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Ricci Flow

$$(g_{ij})_t = -2R_{ij}$$

g_{ij}: Riemannian tensor R_{ij}: Ricci tensor

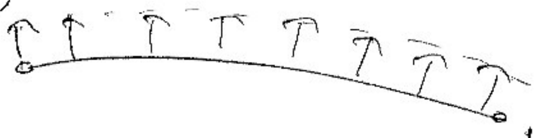
1982 Richard Hamilton – short time existence (smooth)

later: $n=3$, $R_{ij} > 0$ initially $\Rightarrow$ Solution blows up in finite time
and $\Downarrow$
Renormalized Ricci flow (constant volume = 1) $\Rightarrow$ Solution converges exponentially fast to a metric of constant curvature
$S^3/\Gamma \leftarrow M^3$

"Technique which led to this remarkable result":

Hamilton observed Ricci flow is analagous to the heat equation (parabolic)

Look at a length minimizing geodesic joining two pts $\gamma : [0, L] \to M^3$
$V \parallel$ along γ



Study variation $\frac{d}{ds} \int_0^L \left|\frac{\partial}{\partial t} \exp_{\gamma(t)}(sV(t))\right| ds$

Essentially like a heat eqn for the length of <u>short</u> geodesics.

Evolution Eqn for the curvature (Hamilton)

$$\frac{\partial}{\partial t}\left(R_{\cdots}\right) = \Delta_g R_{\cdots} + Q_{\cdots},$$

curvature tensor $R_{\cdots}$

(also parabolic)

Curvature is getting more positive along the flow

scalar curvature R satisfies

$$\frac{\partial}{\partial t} R = \Delta R + 2\,(R_{ij})^2$$

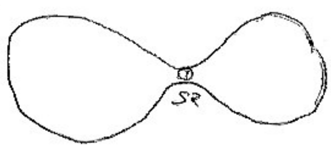
Warning $n > 3 \rightarrow$ Positive Ricci is not preserved only the positivity of the curvature operator

<u>Aftewards</u>
Hamilton + Others tried to extend Ricci flow technique without assumptions of positivity of Ricci curvature.

In general

Flow doesn't exist ∀ time even renormalized except in dimension 2 (Hamilton).

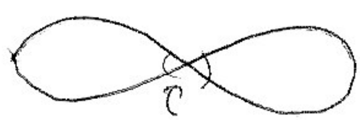
Einstein Metrics are limits when?
(note in dim = 3, this implies limit has const sect curv)



→ pinching
⇒ Singularity develops (in a short time)

How to continue through the singularity?

Hamilton 1996, n=4, positivity on certain curvatures,



remove singularity

glue → restart flow

(Does this only deal w/ pt singularities?)

Ultimately hope flow stops & reassemble initial manifold using connected sums.

Naively Want: $n=3$ arbitrary (M^3, g_0) run Ricci flow, cut & paste to remove singularity, & get $M^3 \approx$ Connected sums of manifolds of constant sectional curvature

Topologists: Geometric decomposition using incompressible tori. (not connected sums?)

Hamilton 1999:

$n=3$, analyze Ricci flow if it exists $\forall$ time with bounded normalized curvatures $\Longrightarrow$ Geometrization of such M^3.

* ———— *

Now a new technique which works in all dimensions - Perelman (complements Hamilton's Max Principle (& inspired by it))

(Integral Monotonicity Formulas

(Ptwise!))

(Warning: (In response to a question) Once you know M^3 has Ricci $\geq$ below then M^3 has some topological finiteness)

In general

Does a parabolic flow come from a gradient flow or not?

Ricci flow does in a certain sense!

$$\mathcal{F}(g_{ij}, f) = \int_M (R + |\nabla f|^2) e^{-f} dV$$

$f: M \to \mathbb{R}$

g_{ij} metric on M

scalar curvature

Riem volume

Look at the variational formula for $\mathcal{F}$.

$$\delta \mathcal{F}(v_{ij}, h) = \int_M [-v_{ij}(R_{ij} + \nabla_i \nabla_j f) + (\tfrac{1}{2}v - h)(2\Delta f - |\nabla f|^2 + R)e^{-f}dV$$

$e^{-f}dV$

$$v_{ij} = \delta g_{ij} \qquad h = \delta f \qquad v = g^{ij} v_{ij}$$

Consider variations where $\frac{v}{2} - h = 0$.

$$\delta \mathcal{F}(v_{ij}, h) = \int_M [-v_{ij}(R_{ij} + \nabla_i \nabla_j f)(2\Delta f - |\nabla f|^2 + R)e^{-f}dV$$

$e^{-f}dV$

So the variational flow in this direction

$$\underbrace{\frac{d}{\partial t} g_{ij} = -2(R_{ij} + \nabla_i \nabla_j f)}$$

Differs from Ricci flow by a diffeomorphism

Does f change in time?

