Isospectral and Isoscattering Manifolds: A Survey of Techniques and Examples

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Abstract. The method of torus actions developed by the first and third authors yields examples of isospectral, non-isometric metrics on compact manifolds and isophasal, non-isometric metrics on non-compact manifolds. In contrast to most examples constructed by the Sunada method, the resulting examples have different local geometry. In this review, we discuss insights into the inverse spectral problem gained through both of these approaches.

1. Introduction

One of Robert Brooks' ongoing research interests—and an area in which he made fundamental contributions—was the inverse spectral problem on Riemannian manifolds. Through ingenious constructions, he helped produce examples of Riemannian manifolds which were sufficiently symmetric to have the same spectral and scattering data, and yet were not isometric. Such examples illuminate the inverse spectral problem first of all by proving non-uniqueness, and secondly by helping to isolate geometric properties of Riemannian manifolds which are not determined by spectral data. In the present paper, we will review recent constructions of compact manifolds which are isospectral, and complete non-compact manifolds which are 'isoscattering' in a sense that we will make precise. We will emphasize recent progress in understanding 'isoscattering' manifolds and recent applications of the method of torus actions, due to the first and third authors, to 'isoscattering' problems.

We will consider the following inverse problems from the contrarian point of view of trying to construct counterexamples.

Problem 1.1. (*Inverse spectral problem for compact manifolds*) *If (M, g) is a closed Riemannian manifold, the spectrum of the Laplacian consists of an infinite sequence $\{\lambda_j\}_{j=0}^{\infty}$ of nonnegative eigenvalues, and is described by the counting function*

$$N(\lambda) = \#\{\lambda_j : \lambda_j \leq \lambda\}.$$

2000 *Mathematics Subject Classification.* Primary 58J53; Secondary 58J50.
Gordon supported in part by NSF grant DMS-0306752.
Perry supported in part by NSF grant DMS-0100829.
Schueth supported in part by the DFG Priority Programme 1154.

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If (M, g) is a compact manifold with boundary and Dirichlet or Neumann conditions are imposed, the spectrum is again an infinite sequence of eigenvalues. Find manifolds (M_1, g_1) and (M_2, g_2) with the same spectrum.

We will call closed manifolds with the same spectrum (including multiplicities) *isospectral*. For compact manifolds with boundary, one may refer to *Dirichlet isospectral* or *Neumann isospectral* manifolds. For the constructions we will consider, the manifolds will be both Dirichlet and Neumann isospectral, so we will sometimes simply say "isospectral". (There is, however, one example known [**5**] of compact manifolds which are Neumann but not Dirichlet isospectral.)

PROBLEM 1.2. (*Inverse scattering problem, 'absolute' version*) *Let (M, g) be a non-compact, complete Riemannian manifold with Laplacian Δ. Let $R(z) = (\Delta - z)^{-1}$ be the resolvent operator, and suppose that the Laplacian has only continuous spectrum in $[c, \infty)$ for some $c \geq 0$. Thus $R(z)$, as a function from $L^2(M, dg)$ to itself, is analytic in $\mathbb{C}\backslash[c, \infty)$. Suppose that $R(z)$, viewed as a map from $\mathcal{C}_0^\infty(M)$ to $\mathcal{C}^\infty(M)$, admits a meromorphic continuation to a Riemann surface which covers the cut plane. Poles of the meromorphically continued operator are called* scattering resonances (*or sometimes* scattering poles). *Find complete, non-compact manifolds (M_1, g_1) and (M_2, g_2) with the same scattering resonances.*

We will call manifolds with the same scattering resonances (including multiplicities) *isopolar*.

PROBLEM 1.3. (*Inverse scattering problem, 'relative' version*) *Let (M, g_0) be a non-compact, complete Riemannian manifold of dimension n and suppose that g is a compactly supported perturbation of g_0 which is also a complete metric on M. Let H_0 be the Laplacian on (M, g_0) and let $H = \tau\Delta\tau^*$ where Δ is the Laplacian on (M, g) and $\tau : L^2(M, dg) \to L^2(M, dg_0)$ be the natural isometry. There is a real-valued, locally integrable function ξ on $\mathbb{R}$ with the property that*

$$\operatorname{Tr}(f(H) - f(H_0)) = -\int f'(\lambda)\xi(\lambda)\,d\lambda$$

for all smooth functions f which vanish sufficiently rapidly at infinity. The metric g_0 and operator H_0 are referred to as the background metric and the reference operator and remain fixed throughout the discussion. The function ξ is called the scattering phase for the pair (H, H_0), and is analogous to the counting function $N(\lambda)$ in Problem 1.1. Find metrics g_1 and g_2 on M so that the pairs (H_1, H_0) and (H_2, H_0) have the same scattering phase.

We will call such pairs of metrics *isophasal* and, when the common, fixed reference metric g_0 is understood, we will also refer to g_1 and g_2 as "isophasal metrics" and to (M, g_1) and (M, g_2) as "isophasal manifolds."

The scenario outlined in Problem 1.2 happens, among other examples, for metric perturbations of $\mathbb{R}^n$ and quotients of real hyperbolic space by geometrically finite discrete groups. We will discuss the mathematical and physical meaning of scattering resonances in §2 of what follows. Relative scattering as discussed in Problem 1.3 makes sense for *any* complete manifold, as was shown in a striking paper of Gilles Carron [**20**].

