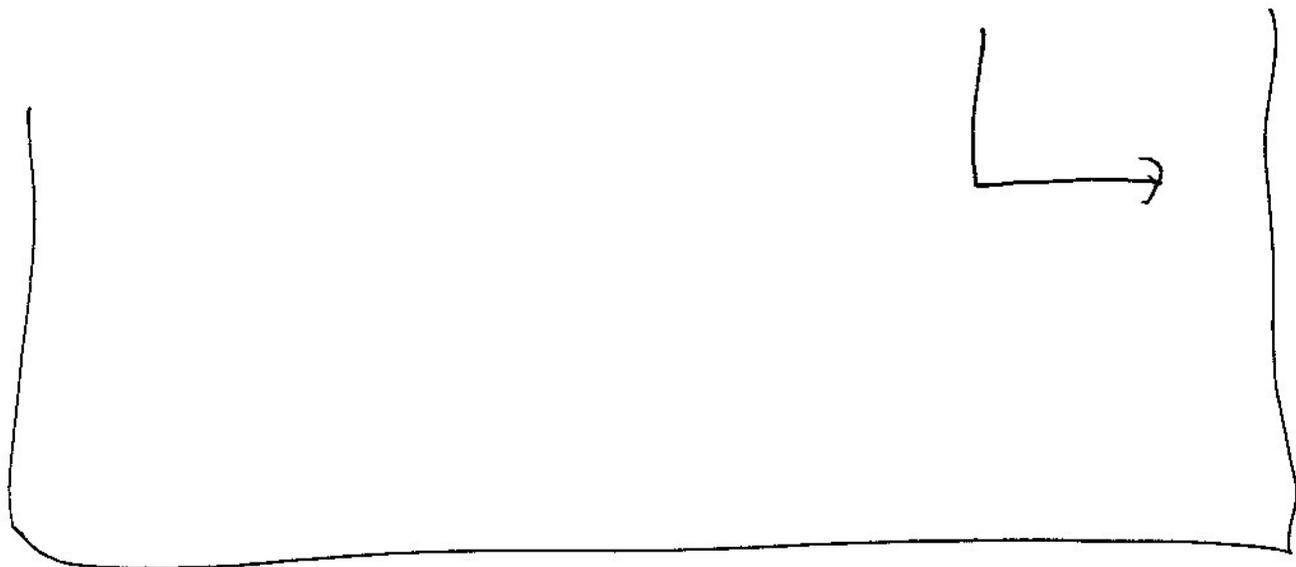


Discussion After III

Asking about Hamilton's argument regarding hyperbolic pieces & their growth under the Ricci flow. (Hyperbolic Rigidity says hyperbolic manifolds are only surrounded by ones with fewer cusps (not the same number)) so if you examine a piece w/ least # of cusps to contradict this issue (hyperbolic pieces can't blend into each other I guess)

For a ball to lose hyperbolicity it would need to lose volume first but it would need to lose hyperbolicity first (or contradict Bishop Gromov).

This is in detail in Hamilton's paper.



Compactness Thm:

$$-\infty < t \leq 0$$

$$R_m \geq 0 \quad \dim = 3$$

$$|R_m| \leq C(t) \quad (\text{which } \Rightarrow |R_m| \leq C)$$

$\& \mathcal{X}$ noncollapsed $\forall t$

then

$$(x_i, t_i) \in M_i \quad R(x_i, t_i) = 1 \quad \text{achieved by scaling}$$

then a subseq converges smoothly
to a limit which shares those
properties

In any dim, a subseq conv smoothly
to something but we don't get $C(t)$
(except in dimension 3)

Compactness Holds even with different \mathcal{X} :

Fix \mathcal{X} & rescale, \mathcal{X} constant, so
 C^∞ limit, so if \mathcal{X} changed then
a contradiction of sorts.

∞∞

Proof

Is based on

Lemma: The solns cannot have Euclidean volume growth

K noncollapse is preserved under rescaling, the curv bound is not

Shift so $(x_i, 0) \in M$.