Yes: $\boxed{\frac{v}{2} - h = 0}$ implies that $\boxed{\frac{\partial}{\partial t} f = -\Delta f - R}$

Lets rewrite

$$\mathcal{F}(g_{ij}) = \int_M (R + |\nabla f|^2)\, dm$$

fixed measure

define f so that $\boxed{dm = e^{-f} dV}$

volume form of g

Fix a measure, find an f ~~&~~ compute the flow

Fix a different measure ⟶ still a diffeomorphic flow.

(Like gauge theory)

Note $\mathcal{F}(g_{ij}) = \int_M (R + |\nabla f|^2)\, dm$

where f : $dm = e^{-f} dV$ (f is a function of g_{ij} pt-wise $-\log(\det g_{ij})$)

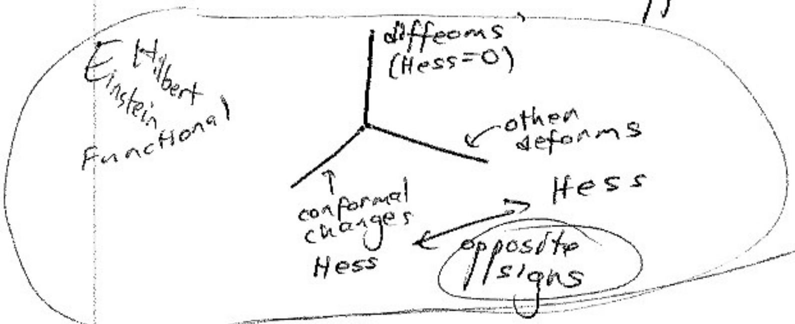
Warning doesn't necessarily exist even for a short time due to $f_t = -\Delta f - R$ difficulties. (like a backwards heat eqn!)

Consider Einstein Hilbert functional

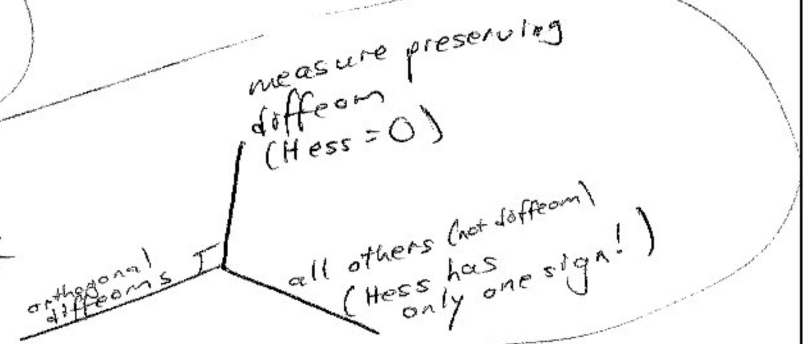
$$EH = \int_M R\, dV$$

Gradient flow is $\frac{\partial}{\partial t}(g_{ij}) = R g_{ij} - 2R_{ij}$

this doesn't exist for a short time for most g_{ij} because it is diffeom invariant,



$\mathcal{F}$ in contrast



$\mathcal{F}$ is only invariant wrt measure preserving diffeomorphisms.

~~Once we~~ ho

Once you have a gradient flow you can get a monotonicity formula

$$\left.\begin{aligned} \frac{\partial}{\partial t}(g_{ij}) &= -2(R_{ij} + \nabla_i \nabla_j f) \\ \frac{\partial}{\partial t} f &= -\Delta f - R \end{aligned}\right\} \Rightarrow \frac{\partial}{\partial t}\mathcal{F} \geq 0$$

But this ~~flow~~ may not exist for a short time, so look at Ricci flow with an appropriate f

$$\left.\begin{aligned} \frac{\partial}{\partial t}(g_{ij}) &= -2R_{ij} \\ f_t &= -\Delta f + |\nabla f|^2 - R \end{aligned}\right\} \frac{\partial}{\partial t}\mathcal{F} \geq 0$$

Change of variables $u = e^{-f}$

$$\frac{\partial u}{\partial t} = -\Delta u + Ru$$

"Adjoint Heat Eqn"