For Problem 1.1, there is a vast literature of examples of isospectral manifolds. See [**27**] for a survey of examples prior to 2000. We will not attempt to survey all the examples here but rather will emphasize the techniques and mention primarily recent examples. There are, roughly speaking, three methods for constructing examples of isospectral manifolds.

(a) *Explicit Construction*: Recent examples constructed by explicit computations include isospectral flat manifolds with surprising spectral properties ([**43**], [**44**], [**45**]), the first examples [**58**] of isospectral manifolds with boundary having different local geometry (these partially motivated and were later reinterpreted by the torus action method below) and the first examples of pairs of isospectral metrics on balls and spheres [**59**].
(b) *Representation-Theoretic Construction*: Representation theoretic methods, especially the celebrated Sunada technique [**56**], have provided the most systematic and widely used methods for constructing isospectral manifolds with the same local, but different global, geometry.
(c) *Torus Actions.* This method generally produces isospectral manifolds with different local geometry.

Among the many examples constructed by Sunada's method are Riemann surfaces of every genus greater than or equal to four [**17**], including huge families of mutually isospectral surfaces in high genus [**10**], and examples of isospectral plane domains [**34**]. As explained in §3 below, the Sunada technique (and other representation-theoretic techniques) produce isospectral quotients $H_1\backslash M$ and $H_2\backslash M$ of a given Riemannian manifold M by discrete groups H_i of isometries; thus the isospectral manifolds are locally isometric. Recently, however, Craig Sutton [**57**] modified Sunada's method to allow the subgroups H_i to be connected, and constructed isospectral simply-connected, normal homogenous spaces that are not locally isometric.

The method of torus actions ([**26**], [**28**], [**29**], [**36**], [**52**], [**53**], [**54**]) was developed to construct isospectral manifolds with different local geometry. The first author used this method to construct continuous families of isospectral metrics on the n-ball and $(n-1)$-sphere for all $n \geq 9$. The third author lowered n to 8 and also obtained pairs of isospectral metrics on the 6-ball and 5-sphere. She also showed that in all cases, one can arrange that the metrics on the balls are Euclidean except on an arbitrarily chosen smaller ball about the origin.

Both the Sunada technique and the method of torus actions have been extended to complete, noncompact manifolds in order to obtain non-isometric manifolds with the same scattering resonances (the isopolar manifolds of Problem 1.2) and, in some cases, the same scattering phase (the isophasal manifolds of Problem 1.3). As explained in §2, isophasality is a stronger condition than isopolarity in contexts where both notions are well-defined (see Remarks 2.2 and 2.3). The examples constructed by these two methods account for all known examples of complete, non-isometric manifolds with the same scattering data. The examples constructed by variants of Sunada's technique include finite-area Riemann surfaces (both isopolar and isophasal–see Bérard [**4**] and Zelditch [**66**]), Riemann surfaces of infinite area (isopolar and isophasal–see Guillopé-Zworski [**38**] and Brooks-Davidovich [**8**]), three-dimensional Schottky manifolds (isopolar–see Brooks-Gornet-Perry [**11**]), and surfaces that are isometric to Euclidean space outside a compact set (isopolar and isophasal–see Brooks-Perry [**12**]).

The generalization of the torus action method to noncompact manifolds is more recent ([**31**] and [**47**]). In [**31**], the first two authors showed the following: Let $\{g_t\}$

be any of the families of isospectral metrics on the unit ball in $\mathbb{R}^n$ constructed in [**28**] or [**54**], modified as in [**54**] so that the metrics are Euclidean outside of a ball of smaller radius about the origin. Extend the metrics to all of $\mathbb{R}^n$ so that they are Euclidean outside of the small ball. Then the resulting metrics are non-isometric but are both isophasal and isopolar. In [**47**], the last two authors show how to use a similar construction to obtain non-isometric, isophasal and isopolar families of metrics on $\mathbb{R}^n$ which are hyperbolic off a small ball or more generally are perturbations of complete metrics which admit an $\mathrm{O}(n)$ action by isometries.

In what follows we first review basic notions of spectral and scattering theory for the Laplacian on a Riemannian manifold (§2), recall the Sunada method (§3), and discuss the method of torus actions (§4). Finally, we pose several open problems (§5).

2. Spectral and Scattering Theory for the Laplacian

The Laplace-Beltrami operator on a Riemannian manifold is most easily defined via the method of quadratic forms. For a closed manifold, we denote by $\mathcal{H}^1(M,g)$ the completion of $\mathcal{C}^\infty(M)$ in the inner product

$$\langle\varphi,\psi\rangle = \int_M \nabla\varphi\cdot\overline{\nabla\psi}\,dg + \int_M \varphi\overline{\psi}\,dg. \tag{2.1}$$

For a complete, non-compact manifold, we denote by $\mathcal{H}^1_0(M,g)$ the completion of $\mathcal{C}^\infty_0(M)$ in the same inner product. If M is a compact manifold with boundary, we denote by $\mathcal{H}^1_D(M,g)$ the completion of $\mathcal{C}^\infty_0(M)$ in the inner product (2.1), and by $\mathcal{H}^1_N(M,g)$ the completion of $\mathcal{C}^\infty(M)$ [1] in the same inner product.