Take largest ball $B(x_i, r)$ s.t. $\max \text{curv} \leq r^{-2}$ (note small r , so just the n .)

for each x_i find z_i s.t. $R(z_i, 0) d^2(x_i, z_i) = 1$
~~Take when d is as large as possible~~

Let $z \in B(z_i, \frac{1}{2}d)$

Claim

$$R(z) \leq C R(z_i) \quad \forall z \in B(z_i, \frac{1}{2}d)$$

If not true then $R(z) \gg R(z_i)$
 take largest such z at time $t=0$
 (one with almost max curvature in its nbhd)

$\forall \epsilon > 0$ if $\exists \bar{z} \in B(z_i, \frac{3}{4}d)$ s.t. $R(\bar{z}) > CR(z_i)$
 $\bar{Q} = R(\bar{z}) > CR(z_i)$
 $R(\bar{z})$ is large and

$$\forall y \in B(\bar{z}, A \bar{Q}^{-1/2}) \quad R(y) \leq 2R(\bar{z})$$

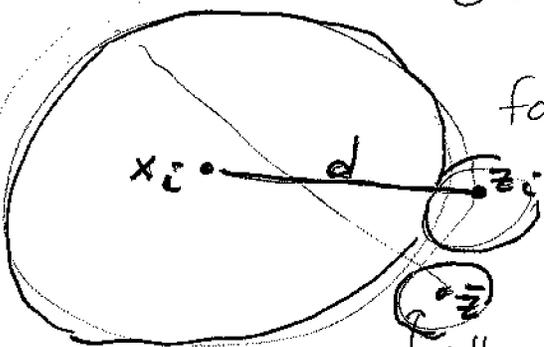
(here we have control on curvature)

then by a blowup argument this large neighborhood of \bar{z} looks like this
 soln (A is large), yet $\text{Vol}(B(\bar{z}, A \bar{Q}^{-1/2}))$

is small because solns don't have Eucl. volume growth - $\left(\frac{\text{Vol}(B(r))}{r^3} \rightarrow 0 \right)$
 actually uniformly $\forall \epsilon$.

forces $\text{Vol}(B_d(z_i, x_i)) < \epsilon d^3$ (by $\text{less for Eucl vol growth}$)

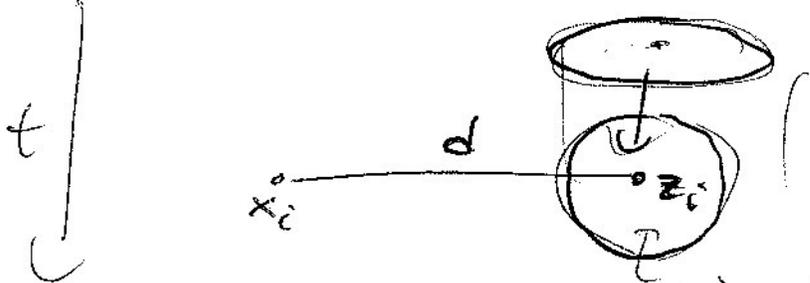
+ curv bound in z_i
 So a contradiction if $\epsilon < K$.



large ball in an ancient soln

So in fact $R(z) \leq C R(x_i) \quad \forall z_i \in B(z_i, \frac{1}{2}d)$

curv controlled in here



$$R(z) \leq C R(z_i) \quad \forall z \in B(z_i, \frac{1}{2}d)$$

$R(z, t) \leq C R(z_i, 0)$ using ^{earlier times} Harnack's ^{deep} Harnack inequality

Estimate of $S_H d$,
 If you bound $R(z, t)$ on a cylinder, then you bound all its derivatives at later time (standard parabolic).

