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In 1991, Gromov introduced the Gromov-Hausdorff distance between compact Riemannian manifolds. Applying the Bishop-Gromov Volume Comparison Theorem, he proved that sequences of Riemannian manifolds, M^m, with uniform upper bounds on their diameter and lower bounds on their Ricci curvature have subsequences which converge in the Gromov-Hausdorff sense to compact geodesic metric space.

Fukaya refined this notion, defining metric measure convergence, in which the measures converge as well. Cheeger-Colding proved that manifolds with lower Ricci curvature bounds converge in the metric measure sense to rectifiable metric measure spaces satisfying the Bishop-Gromov Volume Comparison Theorem. One key consequence was the convergence of the Laplace spectrum.

In 2004, Ilmanen proposed the necessity of a new form of convergence for Riemannian manifolds, one for which sequences of three dimensional spheres with increasingly many splines and positive scalar curvature, as depicted in Figure 1, would converge. Such sequences do not converge in the Gromov-Hausdorff sense.

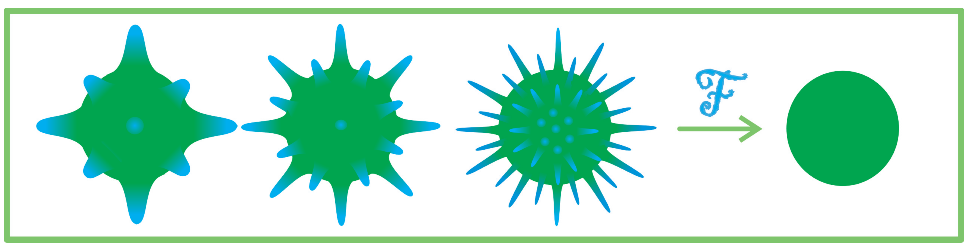

Figure 1. Ilmanen's Example

Recently Sormani-Wenger introduced the Intrinsic Flat Distance between compact oriented Riemannian manifolds applying Ambrosio-Kirchheim's notion of integral currents on metric spaces [1]:

$$d_{\mathcal{F}}(M_1^m, M_2^m) = \inf\left\{ d_F^Z(\varphi_{1\#}[M_1], \varphi_{2\#}[M_2]) : \varphi_i : M_i \to Z \right\} \tag{1}$$

where the infimum is again taken over all metric spaces, Z, and all isometric embeddings, $\varphi_i : M_i^m \to Z$, and where d_F^Z is the Flat Distance between the submanifolds $\varphi_i(M_i)$ viewed as integral currents $\varphi_{i\#}[M_i]$ in the metric space Z [12]. Here, as in Gromov, an isometric embedding is a map $\varphi : X \to Z$ such that

$$d_Z(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2) \;\; \forall x_1, x_2 \in X. \tag{2}$$

Recall that the Flat Distance between integral currents on Euclidean space was first introduced by Federer-Flemming based on work of Whitney. Intuitively, it measures the amount of volume between the two submanifolds.

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To estimate the Intrinsic Flat distance between a pair of oriented Riemannian manifolds one needs only find a pair of isometric embeddings, $\varphi_i : M_i^m \to Z$, into a common complete metric space, Z. When one finds a filling submanifold, $B^{m+1} \subset Z$, and an excess boundary submanifold, $A^m \subset Z$, such that

$$\int_{\varphi_1(M_1)} \omega - \int_{\varphi_2(M_2)} \omega = \int_B d\omega + \int_A \omega, \tag{3}$$

then the Intrinsic Flat distance is bounded by

$$d_{\mathcal{F}}(M_1^m, M_2^m) \le \mathrm{Vol}_{m+1}(B^{m+1}) + \mathrm{Vol}_m(A^m). \tag{4}$$

Generally the filling manifold can have corners and the excess boundary manifold may have many components. One can easily see that Ilmanen's Example converges to a sphere using these estimates. Techniques for estimating the Intrinsic Flat distance appear in the Appendix to [12], in [5] and in [3].

More generally, the Intrinsic Flat distance is defined between pairs of integral current spaces [12]. An integral current space, (X, d, T) is a countably $\mathcal{H}^m$ rectifiable metric space (X, d) with an integral current structure, $T \in \mathbf{I}_m(\bar{X})$, such that set(T) = X where set(T) denotes the set of positive density as in Ambrosio-Kirchheim [1]. The integral current structure encodes both an orientation and a measure $\|T\|$. The fact that X = set(T) guarantees that X is countably $\mathcal{H}^m$ rectifiable, has the correct dimension and does not include cusp singularities [12].