If $\frac{\partial h}{\partial t} = \Delta h$ then $\int_M h u \, dV = \text{const.}$

Now get rid of f (or u) in the monotonicity formula by minimizing over f:

Let $\lambda(g_{ij}) = \inf_f \mathcal{F}(g_{ij}, f)$ s.t. $\int e^{-f} dV = 1$

then under Ricci flow $\frac{d}{dt}\lambda \geq 0$ (Details? inf exists by next page →)

$$\lambda(g_{ij}) = \inf_{\substack{f \\ \int e^{-f} dV = 1}} \mathcal{F}(g_{ij}, f) \quad \text{substitute } \omega = e^{-\frac{1}{2}f}$$

$$= \inf_{\substack{\omega \\ \int \omega^2 dV = 1}} \int_M (R\omega^2 + 4|\nabla\omega|^2)\, dV = \inf_{\omega} \frac{\int_M R\omega^2 + 4|\nabla\omega|^2 dV}{\int_M \omega^2 dV}$$

= lowest eigenvalue of $-4\Delta + R$
(Rayleigh Ritz Quotient).

$$\boxed{\text{So } \frac{d}{dt}\lambda(g_{ij}) \geq 0 \text{ where } \lambda = \text{lowest eigenvalue of } -4\Delta + R.}$$

Note λ is not a scale invariant quantity!

This allows Ricci flow to have the freedom to return to a starting metric that has just been rescaled (homothety). (Sort of like being periodic without being a soliton)

So we try to study something scale invariant.

$\bar{\lambda} = \lambda V^{2/n}$ is scale invariant.

In fact $\frac{d}{dt}\bar{\lambda} \geq 0$ if $\bar{\lambda} \leq 0$

So if g has $\bar{\lambda} < 0$ then we cannot come back to ourselves under homothety. (Except in soliton case?)

(Does this correspond to $R < 0$ case?)

$R = 0 \Rightarrow \bar{\lambda} > 0$ since -4Δ is a nonnegative operator.

Now study

$$\omega(g_{ij}, f, \tau) = \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} dV \left[\tau(R+|\nabla f|^2)+f-n\right]$$

scale invariant wrt g + τ (together?)

$$\frac{d}{dt} g_{ij} = -2R_{ij}$$

$$u_t = -\Delta u + Ru \qquad u = (4\pi\tau)^{-n/2} e^{-f}$$

$$\tau_t = -1$$

Then $\frac{d}{dt}\omega \geq 0$

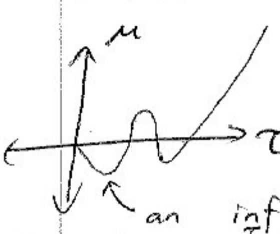
$$\mu(g_{ij}, \tau) = \inf_{\substack{f \\ \int u = 1}} \omega(g_{ij}, \tau) \quad *$$

$\frac{d}{dt} g_{ij} = -2R_{ij}$ + $\tau_t = -1 \implies \frac{d}{dt}\mu \geq 0$

Is μ some kind of eigenvalue?

Yes, for a nonlinear operator, so it is not easy to prove that inf $*$ exists. But can do this.

Trouble is μ depends on the scale parameter, τ,



as follows

continuous

explanation of graph on next page.

an $\inf_\tau \mu$ is a special value + exists + $\boxed{\frac{d}{dt}\inf_\tau \mu \geq 0}$

Rules out homothety issue on page 8.

Let $g_{ij}(0) = \bar{g}_{ij}$ + run Ricci flow on $[0, T]$.

Let $u\Big|_{t=T} = \underbrace{\delta}_{\text{delta function}}$ + $\tau = 0$ at $t = T$

Solve $u_t = -\Delta u + Ru$ $\boxed{\text{backwards?}}$ Yes in time on $[0,T]$

Calculate W + get $W \leq 0$

So μ has to be < 0 on this time interval. $[0, T]$

The behavior of μ at $\tau = 0$ is equivalent ?
to computing w/ Euclidean metric of $\tau = \frac{1}{2}$
why?

minimize:

$$\min_{\int_{\mathbb{R}^n} (2\pi)^{-n/2} e^{-f} = 1} \int_{\mathbb{R}^n} (2\pi)^{-n/2} e^{-f} \left(\tfrac{1}{2} |\nabla f|^2 + f - n \right) \geq 0$$

to be ?