The Laplace-Beltrami operator on M is the positive operator Δ_M associated to the quadratic form

$$\mathfrak{q}(\varphi,\psi) = \int_M \nabla\varphi\cdot\overline{\nabla\psi}\,dg,$$

with form domain given by:

- $\mathcal{H}^1(M,g)$ if M is compact and without boundary,
- $\mathcal{H}^1_D(M,g)$ if M is compact, $\partial M\neq\emptyset$ and Dirichlet boundary conditions are imposed,
- $\mathcal{H}^1_N(M,g)$ if M is compact, $\partial M\neq\emptyset$ and Neumann boundary conditions are imposed, and
- $\mathcal{H}^1_0(M,g)$ if (M,g) is non-compact and complete.

The spectral theory of the operator Δ_M determines the behavior of solutions to the wave equation on (M,g). Consider the initial value problem for a function $u:\mathbb{R}\to L^2(M,g)$ and initial datum $\psi\in L^2(M,g)$:

$$\begin{aligned} u_{tt} &= -\Delta_M u \\ u(0) &= \psi \\ u_t(0) &= 0. \end{aligned} \tag{2.2}$$

Formally, the solution to this equation is

$$u(t) = \cos\left(t\sqrt{\Delta_M}\right)\psi$$

[1] If M is a manifold with boundary, $\mathcal{C}^\infty(M)$ denotes the restrictions of $\mathcal{C}^\infty$ functions on the double of M, i.e., the manifold obtained by gluing two copies of M along ∂M.

where the solution operator

$$E(t) = \cos\left(t\sqrt{\Delta_M}\right)$$

is defined by the functional calculus for the self-adjoint operator Δ_M.

On the level of functional analysis, the spectral theory of the Laplace operator determines the behavior of solutions in the following way. Recall that if ψ is a vector in $\mathcal{H}$, and A is a self-adjoint operator on $\mathcal{H}$, the linear functional $f \mapsto (f(A)\psi, \psi)$ on real-valued continuous functions f that vanish at infinity may be represented as integration with respect to a Borel measure μ_ψ on the real line:

$$(f(A)\psi, \psi) = \int_{\mathbb{R}} f(\lambda)\, d\mu_\psi(\lambda).$$

This measure is called the spectral measure for ψ with respect to the self-adjoint operator A. For any self-adjoint operator A on a Hilbert space $\mathcal{H}$, the orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_{\text{p.p.}}(A) \oplus \mathcal{H}_{\text{a.c.}}(A) \oplus \mathcal{H}_{\text{s.c.}}(A)$$

holds, corresponding to vectors $\psi \in \mathcal{H}$ for which the spectral measure μ_ψ is pure point, absolutely continuous, or singularly continuous with respect to Lebesgue measure on the line (see [**49**], chapter VII and §VIII.3). Roughly and informally, initial data in $\mathcal{H}_{\text{p.p.}}(\Delta_M)$, $\mathcal{H}_{\text{p.p.}}(\Delta_M)$, and $\mathcal{H}_{\text{s.c.}}(\Delta_M)$ corresponds respectively to bound, escaping, and recurrent orbits for the wave equation. On compact manifolds $\mathcal{H} = \mathcal{H}_{\text{p.p.}}(\Delta_M)$ so that there are only bound orbits. On non-compact manifolds with simple geometry at infinity, we expect that $\mathcal{H} = \mathcal{H}_{\text{p.p.}}(\Delta_M) \oplus \mathcal{H}_{\text{a.c.}}(\Delta_M)$, i.e., all orbits are either bounded or escape to infinity, and there are no recurrent orbits.

2.1. Compact Manifolds. If M is a closed manifold or compact manifold with boundary, the spectrum of the Laplacian consists of discrete eigenvalues λ_j associated to normalized eigenfunctions φ_j. The solution is written in the familiar separation of variables form

$$u(x,t) = \sum_{j=0}^{\infty} (\varphi_j, \psi) \cos\left(t\sqrt{\lambda_j}\right) \varphi_j(x), \tag{2.3}$$

where $(\,\cdot\,,\,\cdot\,)$ is the $L^2(M,g)$-inner product. The eigenfunctions φ_j and the numbers $\sqrt{\lambda_j}$ determine standing wave patterns and frequencies of oscillation. These are determined by the geometry of the manifold and encode geometric data.

For purposes of comparison with the non-compact case, it will be useful to note that the numbers λ_j may be obtained as poles of the L^2-resolvent operator

$$R(z) = (\Delta_M - z)^{-1}$$

whose residues project onto the appropriate eigenspaces. The solution operator $E(t)$ is obtained from the resolvent $R(z)$ via the integral formula

$$E(t) = \frac{1}{2\pi i} \int_{\operatorname{Re}(\lambda)=c} \lambda R(-\lambda^2) e^{t\lambda}\, d\lambda, \tag{2.4}$$

where $c < 0$, as follows from the inverse Laplace transform

$$\cos(tx) = \frac{1}{2\pi i} \int_{\operatorname{Re}(\lambda)=c} \frac{\lambda}{\lambda^2 + x^2} e^{t\lambda}\, d\lambda, \tag{2.5}$$

true for any $c < 0$, together with the spectral theorem for self-adjoint operators. One can recover the formula (2.3) from (2.5) and the meromorphy of the resolvent operator.