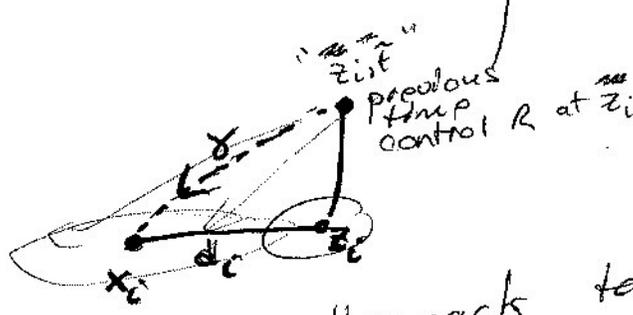
$$\text{In fact } 0 \leq R_t(z, t) \leq C R^2(z_i, 0)$$

$$C R^{-1}(z_i, 0) \leq t \leq 0$$

So going up a bit has curvature drop a bit but not much so $R(z_i, \tilde{t}) \geq \frac{1}{2} R(z_i, 0)$

$$\tilde{t} \approx R^{-1}(z_i, 0) z_i$$

$$|\delta| \approx c d^{-1}$$



now use Harnack to control scalar curvature at all pts near x_i
 $R_t + 2C(R, \delta) + 2R_k(\delta, \delta) \geq 0$ use δ from \tilde{z}_i to x_i .

$$\frac{d}{dt} R(\delta(t), t) = R_t + \langle \nabla R, \dot{\delta} \rangle$$

~~scribble~~

$$= R_t + 2 \langle \nabla R, \frac{1}{2} \dot{\delta} \rangle$$

$$\geq -2 \text{Ric}(\frac{1}{2} \dot{\delta}, \frac{1}{2} \dot{\delta})$$

$$= -\frac{1}{2} \text{Ric}(\dot{\delta}, \dot{\delta})$$

Harnack true & χ
Take $\delta = \frac{\dot{\delta}}{2}$

~~scribble~~

$$= -\frac{1}{2} \frac{K^2}{d^2} \text{Ric}(\frac{d}{c} \dot{\delta}, \frac{d}{c} \dot{\delta})$$

$$\geq -4R |\dot{\delta}|^2$$

$$\frac{\dot{R}}{R} = \frac{d}{dt} \log R \geq -|\dot{\delta}|^2 \geq c d^{-2}$$

$$0 < c \leq \frac{R(x_i, 0)}{R(z_i, \tilde{t})}$$

~~scribble~~

$$\tilde{t} \approx -c R^{-1}(z_i, 0)$$

$$\underbrace{R(x_i, 0)}_{\substack{\text{assumed} \\ = 1}} \geq c R(z_i, \tilde{t}) \geq c \frac{1}{2} R(z_i, 0)$$

So curvature at z_i is bounded above

So d_i is bounded below. (indep of i)

So largest $B(x, r)$ s.t. $\max \text{curv} \leq r^{-2}$
has size bounded below.

So now start bounding
curvature in a nbd of x_i
Assume large at y_i , got nearby \bar{y} ,
then use B-G again and
argue \tilde{x}_i ball would be
collapsed.

Thus $\dim M$ $R_m \geq 0$ $|R_m| \leq C(t)$
JC noncollapsing

then after rescaling

$(x_i, 0) \in M_i$ $R(x_i, t) = 1$

control curvature in
ball as above (volume using noncollapsing)

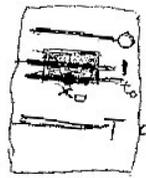
+ get C^∞ convergence.

Now explain Thm 12.1:

Why near a pt w/ large curv it looks like a singularity model.

Thm $\forall \epsilon > 0 \forall K > 0$ and for ϕ s.t $Rm(x,t) \geq -\phi(R(x,t))R(x,t)$ one can find $r_0 > 0$ s.t:

If $g_{ij}(t)$, $0 \leq t \leq T$ is a soln to the Ricci flow on a closed M^3 which has ϕ almost nonneg curv and is K noncollapsed on scales $< r_0$, then



I suppose $t_0 = T$ is ok?

