

# Perelman III

(I'm ~~late~~ <sup>3 minutes</sup> late)

On the board

$$R_t = \Delta R + 2|Ric|^2 \rightarrow$$

mon  $R \uparrow$   
 $R_t \geq -C$

$$(g_{ij})_t = -2Ric_{ij}$$

$$(Ric_{kl})_t = \Delta Ric_{kl} + Q_{ijkl}$$

$$Ric_{kl} u_{ij} u_{kl} \geq 0 \rightarrow$$

$n=3 \quad Ric_{ij} \geq 0$   
 $t=0 \quad |R_m| \leq 1$   
 $t \quad R_m \geq -\phi(Rt)Rt - C$   
 $\phi \sim \frac{1}{\log}$

Gromoll Meyer  $\rightarrow$  Manifold is topologically simple under certain bounds  
 Cheeger-Gromoll  $\rightarrow$

Harnack Ineq by Hamilton 1993

$$Ric_{kl} \geq 0$$

"It was really a miraculous achievement" - Perelman

$$\left[ \Delta Ric_{ij} - \frac{1}{2} \nabla_i \nabla_j R + 2 Ric_{ik} Ric_{kl} - Ric_{ik} Ric_{jk} + \frac{R_{ij}}{2t} \right] u_i u_j$$

$$+ 2 (\nabla_i Ric_{kl} - \nabla_k Ric_{il}) u_i u_{jk} + Ric_{abcd} u_{ab} u_{cd} \geq 0$$

1995 Bennet Chow + his student give a Geometric Interpretation.

Perelman:  $M \times S^N \times \mathbb{R} \quad N \rightarrow \infty$   
 full curvature of this metric is + certain traces of this expression

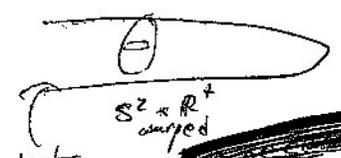
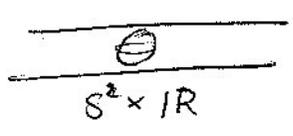
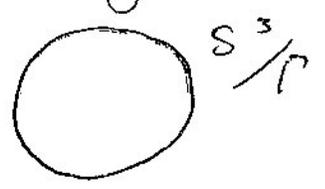
$$\frac{R}{t} + R_t + 2 \langle \nabla R, X \rangle + 2 Ric(X, X) \geq 0$$

$R_t + \frac{R}{t} \geq 0$  scalar curvature cannot decrease too fast.  
 $R_t \rightarrow$  pointwise

" $R \cdot t$  is nondecreasing in these models"  
 (does he mean  $M \times S^N \times \mathbb{R}$ ?)

Thm all singularities <sup>at finite times</sup> (for  $M$ ) are of the

form:  
 after  
 blowup



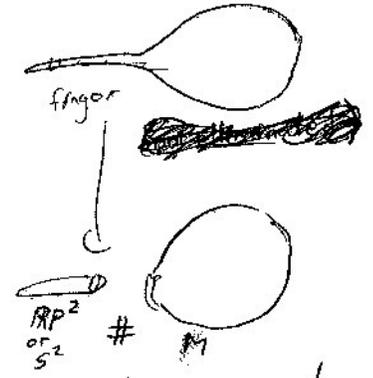
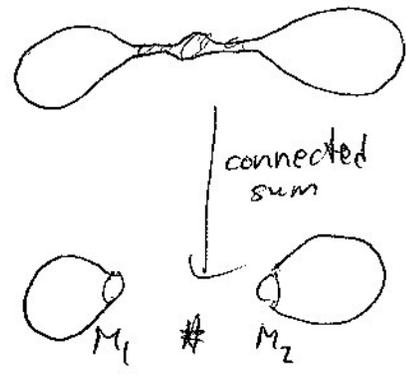
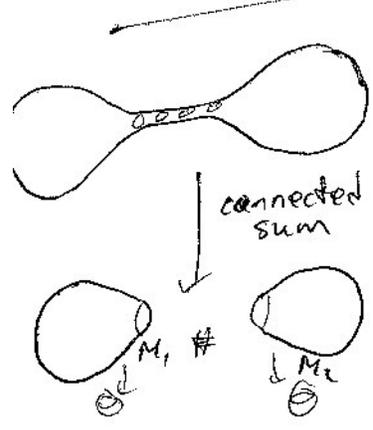
Pf requires:

Pinching Result  
 Harnack Ineq  
 + Noncollapsing

blowup  
 can be  
 done when  
 curvature  
 is large

looks  
 like  
 cylinder  
 at  $\infty$   
 (not nec  
 rotational  
 symmetric  
 away from  $\infty$ )

(Proof that blowups converge discussed during the discussion session)



Thm On any finite time interval

there are only finitely many singularities  
 (no accumulation of singularities) (Pf: uses volume estimate) (forces a loss of volume at each surgery in a given time interval)

Does not claim that surgeries stop after enough time!  
 (This was claimed in Paper I but is now retracted)

To study long term behavior:

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$\epsilon \downarrow$  ~~noncollapsing~~ <sup>control</sup> ~~area~~ forward control

In dim 3 we can also obtain backward control

If  $B(x, r) \times [T - r^2, T]$ ,  $R_m \geq -r^{-2}$ ,  $\forall |B| \geq \alpha r^3$

~~and~~  
then  $B(x, \frac{1}{2}r) \times [T - \frac{1}{2}r^2, T]$

$$|R_m| \leq K_\alpha r^{-2}$$

$K_\alpha$  can be calculated

Pf: Suppose there are pts w/ large curvature, use noncollapsing, so you can find a pt such that near this pt you have a sufficiently big piece of a smaller model, but then the volume of this small ball about this pt is small, which contradicts  $\forall |B| \geq \alpha r^3$  for the larger ball using Bishop-Gromov.

But we really want backward control with only assumptions on  $B(x, r) \times \{T\}$ .

Thm 2  
Assump's

$$B(x, r) \quad |R_m| \leq r^{-2} \quad \text{Vol} \geq A^{-1} r^3 \quad \text{on } [T-r^2, T] \quad 45$$

Conclusion

In  $B(x, Ar)$   $|R_m| \leq Kr^{-2}$  at time  $T$

where  $K$  depends only on  $A$ .

Perelman points out this is an Error: This conclusion is not always true  
Counter Example: no singularities  
  
take a ball over here

Correct Statement for 12.2: (6.3?) (corrected in Paper II section 6)  
If Ricci flow exists on  $[0, T]$

$$\forall A \exists \bar{r} = \bar{r}(A) \text{ s.t. } t \in [T-r^2, T]$$
$$B(x, r) \quad |R_m| \leq r^{-2} \quad \text{Vol} \geq A^{-1} r^3 \quad r < \bar{r} \sqrt{t}$$

at time  $t$

Then in  $B(x, Ar)$  at  $T$

$$|R_m| \leq Kr^{-2}$$

Can in fact make assumptions at time  $t$  but the proof is more tricky.

Important Point: Ricci flow exists on  $[0, T]$

The longer the flow has existed, the larger that we can take  $r$ .

Hamilton 1999'

If Ricci flow exists  $\forall t \in \mathbb{R}$  and if  $|Rm \sqrt{t/3}| < C$  for all time, then for sufficiently large  $t$  the manifold admits a thick-thin decomposition, thick-hyperbolic thin-graph manifolds by Cheeger-Gromoll theory

Now Perelman will discuss Hamilton's proof and how we can avoid the assumption  $|Rm \sqrt{t/3}| < C$ .