2.2. Non-Compact Manifolds: Scattering Resonances. If (M, g) is not compact but has "simple geometry at infinity,"[2] the Laplace operator may have *no* eigenvalues, corresponding to the fact that energy may "leak out" of any bounded region. If we examine the behavior of a solution to (2.2) on a non-compact manifold M with simple geometry at infinity, but restrict attention to a compact subset of M, we find an expansion analogous to (2.3) in which the cosines are replaced by complex exponentials and the eigenvalues are replaced by complex numbers ζ, the *scattering resonances* of the Laplacian, whose real parts determine a frequency of oscillation and whose imaginary parts determine a rate of energy decay for the associated normal mode.[3] For example, in the case of scattering by a compactly supported perturbation in $\mathbb{R}^n$ when n is odd, the expansion[4]

$$\chi \cos\left(t\sqrt{\Delta_M}\right)\chi\psi \sim \sum_j \sum_{k=0}^{N_j} c_{j,k} t^k \exp\left(i\zeta_j t\right)\varphi_j(x)$$

holds, where $\chi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ is a cutoff function, the φ_j are resonance eigenfunctions, and the numbers $c_{j,k}$ depend on the initial data ψ. The secular terms (involving powers of t) may arise because the resonances are solutions of a non-self-adjoint eigenvalue problem. More precisely, the resonances are poles of the analytically continued operator

$$\widetilde{R}(k) = \chi\left(\Delta_M - k^2\right)^{-1}\chi,$$

initially defined on the half plane $\mathrm{Im}(k) < 0$ (corresponding to the cut plane $\mathbb{C} - [0, \infty)$ in the $\lambda = k^2$ variable) and extended to the complex k-plane. Like the resolvent of a non-symmetric matrix, the resolvent $\widetilde{R}(k)$ has a Laurent expansion near a given singularity ζ whose polar part takes the form $\sum_{j=1}^{N_\zeta} A_j\,(k-\zeta)^{-j}$. Here the A_j are finite-rank operators and the A_j for $j \geq 2$ are nilpotent. The resonance eigenfunctions φ_j are determined by the finite-rank residues of the resolvent. The *multiplicity* of a scattering resonance ζ is the dimension, m_ζ, of the space $\oplus_{j=1}^{N_\zeta}\mathrm{Ran}(A_j)$. The set of resonances ζ together with their multiplicities m_ζ forms the *resonance set* for Δ_M and constitutes a discrete set of 'scattering' data analogous to the eigenvalues. This resonance set is the subject of the 'absolute scattering' inverse problem, Problem 1.2.

In most cases of interest, the scattering poles for a complete, non-compact manifold M with geometric boundary $\partial_\infty M$ can also be characterized as poles of a scattering matrix $S(z) : \mathcal{C}^\infty(\partial_\infty M) \to \mathcal{C}^\infty(\partial_\infty M)$.

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[2]Examples include perturbations of the Euclidean metric and non-compact locally symmetric spaces.

[3]For an introduction to resonances and a review of earlier literature, see the surveys [**67**] and [**68**].

[4]So-called *resonance wave expansions* of this kind were first proved by Vainberg [**61**] for acoustical scattering by an obstacle in $\mathbb{R}^n$. Resonance wave expansions have also been obtained for certain hyperbolic surfaces by Christiansen and Zworski [**23**] and for scattering on $\mathbb{R}^n$ by a compactly supported perturbation by Tang and Zworski [**60**]. Although resonance wave expansions are expected to hold in "reasonable" scattering situations, the proof involves delicate estimates on the meromorphically continued resolvent and subtle remainder estimates.

2.3. Non-Compact Manifolds: Scattering Phase. Relative scattering theory compares solutions of an evolution equation such as the wave equation (2.2) to solutions of the same equation for a simpler, 'unperturbed' system. For example, suppose that $(M, g) = (\mathbb{R}^n, g)$ where g is a metric on $\mathbb{R}^n$ which differs from the Euclidean metric g_0 only on a compact set. A natural comparison problem is then the wave equation for the Laplacian Δ_0 on $(\mathbb{R}^n, g_0)$.

Thus, comparison or relative scattering theory is very naturally a branch of perturbation theory for linear operators. One of the most fruitful versions of scattering theory at the level of operator theory is the trace-class scattering theory pioneered by Kato, Birman, Krein, and others; Yafaev's monograph [**64**] gives a very complete survey. For basics of scattering theory and a more concise review of the trace-class theory of scattering, see [**50**]. The trace-class theory concerns spectral and scattering theory for pairs of operators (A, B) for which $\varphi(A) - \varphi(B)$ belongs to the trace class for some monotone function φ. The following theorem of Carron [**20**] (actually proved in somewhat greater generality there) shows that we can apply trace-class scattering to many geometric situations.

Theorem 2.1. *Suppose that (M, g_0) is a complete Riemannian manifold and that g is another complete metric on M with the property that $g - g_0$ is compactly supported. Let Δ_0 and Δ be the respective Laplacians on (M, g_0) and (M, g), let $\tau : L^2(M, g) \to L^2(M, g_0)$ be the natural isometry, let $H_0 = \Delta_0$, and let $H = \tau\Delta\tau^*$. Then for any integer $k > n/2$ and any $z \in \mathbb{C} - \mathbb{R}$, the operator*

$$(H - z)^{-k} - (H_0 - z)^{-k}$$

is a trace-class operator.