When M_j^m are a noncollapsing sequence of compact Riemannian manifolds with nonnegative Ricci curvature, $\mathrm{diam}(M_i) \le D$ and $\mathrm{Vol}(M_i) \ge V_0$ then the Gromov-Hausdorff and Intrinsic Flat limits agree [11]. More generally the Gromov-Hausdorff limit (if it exists) may contain the Intrinsic Flat limit as a proper subset. The Intrinsic Flat limit may also be the 0 space. This occurs, for example, when we have a collapsing sequence with $\mathrm{Vol}(M_j) \to 0$ or due to cancellation [12].

Wenger has proven a compactness theorem: sequences of compact oriented manifolds, M_j^m, with $\mathrm{diam}(M_j) \le D$, $\mathrm{Vol}_m(M_j) \le V$ and $\mathrm{Vol}_{m-1}(\partial M_j) \le A$ have subsequences which converge in the Intrinsic Flat sense to an integral current space [13]. We conjecture that *if M_j are three dimensional, with positive scalar curvature and no interior closed minimal surfaces, then there is no cancellation; so the limit space is not the 0 space unless $\mathrm{Vol}(M_j) \to 0$* [12]. One approach to proving this conjecture would involve estimating the filling volumes of spheres [11].

There are a number of consequences of Intrinsic Flat convergence following immediately from the work of Ambrosio-Kirchheim: including, in particular, the lower semicontinuity of mass (or volume). These are explored in [7]. When one also assumes the mass converges, one has convergence of the measures and additional consequences [8].

Applications of the Intrinsic Flat convergence are explored in work of Lakzian-Sormani [3], Li [6] and Lee-Sormani [5] [4]. In [3] we study smooth convergence away from singular sets using Intrinsic Flat convergence to understand the Gromov-Hausdorff limits. This has applications to the investigation of Kahler-Einstein manifolds as seen in work of Li [6]. Further applications are being investigated by Lakzian.

In the work of Lee-Sormani, we explore the stability of the Positive Mass Theorem. Recall that the Schoen-Yau Positive Mass Theorem states that an asymptotically flat Riemannian manifold, M^3, with nonnegative scalar curvature has nonnegative ADM mass, and if $m_{ADM}(M^3) = 0$ then M^3 is Euclidean space [SY]. We propose the following conjecture: *if a sequence of asymptotically flat manifolds, M_j^m, has nonnegative scalar curvature and no interior closed minimal surfaces and has $m_{ADM}(M_j^3) \to 0$ then it converges in the* *pointed Intrinsic Flat sense to Euclidean space if the points are chosen on CMC surfaces of constant area*. See Figure 2. We prove this theorem in the rotationally symmetric case in [5]. In fact we have volume convergence in this case. We examine a similar theorem concerning the Penrose Inequality in [4]. One may also wish to extend the results in [9] using the Intrinsic Flat Distance.

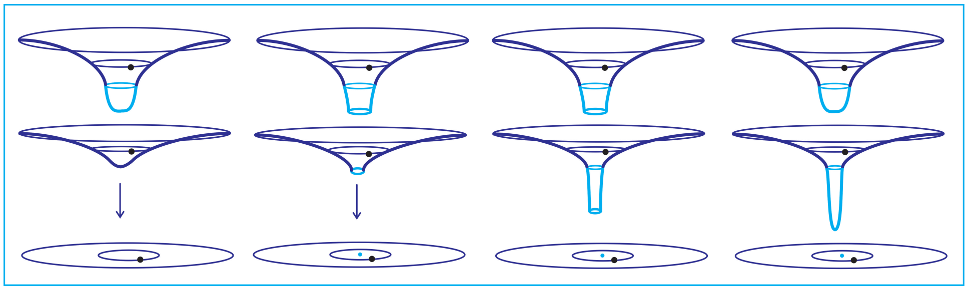

FIGURE 2. Stability of the Positive Mass Theorem

Other possible notions of distances between Riemannian manifolds and convergence of Riemannian manifolds are proposed in [10]. A scalable Intrinsic Flat distance is being developed by Basilio. A notion of area convergence is being developed by Burago-Ivanov [2]. One may speculate on how to define an intrinsic varifold convergence for manifolds [10]. For all of these notions of convergence, one may wish to understand sequences of conformal Riemannian manifolds: *examining which forms of convergence of* $(M, e^{f_j} g) \to (M, e^f g)$ *correspond with which forms of convergence of* $f_j \to f$ *as functions on* (M, g).

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