$$R_t = \Delta R + 2|Ric|^2 = \Delta R + \frac{2}{3}R^2 + 2|Ric|^2$$

$$\frac{d}{dt} R_{min} \geq \frac{2}{3} R_{min}^2 *$$

$$R_{min} \geq -\frac{3}{2} \frac{1}{t + const}$$

↳ depends on initial data

(If  $R_{min} > 0$  then it blows up in finite time)  
by \*

$$\frac{\partial}{\partial t} Vol = -\int R$$

So  $V \cdot (t + const)^{-3/2}$  is nonincreasing

(Can normalize the metric so the volume stays bounded and decreasing)

let  $\hat{R} = R \sqrt{t/3}$

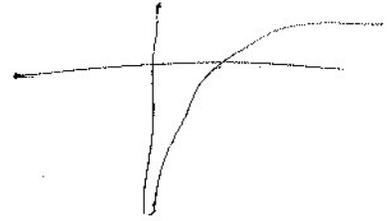
$$\frac{d}{dt} \hat{R} \geq \frac{2}{3} \hat{R} \underbrace{f(R_{min} - \hat{R})}_{\geq 0}$$

↳ in interesting case

So  $\hat{R}$  is negative & increasing.

Jeff Cheeger:  
How do you control the topology of  $M$  which collapse in finite time?  
Perelman: They are either spherical space forms or are  $S^2 \times S^1$ . From the analysis of singularities.

There are two cases:



$$\frac{d}{dt} \log(\hat{R}) \geq -\frac{2}{3} f(R_{min} - R)$$

I  $\hat{R} \rightarrow 0$  exceptional discuss later

II  $\hat{R} \rightarrow \neq 0$

then this is integrable

and  $R_{min}$  is of the order  $\sim \frac{1}{t}$  (else volume grows slower than  $t^{3/2}$ )

but  $R_{min}$  needs to be smaller than  $\hat{R}$

So  $R$  is close to  $R_{min}$

~~Notes~~ Vol grows like  $t^{3/2}$  ~~later~~ A case we will deal with ~~later~~



Suppose  $\exists t_i \rightarrow \infty$

$$B(x_i, \sqrt{t_i}) \quad \text{Vol} > c(\sqrt{t_i})^3$$

$$\text{yet } |R_m| \leq c t_i^{-1} = \frac{c}{(\sqrt{t_i})^2}$$

Study  $t_i^{-1} g_{ij}(t_i)$  & look at limit  $(\infty)$   
( $|R_m|$  bounded, Vol (bounded))

Assumption  
a neighborhood  
of thick  
pieces

So limit has const scalar curv =  $\frac{-3}{2}$

& can apply strong max principle

$$|\text{Ric}|^2 = 0$$

So the metric is Einstein

dim 3  $\Rightarrow$  hyperbolic

Now Perelman wants to use the technical statements from 12.2 p 45 to remove Hamilton's assumption  $|Rm| \leq C$   $0 < t < \infty$ .

Suppose

$$\bar{x}, t \quad \rho = \rho(x), \quad B(x, \rho), \quad \min Rm = -\rho^{-2}$$

Anderson: Can  $\rho \gg \text{diam}$ ? Yes, if so can rescale so  $\text{diam} = 1$ ,  $Rm \geq -\epsilon$ , then the manifold collapses w/ unif lower cur bound or it converges to a Ricci flat manifold.

The topology of such manifolds are all graph manifolds. (dim 3 I believe).

More difficult when  $\rho \ll \sqrt{t}$

Want to use 12.2,  $\min Rm = -\rho^{-2}$  gives us  $Rm \geq -r^{-2}$  but we don't know if  $\text{Vol } B \geq \alpha r^3$

If we do have  $\text{Vol}(B) \geq \alpha r^3$  then by 12.2

we conclude  $|Rm| \leq Kr^{-2}$

In fact  $Rm \geq -\epsilon r^2$  if  $r \ll \sqrt{t}$  because

$t \cdot Rm \geq -\rho(t) + R - \text{const}$  (pinching)

but then  $\min Rm < -\rho^{-2}$  so a  $\otimes$ . Thus  $\text{Vol } B \not\geq \alpha r^3$ .