For such pairs (H, H_0), the trace-class theory guarantees that for every solution of the initial value problem

$$\begin{aligned} u_{tt} &= -Hu \\ u(0) &= \psi \\ u_t(0) &= 0 \end{aligned}$$

with initial data ψ in the absolutely continuous spectral subspace for H, there are corresponding initial data $\psi_\pm$ and solutions $u_\pm$ of the equation

$$\begin{aligned} u_{tt} &= -H_0 u \\ u(0) &= \psi_\pm \\ u_t(0) &= 0 \end{aligned}$$

with the property that $\lim_{t\to\pm\infty} \|u(t) - u_\pm(t)\| = 0$. That is, the solutions $u_\pm$ of the unperturbed equation are asymptotic to the solution u of the perturbed equation. The scattering operator is the map $S(H, H_0) : \psi_- \to \psi_+$ from the "past" to the "future" asymptote.

The operator $S(H, H_0)$ commutes with H_0. Thus, in a spectral representation for H_0, $S(H, H_0)$ acts by unitary operators $S(\lambda)$ on Hilbert spaces $\mathcal{H}_\lambda$ that arise in the spectral decomposition of H_0; here $\lambda \in [0, \infty)$ is a spectral parameter.[5]

[5] For example, if H_0 is the Euclidean Laplacian, $S(H, H_0)$ acts on $L^2(\mathbb{R}^n)$. Let $\mathcal{H}_\lambda = L^2(S^{n-1})$, and think of the space $\mathcal{H} = L^2((0, \infty) \times S^{n-1})$ as the 'constant fibre direct integral'

$$\int_{(0,\infty)}^{\oplus} \mathcal{H}_\lambda \, d\lambda.$$

In geometric situations, $S(\lambda)$ can be viewed as an operator from $\mathcal{C}^\infty(\partial_\infty M)$ to itself: elements of $\mathcal{C}^\infty(\partial_\infty M)$ should be thought of as 'radiation patterns' for a wave of energy λ, and $S(\lambda)$ as a map from incoming to outgoing radiation patterns. The trace-class theory guarantees that $S(\lambda) - I$ is a trace class operator, so that the operator determinant $\det S(\lambda)$ is well-defined. Since $S(\lambda)$ is unitary, it follows that

$$\det S(\lambda) = \exp(2\pi i \sigma(\lambda))$$

for a function σ on the real line. The function $\sigma(\lambda)$ is called the *scattering phase* and is determined by the *pair* (H, H_0).[6]

A fundamental result of Birman and Krein relates the scattering phase to the spectral shift function (SSF) for the pair (H, H_0). Under the trace-class condition in Theorem 2.1, there is a measurable, real-valued, and locally integrable function ξ on $\mathbb{R}$ with the property for all admissible functions f (including $\mathcal{C}_0^\infty(\mathbb{R})$ functions and the function $f(\lambda) = \exp(-t\lambda)$), the trace formula

$$\operatorname{Tr}\left(f(H) - f(H_0)\right) = -\int f'(\lambda)\xi(\lambda)\, d\lambda$$

holds. The celebrated Birman-Krein formula states that

$$\det S(\lambda) = \exp(2\pi i \xi(\lambda)). \tag{2.6}$$

Setting $f(\lambda) = \exp(-t\lambda)$ yields a 'relative heat trace'

$$\begin{aligned} H(t) &= \operatorname{Tr}\left(\exp(-tH) - \exp(-tH_0)\right) \\ &= -t \int \exp(-t\lambda)\, \xi(\lambda)\, d\lambda. \end{aligned}$$

which is thus determined by the scattering phase. It is not difficult to see that if pairs of operators (H_1, H_0) and (H_2, H_0) have the same relative heat trace, then the spectral shift functions are equal almost everywhere.

The scattering phase is the subject of the 'relative scattering inverse problem,' Problem 1.3.

Remark 2.2. When (M, g_0) is Euclidean space and g is a compactly supported perturbation of g_0, the scattering phase $\sigma(\lambda)$ has a meromorphic extension to a double cover (n odd) or an infinite cover (n even) of the cut plane whose poles are exactly the scattering resonances. Thus the scattering phase determines the scattering resonances. On the other hand, the scattering resonances determine $\sigma(\lambda)$ only up to finitely many parameters, in analogy to the fact that the zeros of an entire function of finite order determine an entire function only up to an overall factor

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If $\mathcal{F}$ is the Fourier transform, then the map

$$\mathcal{V}f(\lambda, \omega) = 2^{-1/2}\lambda^{(n-2)/4}\left(\mathcal{F}f\right)(\lambda^{1/2}\omega)$$

from $L^2(\mathbb{R}^n)$ to $\mathcal{H}$ (here $\lambda \in (0,\infty)$ and $\omega \in S^{n-1}$) gives the spectral representation for H_0. The operator $\mathcal{V}S(H, H_0)\mathcal{V}^{-1}$ acts on $\mathcal{H}$ as

$$\int_{(0,\infty)}^{\oplus} S(\lambda)\, d\lambda$$

where $S(\lambda)$ is a unitary operator on $\mathcal{H}_\lambda$ and $S(\lambda) - I$ is a trace-class operator on $\mathcal{H}_\lambda$.

[6] As we have defined it, σ is defined only modulo the integers. The Birman-Krein formula (see (2.6)) expresses the scattering phase in terms of the spectral shift function for the pair (H, H_0), which *is* uniquely determined.

which is the exponential of a polynomial (if n is odd, the additional parameters are exactly the coefficients of a polynomial in such an exponential factor).