$\forall (x_0, t_0)$ w/ $t_0 \geq 1$ and $Q = R(x_0, t_0) \geq r_0^{-2}$, the soln in $\tilde{E}(x, t) = \{ \text{dist}_{t_0}^2(x, x_0) < (\epsilon Q)^{-1}, t_0 - (\epsilon Q)^{-1} \leq t \leq t_0 \}$

is, after scaling by a factor Q , ϵ close to the corresponding subset of some ancient soln, satisfying assumptions of 11.1

Consider a smooth soln on $t \in [0, T]$, $\epsilon > 0$

$\exists r > 0 : Q = R(x, t) > r^{-2}$
dep on T & init data

$B(x, \epsilon^{-1} Q^{-1/2})$
 after scaling metric by Q

is ϵ close to $B(\bar{x}, \epsilon^{-1})$ in an ancient soln

Note he's abandoned the lower bound on $Km \geq 0$ curv ~~required~~ in his Compactness Theorem

So he has difficulty.

Note $R_{ij} \geq -H$ on initial g_0 doesn't imply R_{ij} controlled at later time.

Recall lemmas about Ancient Solutions

$\exists \eta > 0 \quad |DR| < \eta R^{3/2} \quad |R_t| < \eta R^2$ ← scale invariances

Now let us prove Thm 12.1 by contradiction.

~~Assume~~ initial data $|R_m| \leq 1 \quad \text{Vol}(B(1))$
(scale as nec)

So now τ will depend only on T .

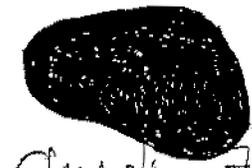
Fix T , $\exists r_k \rightarrow 0 \quad \exists t_k \in [0, T]$

s.t. $R(x_k, t_k) \geq r_k^{-2}$ (first such time)
for some x

note t_k is not $\rightarrow 0$ because of small time existence.

take x_k s.t. $R(x_k, t_k)$ is largest $\geq r_k^{-2}$

set $Q_k = R(x_k, t_k) \geq r_k^{-2}$

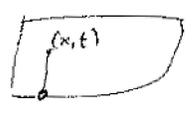


Claim: $\exists (x_0, t_0) \quad Q_0 = R(x_0, t_0) + r_k^{-2}$

Then $R(x, t) \leq 8 Q_0$ in $B(x_0, \frac{1}{8} \eta^{-1} Q_0^{-1/2})$
 $t \in (t_0 - \frac{1}{8} \eta^{-1} Q_0^{-1}, t_0)$

If not \exists pt w/ large curv $R(x, t)$

So curv at (x, t_0) is large by Harnack



the gradient estimate $R(x_0, t_0) > Q_0$ which is a contradiction

Anderson: Lets reconvene on Monday.

Questions I have

1

Perelman does not prove that the surgeries stop in a finite number of steps. He claims, but I'm not sure where he proves this, that the number of surgeries in a finite time interval is finite (no accumulation point in time for surgeries). He has not yet proven this. I'll ask him on Monday.

~~not~~
(no time to ask)
(2nd week)

2

The reason he doesn't need the surgery to stop is that in his program, whether flowing or taking surgeries, he eventually gets a thick thin decomposition. The decomposition could conceivably change if he continued the flow, but he stops as soon as he has one with a sufficiently large $\lambda V^{2/3}$. He needs this to be close to the supremum on the manifold. It is not clear to me why we should approach this supreme value, unless he takes the initial metric close to this value, but then I don't see why after surgery it should be close to the supreme value for the new manifold. Another question to ask him next week.

see page 76
(he does not do this)

Once it is close to the supreme value and has a thick thin decomposition, he argues that the tori are noncompressible. If not he uses Anderson's construction of a new metric, and shows that the new metric would have a larger lambda that would cause it to jump over the supremum.

3

I did understand at one point why it gets close to a thick thin decomposition in finite time. Now I can't quite remember. There's a big point about thick parts remaining thick (hyperbolic) and only moving slowly. So once things become thick they stay thick, and eventually all is included in thick parts that isn't actually thin. The thin parts are dealt with using convergence theory (which is smooth convergence not just GH). I believe this is crucial, and is also used elsewhere to study the singularities. He's going to discuss this convergence theory on Monday.

(no time to ask)
(2nd week)