So now we know  $B(x, \rho)$   $\min R_m = -\rho^{-2}$

$$\rho \ll \sqrt{t} \rightarrow \text{Vol } B \ll \rho^3$$

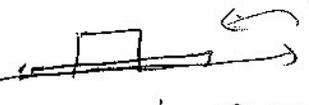
If  $\rho$  is not  $\ll \sqrt{t}$ , then  $R_{\min} \sim t^{-1}$

If Vol is small  $\rightarrow$  collapsing

If Vol is large

$B(x, \sqrt{t})$  Vol  $> c(\sqrt{t})^3$  ( $R_m < ct^{-1}$ )  
just as in Hamilton

& get thick thin decomposition.

Thick part use  Theorems which spreads out good behavior.

So if we have a substantial almost hyperbolic piece then we get an even larger hyperbolic piece.

Now hyperbolic pieces are growing & limit is complete finite volume hyperbolic manifold.

(possibly disconnected w/ cusps).

Show once  $M_t$  is close to a large piece of limit space, it has a larger & larger piece (may move but not shrink)

Hamilton's argument here goes through unaltered

Then Hamilton looks at incompressible tori  
finds an area of a disk & it disappears  
in finite time  $\Rightarrow \otimes$

This argument (1999) doesn't use  
 $|Rm| \leq C$  so it works as  
is & it is unaffected by  
surgery because.

(I missed this)  
check Hamilton's 1999 paper.

We obtain the following result:  
for large times  $M$  admits  
a thin thick decomposition

graph  
indefinite  
complete  
noncompact  
finite volume  
hyperbolic

tori which separate them are  
incompressible

which implies Thurston's Geometrization  
Conjecture

How to solve Geometrization without using minimal surfaces:

We use external theory:

$$\bar{\lambda} := \lambda V^{2/3} \quad \text{where } \lambda \text{ is the 1st nonzero eigenvalue (eigenvalue) of } -4\Delta + R$$

↑  
 scale invariant and nondecreasing under Ricci flow.

Try to find the extremal metric for  $\bar{\lambda}$ .

I thought he said this metric was achieved by Ricci flow + that he can arrange the surgeries in such a way that  $\bar{\lambda}$  is still nondecreasing.

However, when I spoke with Perelman later he said one should start with a metric on  $M^3$  which is sufficiently close to the extremal metric for  $\bar{\lambda}$ . (This extremal metric is not necessarily attained)

Three cases:

\* If  $\bar{\lambda} > 0$  then  $\lambda > 0$  +  $\frac{d}{dt} \lambda \geq \frac{2}{3} \lambda^2 \Rightarrow$  Solution dies in finite time (same as scalar  $> 0$ )  
 so  $\bar{\lambda} \leq 0$ .

\*\* If  $\sup \bar{\lambda} = 0$  then  $M^3$  is a graph manifold without positive scalar curvature. (the thick parts give volume bounded away from 0).  
 [Conversely: Cheeger-Gromov show graph manifolds have collapsing metrics so  $\sup \bar{\lambda} = 0$  on such]

\*\*\* If  $\sup \bar{\lambda} < 0$  then  $\bar{V} = (-\frac{2}{3} \bar{\lambda})^{3/2}$   $\bar{V} = \min V$   
 $M^3 = \#_{R>0} \dots \# M_1 \# M_2 \# \dots \# M_k$   
 $\text{sec} = -\frac{1}{4} \bar{V}$  hyperbolic manifolds with incompressible cusps.

Note  $\sup \bar{\lambda}$  is not actually achieved.

How to show the tori in this decomposition are incompressible:

Can use a Hamilton method or an Anderson method

Andersons: If torus in cusp is compressible

then you can change the metric on  $M^3$  by editing  $U^3 \subset M^3$  which compresses the torus ( $U^3 = \text{Dehn surgery}$ )

This new metric has larger curvature so  $\bar{\lambda}$  goes up by a definite amount which can go over  $\sup \bar{\lambda}$  if the original metric on  $M^3$  is close enough to the sup.

This doesn't use hyperbolic rigidity nor minimal surface theory.

Note the Classification  $\# + \# + \# + \#$  corresponds to the one obtained in the Yamabe Invariant!

End of Lecture III