REMARK 2.3. When (M, g) is a Riemann surface with finite geometry and infinite area, there is also a natural 'comparison scattering theory' as explained in §3. In these cases, the corresponding scattering phase can be continued to the complex plane and has poles at the scattering resonances. One can show that the continued scattering phase is an entire function of finite order so that it is determined by its zeros up to finitely many parameters. For pairs of isopolar manifolds, the two scattering phases have the same poles when analytically continued; for pairs of isophasal manifolds, the scattering phases are actually the same, a stronger condition.

3. The Sunada Technique

Sunada's technique [**56**] reduces the problem of constructing isospectral or isoscattering manifolds to finding a geometric model for a triple of finite groups (G, H_1, H_2) (sometimes called a *Sunada triple*) that obeys a simple conjugacy condition.

DEFINITION 3.1. Let G be a finite group and let H_1 and H_2 be subgroups of G. We will say that H_1 is *almost conjugate* to H_2 in G if each G-conjugacy class $[g]_G$ intersects H_1 and H_2 in the same number of elements.

REMARK 3.1. The almost conjugacy condition is equivalent to a representation theoretic condition as follows. The left multiplication of G on the cosets in G/H_i gives rise to a natural action of G on the finite-dimensional vector space $L^2(G/H_i)$. The subgroups H_1 and H_2 of G are almost conjugate if and only if $L^2(G/H_1)$ and $L^2(G/H_2)$ are isomorphic as G-modules.

Recall that a group G acts *freely* on a manifold M if the only $g \in G$ with a non-trivial fixed point set is the identity element. A group action is called *effective* if no nontrivial group element acts as the identity.

THEOREM 3.2. *Let H_1 and H_2 be almost conjugate subgroups of a finite group G. Let (M, g) be a compact Riemannian manifold on which G acts on the left by isometries. Assume that H_1 and H_2 act freely. Then*

$$\operatorname{spec}(H_1 \backslash M, g) = \operatorname{spec}(H_2 \backslash M, g).$$

There are many proofs of this theorem, each one simple and elegant. (See the survey [**7**] for a full discussion and references.) Pierre Berard [**4**], motivated by an example of Peter Buser, developed a proof by "transplantation" in which eigenfunctions on one manifold can be explicitly transplanted to eigenfunctions with the same eigenvalue on the other manifold. See also Zelditch [**65**] for an independent construction of transplantation. In our presentation below, we give a simplified version of the transplantation argument by Robert Brooks and Orit Davidovich [**8**] (see also [**11**] and [**12**] for similar constructions in different geometric contexts and [**30**] for an expanded treatment of transplantation).

The transplantation argument is based on the representation-theoretic version of the almost conjugacy condition given in Remark 3.1. Any G-module isomorphism τ between $L^2(G/H_1)$ and $L^2(G/H_2)$ is uniquely determined by the image $c = \tau(\chi_{H_1})$, where $\chi_{H_1} \in L^2(G/H_1)$ is the map that takes the value 1 on the coset H_1 and zero elsewhere. We may view c as a function on G satisfying $c(gh) = c(g)$ for all $g \in G$ and $h \in H_2$. Under the hypothesis of Theorem 3.2, identify $\mathcal{C}^\infty(H_i \backslash M_i)$ with $\mathcal{C}^\infty(M)^{H_i}$, the space of smooth functions on M invariant under the action of H_i. Then one shows that the map $T : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ given by

$$T(f)(x) = \sum_{g \in G} c(g) f(g \cdot x) \tag{3.1}$$

is a linear isomorphism from $\mathcal{C}^\infty(M)^{H_1}$ to $\mathcal{C}^\infty(M)^{H_2}$ intertwining the Laplacians.

Given a finite group G and almost conjugate subgroups H_1 and H_2, one can easily obtain examples of compact Riemannian manifolds M on which G acts by isometries in such a way that Theorem 3.2 can be applied. Indeed let M_0 be any compact Riemannian manifold whose fundamental group admits a surjection $\phi : \pi_1(M_0) \to G$, and let M be the Riemannian covering with fundamental group $\ker(\phi)$. Then G acts freely by isometries on M. (This condition is stronger than what is needed in Theorem 3.2. There we require that H_1 and H_2 act freely so that the quotients are manifolds but not that G act freely. In the more general case, $M_0 = G \backslash M$ is an orbifold.) In this way one obtains manifolds M_1 and M_2 which are isospectral and are covers of M_0, hence are locally isometric.

As noted in the introduction, Sunada's theorem has been used extensively to construct isospectral, non-isometric compact manifolds (see for example the first author's survey paper [**27**]), and there are a number of adaptations to the noncompact setting (see [**4**], [**8**], [**11**], [**12**], [**38**], [**65**]). Here we review one such adaptation, given by Brooks and Davidovich [**8**], to construct isopolar and isophasal Riemann surfaces with cusps and/or funnels.

Let M be a complete Riemann surface of infinite area and finite geometry, and let M^C be a conformal compactification. M^C has one boundary circle for each funnel and one boundary point for each cusp. Thus if M has N_c cusps and N_f funnels, a smooth function on ∂M^C is an element of $\mathbb{C}^{N_c} \oplus \left(\oplus_{j=1}^{N_f} \mathcal{C}^\infty(S^1) \right)$. Choose a *defining function* on M^C; this is a function ρ which is positive on M^C except at the boundary circles and cusp points, where it vanishes exactly to first order. Clearly, any two defining functions ρ and ρ' are related by $\rho = w\rho'$ for a strictly positive function $w \in \mathcal{C}^\infty(M^C)$.

