric product.* An example where the latter occurs is the infinite flat moebius strip, whose double cover is a standard cylinder. [So5]

After I posted the preprint [So5], Zhongmin Shen contacted me. He believed we could use the result to prove an old conjecture of Yau that he had been working on: *the n-1 homology of a complete noncompact Riemannian manifold with positive Ricci curvature is trivial* [Yau3]. Yau had proven this result for real homology using harmonic functions. However real homology does not detect holes of finite order. Shen had proven the conjecture for integer homology when the manifold is assumed to have compact Busemann level sets using Morse Theory [Sh1]. Kobayashi and Itokawa had applied currents to control the integer homology for most manifolds with nonnegative Ricci curvature and had explicitly conjectured the precise groups, $H_{n-1}(M,Z)$, for all complete noncompact Riemannian manifolds with $Ricci \geq 0$ [ItKo]. Shen and I were able to completely classify $H_{n-1}(M,Z)$ for all M^n with $Ricci \geq 0$ exactly as predicted by Itokawa-Kobayashi by applying the techniques of the loops to infinity paper and long exact sequences [SoSh1].

The original Milnor Conjecture remains open. Recently Zhongmin Shen and I wrote a survey paper describing the partial solutions of this conjecture [SoSh2]. Here we proposed a possible counter example to this conjecture. We described the dyadic solenoid complement, a topological manifold discovered by Whitehead soon after he realized that he had published an incorrect proof of the Poincare conjecture [W]. It is not known whether such a manifold can be endowed with a metric of nonnegative Ricci curvature, however, we do note that it satisfies all the topological properties such a manifold must satisfy, including properties from Milnor's paper [Mil] as well as the loops to infinity property [Sor5].

Last May, my student Michael Munn completed a thesis last on Riemannian manifolds with nonnegative Ricci curvature and maximal volume growth. For each k, he found an inductively defined constant $C_{k,n}$ depending on k and the dimension n, such that *if the volume growth is more than $C_{k,n}r^n$ then the kth homology of the manifold is trivial.* His proof is based on work of Perelman, who showed that for C sufficiently large, the manifold is contractible. Michael has been awarded an NSF International Postdoc to work with Topping at Warwick for the next two years.

3 The Topology of Limit Spaces

By Gromov's compactness theorem, a sequence of compact Riemannian manifolds, M_j^n, with nonnegative Ricci curvature has a subsequence which converges in the Gromov-Hausdorff sense to a geodesic metric space, Y [Gr]. By work of Cheeger-Colding, a further subsequence converges in the metric measure sense, endowing the limit space with a measure that satisfies Bishop-Gromov. In their famous three part paper they proved a number of beautiful results concerning these limit spaces, Y, yet they did not control the global topology of the spaces ChCoI-III]. In fact, Colding's student, Menguy had constructed a limit space, Y, with infinite topological type [Men2]. His example is a four dimensional limit space with locally infinite second homology.

Spaces with infinite topological type may not have universal covers. In fact, the Hawaii ring, a collection of infinitely many circles of radii decreasing to 0 joined at a common point, has locally infinite topological type and no universal cover. Nevertheless, Guofang Wei and I felt that limit spaces of manifolds with nonnegative Ricci curvature could not have so many tiny noncontractible loops. We proved that *a limit space, Y, must have a universal cover* [SoWei1] [SoWei2].

It should be noted that universal covers $\tilde{M}_j$ of the manifolds M_j always have a subsequence which converges in a Gromov-Hausdorff sense to some metric space Z. The difficulty is that Z need not cover Y. For example, M_j could be $\frac{1}{j} \times 1$ flat tori converging to a circle, Y. The all the $\tilde{M}_j$ and Z are the Euclidean plane. However, the universal cover of the circle, Y, is a line not a plane. Nevertheless, Wei and I discovered that *if M_j and Y are compact then the universal cover of the limit space is the limit of other covering spaces of the M^j.* In this example the line is a limit of increasingly thin cylinders which are covering spaces of the tori.

In our first paper, Wei and I focused on such compact metric spaces, defining a notion of *delta cover* we denoted $\tilde{M}^\delta$. These covers do not detect topology of M on a scale smaller than the given δ. We proved that a subsequence of δ covers for a fixed δ, $\tilde{M}_j^\delta$, converge to a covering space, Y^δ, of the limit space, Y. We then took delta smaller and smaller and proved that the covers Y^δ eventually stabilized using properties proven by Cheeger-Colding and my notion of uniform cut points. So for delta small enough, Y^δ was the universal cover of Y. Since it was the limit of $\tilde{M}_j^\delta$ which have nonnegative Ricci curvature, $\tilde{Y}$ had all the properties of a limit space [SoWei1].

In our second paper we considered complete noncompact M_j converging to complete noncompact metric spaces Y in the pointed Gromov-Hausdorff sense. Here we needed to restrict ourselves to compact subsets and used a concept we called *reletive delta covers.* We once again were able to prove the existence of a universal cover for Y. This time, however, the universal cover was found using proof by contradiction and was not constructed as a global limit. Nevertheless, we were able to lift all the properties of the limit space to the universal cover. We also extended the work of Milnor and Anderson to limit spaces, as well as my partial solution to the Milnor conjecture [SoWei2].