Shown by the Logarithmic Sobolev Inequality

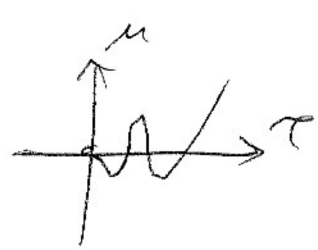
$d\mu$ = Gaussian measure on $\mathbb{R}^n$

$$d\mu = (2\pi)^{-n/2} e^{-\frac{1}{2}|x|^2} dx_1 \text{---} dx_n$$

$$\boxed{\int |\nabla \varphi|^2 d\mu \geq \tfrac{1}{2} \int \varphi^2 \log \varphi^2 d\mu}$$

minimum is 0 when $f = \frac{|x|^2}{c_n}$ ← something?

So now we have justified



(at least behavior near $\tau = 0$)

In Statistical Mechanics

$$\tau,\ \beta = \tau^{-1},\quad Z = \int_{\mathbb{R}} \exp(-\beta E)\, d\omega(E)$$

distribution over $\mathbb{R}$ of values of energy
called density of states.

$$\langle E \rangle = -\frac{\partial}{\partial\beta}\log(Z)$$

average energy

$\int_{\mathbb{R}} E\exp(-\beta E)\,d\omega(E)$

Entropy $S = \beta\langle E\rangle + \log Z$

$= \sum_i -p_n \log p_n$

p_n = states

Dispersion $\sigma = \langle E - \langle E\rangle|^2 \rangle$

$$= \frac{\partial^2}{(\partial\beta)^2}\log Z$$

Now relate Ricci flow to this picture 12

$$(g_{ij})_\tau = 2(R_{ij} + \nabla_i \nabla_j f)$$

$$dm = u\,dV$$

$$u = (4\pi\tau)^{-n/2} e^{-f}$$

$$\log Z = \int \left(-f + \frac{n}{2}\right) dm$$

"Is this compatible with the Statistical Mechanics? I make no such claim. This is just a formal analogy" - Perelman

Let $\beta = \tau^{-1}$

$$\langle E \rangle = \tau^2 \int_M \left(R + |\nabla f|^2 - \frac{n}{2\tau}\right) dm$$

$$S = -\int_M \left(\tau(R + |\nabla f|^2) + f - n\right) dm = -W(g_{ij}, f, \tau)$$

I think

$$\sigma = 2\tau^4 \int_M \left|R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}\right|^2 dm$$

$\boxed{\sigma > 0 \Leftrightarrow \text{monotonicity of } S}$?

This is 0 iff you have a gradient shrinking Ricci soliton

$\boxed{\sigma = 0}$

If $u = \delta$ function at $\tau = 0$ then $\langle E \rangle$ is positive.

$$\frac{d}{dt} \mathcal{F} \geq \frac{2}{n} \mathcal{F}^2$$

Question: What can you say about a system with bounded entropy?

In dimension 2, get nice properties for σ which don't hold in dimension 3. 13

Now some physics stipulations:

$$\mathcal{F}(g_{ij}, f) = \int_M \underbrace{(R + |\nabla f|^2) e^{-f} dV}$$

This functional is used in String Theory

Dilaton? Field

Physicists usually add an antisymmetric two form but so far it seems unrelated to Ricci flow.

Gradient flow $(g_{ij})_\tau = 2(R_{ij} + \nabla_i \nabla_j f)$

Ricci flow = 1st approx for 2 dim'l nonlinear Renormalization ~~Group Flow~~ σ model (Friedman, Annals of Physics 1985)

Metaphysical level:

Regarding physics at many scales with different coupling constants for each scale. Hierarchy of scales (ultraviolet ⟷ infrared)

distance small ⟷

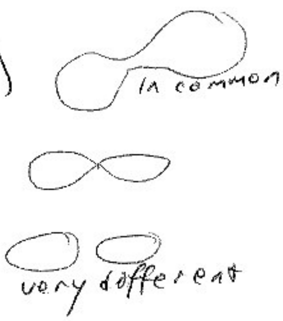
energy large ⟷

uncertainty principle

Calculate at a higher energy scale & then average to find out behavior at a lower scale.

W is this expectation. Z surpresses contribution of higher energies.

Paradox:
Higher Energy (Strong Microscope)



Global Level
(I'm not sure what he's getting at here.)

* —————————————————— *
Finish for today

Wednesday
Monotonicity ~ Bishop Gromov
Geometric Results.
Applications → (no local collapse).