Let Δ be the positive Laplacian on M with its natural hyperbolic metric. To define the scattering matrix, we first need a uniqueness result for generalized eigenfunctions (see, for example [**41**]). This result should be viewed as an analogue of uniqueness of solutions for the Dirichlet problem on a bounded domain.

PROPOSITION 3.3. *For s with* $\mathrm{Re}(s) = 1/2$ *and* $s \neq 1/2$, *let* $f_+ \in \mathcal{C}^\infty(\partial M^C)$. *Then there exists a unique solution* $u \in \mathcal{C}^\infty(M)$ *of the eigenvalue equation*

$$(\Delta - s(1-s))u = 0$$

with the property that

$$u \sim \rho^{1-s} f_+ + \rho^s f_- + \mathcal{O}(\rho)$$

as $\rho \downarrow 0$ *for a function* $f_- \in \mathcal{C}^\infty(M^C)$.

The uniqueness of u implies that the map $S(s) : f_+ \to f_-$ is well-defined; this map is called the scattering matrix, a map from $\mathcal{C}^\infty(\partial M^C)$ to itself. The functions $f_\pm$ can be thought of as incoming and outgoing radiation patterns. Although initially defined for $\mathrm{Re}(s) = 1/2$, the scattering matrix extends to a meromorphic operator-valued function of s. Roughly and informally, the poles of $S(s)$ coincide with the poles of the resolvent (see, for example, [**38**] for a detailed discussion). Two Riemann surfaces are called *isopolar* if the poles of their respective scattering matrices coincide. Although the scattering matrix appears to depend on the choice of defining function, this dependence is trivial and it can be shown that the poles of the scattering matrix are independent of the choice of defining function.

To obtain isopolar surfaces, Brooks and Davidovich began with a complete surface M_0 whose fundamental group surjects on a finite group G containing almost conjugate subgroups H_1 and H_2. They let M_0^C be a conformal compactification and chose a defining function ρ_0. Letting M be the complete Riemann surface that covers M_0 with covering group G, then a conformal compactification M^C of M covers M_0^C, and the defining function ρ_0 lifts to a G-invariant defining function ρ on M^C. Let $M_i = H_i\backslash M$. Then $M_i^C := H_i\backslash(M^C)$ is a conformal compactification of M_i and ρ descends to a defining function ρ_i on M_i^C. Identifying $\mathcal{C}^\infty(M_i^C)$ with $\mathcal{C}^\infty(M^C)^{H_i}$ and $\mathcal{C}^\infty(\partial M_i^C)$ with $\mathcal{C}^\infty(\partial M^C)^{H_i}$, then it is straightforward to check that the transplantation map T defined by equation 3.1 both on $\mathcal{C}^\infty(M^C)$ and on $\mathcal{C}^\infty(\partial M^C)$ intertwines the scattering matrices of M_1 and M_2. Thus these manifolds are isopolar. Brooks and Davidovich used this method (with very carefully chosen M_0) to construct isopolar surfaces of various genera with various numbers of ends and also to construct isopolar congruence surfaces.

As shown in [**38**], it is possible to define relative scattering from a Riemann surface of infinite area and finite geometry. For such a Riemann surface M,

$$M = Z \cup \left(\cup_{i=1}^{N_C} C_i\right) \cup \left(\cup_{j=1}^{N_f} F_j\right)$$

where C_i are cusps and F_j are funnels. The absolute scattering operator $S(s)$ acts on $\mathcal{C}^\infty(\partial M^C) = \mathbb{C}^{N_c} \oplus \left(\oplus_{j=1}^{N_f} \mathcal{C}^\infty(S^1)\right)$; if

$$S_0(s) = \mathbf{1} \oplus \left(\oplus_{j=1}^{N_f} S_{F_j}(s)\right)$$

where $\mathbf{1}$ is the identity on $\mathbb{C}^{N_C}$ and $S_{F_j}(s)$ is the scattering matrix for a hyperbolic half-funnel with Dirichlet conditions, then the relative scattering matrix is $S_{\text{rel}}(s) = S(s)S_0(s)^{-1}$ and compares wave motion on M with wave motion on a disjoint union of funnels. The scattering phase is given by $\det(S_{\text{rel}}(s)) = \exp(2\pi i\xi(s))$. The counterexamples constructed by Brooks and Davidovich (as well as earlier examples constructed by Guillopé and Zworski in [**38**]) are also isophasal.

4. The Method of Torus Actions

By a torus we will always mean a compact connected abelian Lie group. At the beginning of the development of the so-called method of torus actions lay the first author's observation of the general principle expressed in the theorem below. She used this idea to obtain the first examples of pairs of *locally nonisometric* isospectral closed manifolds (certain two-step nilmanifolds). (Earlier Z. Szabo had announced the first examples of locally nonisometric, isospectral manifolds with boundary, later published in [**58**].)