4 The Covering Spectrum and the Length Spectrum

As mentioned earlier, Cheeger-Colding had developed the notion of metric measure convergence defining a measure on the limit, Y, of a sequence of Riemannian manifolds, M_j^m with nonnegative Ricci curvature, This measure allowed Cheeger-Colding to define a Laplacian on the limit spaces. Furthermore, they were able to prove that the spectrum of the Laplacian on the manifolds converges to the spectrum of the Laplacian of the limit space [ChCoIII]. Further work in this direction has been completed by Yu Ding [Dng].

In contrast, the length spectrum, which is the collection of lengths of smoothly closed geodesics, does not behave well even under C^∞ convergence: new lengths can suddenly appear in the limit. Under Gromov-Hausdorff convergence, lengths can suddenly disappear in the limit. This is intriguing because the length and the Laplace spectra are related both by the heat equation and the wave equation [CdV]DG]. Another important spectrum studied by spectral geometers is the marked length spectrum, a subset of the length spectrum which consists of the lengths of minimal geodesics associated with elements of the fundamental group of the manifold.

Guofang Wei and I defined the notion of *covering spectrum* on a compact geodesic metric space or Riemannian manifold. It is continuous under Gromov-Hausdorff convergence. That is:

if $\lambda \in CovSpec(Y)$*, then* $\exists \lambda_j \in CovSpec(M_j)$ *such that* $\lambda_j \to \lambda$*, and*
if $\lambda_j \in CovSpec(M_j)$ *and* $\lambda_j \to \lambda$ *then* $\lambda \in CovSpec(Y) \cup \{0\}$*.*

We proved *the covering spectrum is completely determined by the marked length spectrum and is a subset of the length spectrum.* It can be applied to count the number of generators of the fundamental group. It has a positive minimum iff the space has a universal cover. Our paper was published in JDG [SoWei3]. The spectrum is further studied by spectral geometers Gornet and Sutton in upcoming work.

When I presented this paper in Montreal, Steve Zelditch suggested I find a larger subset of the length spectrum which had similarly good convergence properties. As Wei was busy on another project at the time, I went ahead on this investigation alone. Since the length spectrum is the collection of lengths of smoothly closed geodesics, I decided to focus on its metric properties rather than focusing only on noncontractible loops.

I defined the notion of $1/k$ *geodesics* as geodesic loops which are minimizing on any subinterval of length L/k where L is the length of the geodesic. This lead to a notion of the $1/k$ length spectra. I proved these spectra behave well under Gromov-Hausdorff convergence allowing me to apply work of Colding to prove a variety of gap theorems. The paper also includes a close study of the $1/2$ length spectra: lengths of geodesics which are minimizing halfway around. This spectrum includes the systole. The paper includes a survey of related prior work and a number of open questions, one of which has already been solved and submitted for publication by a student of Burago [Sor-Length].

Guofang Wei and I then turned to the covering spectrum of complete noncompact Riemannian manifolds and locally compact geodesic metric spaces. In this setting, few of our earlier results hold. The covering spectrum is not a subset of the length spectrum because not all holes are surrounded by geodesic loops. It is not continuous under pointed Gromov-Hausdorff convergence because handles can slide off to infinity and simply connected manifolds can break apart in the limit. This can be seen for example with a sequence of longer and longer ellipsoids that converge in the pointed Gromov-Hausdorff sense to a cylinder. We were able to prove a number of nice properties for the covering spectrum on Alexandrov spaces with nonnegative curvature including manifolds with nonnegative sectional curvature, but we wanted to find a new spectrum which was continuous under pointed Gromov-Hausdorff convergence.

Our most recent joint paper, defines the notions of the *cut-off covering spectrum* and the *R cut-off covering spectrum*. These spectra do not detect holes which extend to infinity like the holes in a cylinder. We prove that the limits of simply connected spaces have empty cut-off covering spectra. More generally we prove that whenever λ is in the cut-off covering spectrum of a limit space, it is the limit of λ_j which are in the cut off covering spectra of the M_j. To prove the convergence properties of these spectra under pointed Gromov-Hausdorff convergence, we develop a notion called a δ homotopy and apply the Arzela-Ascoli theorem to portions of nets restricted to compact domains [SoWei4]. The methods are completely different from the work in the compact setting. Not suprisingly, we proved the cut-off covering spectrum of a complete noncompact Riemannian manifold with nonnegative Ricci curvature is empty.

Lately we've begun working on a concept we call the *rescaled covering spectrum* which should record topology at infinity, or the topology of the tangent cone at infinity. Naturally such a spectrum cannot behave well under pointed Gromov-Hausdorff convergence but we believe it may have some applications to Geometric Group Theory. We are also investigating the rescaled covering spectrum on manifolds with nonnegative Ricci curvature.

5 Friedmann Cosmology

While the above work on the convergence of Riemannian manifolds was a development of a purely abstract theory, I was also interested in applications of Gromov-Hausdorff convergence to the physical world. The Friedmann Model of cosmology is a description of the universe as a space of constant sectional curvature crossed with a time direction that satisfies Einstein's equation. This model is applied to date the big bang and study the expansion of the universe. Naturally space does not have constant sectional curvature and so one must question the basic assumptions in this model. The justification of the model is that space looks the same in all directions: that it is *locally isotropic*. Riemannian manifolds which are locally isotropic around each point have sectional curvatures which only depend on the point and not on the plane. Thus by Schur's lemma, they in fact have constant sectional curvature.

In reality space is not locally isotropic. There is weak gravitational lensing (caused by dust and distant gravity sources) and there is strong gravitational lensing (caused by concentrated masses like stars and black holes). In order to justify Friedmann's model, one needs to show that a space which is almost locally isotropic is also almost a space of constant sectional curvature. One needs stability. However, Schur's Lemma is not stable. Gribcov has provided examples demonstrating that manifolds with almost constant sectional curvature at each point need not be C^∞ close to manifolds of globally constant sectional curvature [Grib]. Yet intriguingly physicists had run models with computer simulations and the outputs had been close.

I investigated and proved that manifolds which are almost locally isotropic in a way which allows for both weak and strong gravitational lensing are Gromov-Hausdorff close to spaces of constant sectional curvature. Thus the physical assumption the cosmologists are making is stable if one allows for Gromov-Hausdorff variation of the space like universe. This is perhaps my favorite paper in recent years and it involved the development of a new concept called an exponential length space. Interestingly this paper does not use methods of Cheeger-Colding but old theorems of Busemann and Birkhoff [Bu][Br]. The smoothness of the limit spaces is proven not using regularity thoery but by proving the universal cover of the limit space must be a standard sphere, Euclidean space or Hyperbolic space. This paper was published in GAFA [Sor6]

6 Conjugate Points in Geodesic Spaces

Krishnan Shankar saw the Friedmann Cosmology paper and felt that some of the theory developed in it could be useful in other settings. One notion, for example, was the notion of a conjugate point, which he noted constrasted nicely with a notion Alexander-Bishop applied in their work on Alexandrov Spaces [AB1][AB2]. He felt the notion could be further generalized to any geodesic space.

On manifolds, conjugate points are defined using the differentiability of an exponential map. The notion had first been extended by Rinow to the nonsmooth setting to spaces that nevertheless had exponential maps. In my prior work and in Alexander-Bishop there were also exponential maps. However, on arbitrary geodesic spaces, there are only minimizing geodesics between pairs of points. Geodesics may branch and reconvene. They need not be locally unique and they need not extend for all time. Nevertheless we found a variety of valid extensions of the notion of a conjugate point and proved a few classical theorems from Riemannian Geometry in this larger class of spaces including Klingenberg's long homotopy lemma. We also surveyed applications of our notions to Alexandrov spaces, proving a relative Rauch comparison theorem. This paper will appear in Advances in Mathematics [SnkSor].

7 A new convergence for Riemannian Manifolds

A few years ago Tom Ilmanen suggested I develop a new notion of convergence for Riemannian manifolds which is weaker than Gromov-Hausdorff convergence: a notion of convergence which would be weak enough to have a compactness theorem for manifolds with positive scalar curvature and no interior minimal surfaces. He described the example of a sphere with increasingly many tiny but deep gravity wells which intuitively should converge to a hairy sphere (a sphere with infinitely many hairs).

Stefan Wenger and I have developed the notion of such a distance using the flat norm between currents and the work of Ambrosio-Kirchheim. We have defined the *intrinsic flat distance* between two compact oriented Riemannian manifolds with boundary. The spaces are a distance zero apart iff there is an orientation preserving isometry between them. The sequence in Ilmanen's example, converges to a standard sphere. The limit spaces are weaker than Gromov-Hausdorff limits in the sense that they may not be geodesic spaces, and they may have cancellation. We call the limit spaces, *current spaces*, and prove a number of rectifiability properties for them using Ambrosio-Kirchheim. This preprint in progress will be ready in January. We are adding examples [SorWen1].

Wenger has proven a compactness theorem for this distance: sequences of oriented Riemannian manifolds with boundary that have a uniform upper bound on diameter, on volume and on the volumes of their boundaries, have converging subsequences. The difficulty is that regions in the manifolds may cancel and the limit space may be the 0 space [Wen].

In our second paper in progress we examine when cancellation can or cannot occur. We have examples of sequences of manifolds with positive scalar curvature with increasingly many tinier and tinier holes, which converge to the zero space. We prove that if we add a condition of uniform local contractibility used by Greene-Petersen, then this cancellation does not occur. When there is an assumption of nonnegative Ricci curvature on the spaces, then we can show using work of Cheeger-Colding and Perelman that the Gromov-Hausdorff limits and the intrinsic flat limits agree [SorWen2].

We conjecture that sequences of compact manifolds with positive scalar curvature and no interior minimal surfaces, that satisfy the volume and diameter bounds of Wenger's compactness theorem, will not have cancellation except in regions where the volume collapses. We believe this new theory of convergence will have many applications.

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