Theorem 4.1. [**26**] *If a torus T acts on two closed Riemannian manifolds (M, g), (M', g') freely and isometrically with totally geodesic orbits, and if the quotients of the manifolds by any subtorus W of codimension at most one are isospectral when endowed with the submersion metrics g^W, ${g'}^W$, then (M, g) and (M', g') are isospectral.*

The proof is quite simple: We use the fact that if $M \to \overline{M}$ is a Riemannian submersion with totally geodesic fibers, then the spectrum of $\overline{M}$ coincides with the spectrum of the Laplacian on M restricted to functions constant on the fibers. Using Fourier decomposition with respect to the T-action and the fact that the T-orbits are totally geodesic (and hence that the W-orbits are also totally geodesic for each subtorus W), one thus shows that

$$\operatorname{spec}(M,g) = \operatorname{spec}(M/T, g^T) \cup \bigcup_W \big(\operatorname{spec}(M/W, g^W) \setminus \operatorname{spec}(M/T, g^T)\big),$$

where multiplicities are taken into account and W runs through the set of all subtori $W \subset T$ of codimension one. Since the analogous decomposition of the spectrum also holds for (M', g'), the theorem follows immediately.

In view of this, the theorem above seems almost tautological. Its usefulness, however, lies in the fact that there are lots of examples in which the submersion quotients $(M/W, g^W)$ and $(M'/W, g'^W)$ are actually *isometric* (and thus trivially isospectral), but still the "big" isospectral manifolds (M,g), (M',g') are nonisometric.

Particularly simple examples of this kind occur in the case where $M = M' = N \times T$ for some closed manifold N and the T-action is the canonical action on the second factor. Fix a Riemannian metric h on N and a translation invariant (i.e., flat) metric on T. The metric on T is specified by an inner product $\langle\, , \rangle$ on the Lie algebra $\mathfrak{t}$. For each $(x,t) \in N \times T$ the tangent space $T_{(x,t)}(N \times T)$ is a direct sum $T_xN \oplus \mathfrak{t}$. Given a $\mathfrak{t}$-valued 1-form λ on N, we construct a metric g_λ on $N \times T$ so that the projection $\pi : (N \times T, g_\lambda) \to (N, h)$ is a Riemannian submersion with fibres isometric to T with its given metric. We specify g_λ at each $(x,t) \in N \times T$ by defining $\mathrm{Hor}_{(x,t)}$ (i.e., the g_λ-orthogonal complement of the tangent space to the T-orbit through (x,t)) to be the graph of $-\lambda_x : T_xN \to \mathfrak{t}$ and requiring:

- $g_\lambda(U,V) = \langle U, V\rangle$ for $U, V \in \mathfrak{t}$,
- $\mathrm{Hor}_{(x,t)} \perp \mathfrak{t}$, and
- $\big(\mathrm{Hor}_{(x,t)}, \langle\,\cdot\,,\,\cdot\,\rangle_g\big) \to (T_xN, \langle\,\cdot\,,\,\cdot\,\rangle_h)$ is an isometry.

Remark 4.2. The pull-back of λ to $N \times T$ by the projection $N \times T \to N$ is a T-equivariant 1-form $\tilde\lambda$ that vanishes on vectors tangent to the T-orbits. Letting g denote the product metric on $N \times T$ defined by the metric h on N and the given flat metric on T, then the metric g_λ defined above may be expressed as $g_\lambda(X,Y) = g(X + \tilde\lambda(X), Y + \tilde\lambda(Y))$ for $X, Y \in T(N \times T)$. (This is one reason for the minus sign in the definition of $\mathrm{Hor}_{(x,t)}$. Also, the connection form on the principal T-bundle $N \times T$ which has the same horizontal distribution as g_λ is given by $(X, Z) \mapsto \lambda(X) + Z$ for all $X \in T_xN$, $Z \in \mathfrak{t}$.)

Suppose now that g_{λ_1} and g_{λ_2} are two such metrics and that there is an isometry F of N with the property that $\lambda_2 = F^*\lambda_1$. Then the map $(F, Id) : (N \times T, g_{\lambda_1}) \to (N \times T, g_{\lambda_2})$ is easily seen to be an isometry.

By the very construction of g_λ we have $g_\lambda^T = h$ for all λ. Let $W \subset T$ be a subtorus of codimension one. Then $T/W \cong S^1$ is a 1-dimensional torus with Lie algebra $\mathfrak{t}/\mathfrak{w}$, and $((N \times T)/W, g_\lambda^W)$ is isometric to $(N \times (T/W), g_{\lambda^W})$, where λ^W is the $\mathfrak{t}/\mathfrak{w}$-valued 1-form on N given by λ, followed by the canonical projection. Here, $\mathfrak{w}$ denotes the Lie algebra of W. We may view λ^W as a real-valued 1-form on N by choosing $\mu \in \mathfrak{t}^*$ with $\ker\mu = \mathfrak{w}$ and identifying λ^W with $\mu \circ \lambda$.

Now let λ' be a second $\mathfrak{t}$-valued 1-form on N. By our observations above, the two submersion quotients $((N\times T)/W, g^W_\lambda)$ and $((N\times T)/W, g^W_{\lambda'})$ will be isometric if the two $\mathfrak{t}/\mathfrak{w}$-valued 1-forms λ^W and ${\lambda'}^W$ are intertwined by an isometry of (N,h). We have thus proved:

THEOREM 4.3. [**53**] *Let λ, λ' be $\mathfrak{t}$-valued 1-forms on N. Assume that for each $\mu\in\mathfrak{t}^*$ there exists $F_\mu\in\operatorname{Isom}(N,h)$ such that $\mu\circ\lambda = F^*_\mu(\mu\circ\lambda')$. Then $(N\times T, g_\lambda)$ and $(N\times T, g_{\lambda'})$ are isospectral.*

In all the applications thus far of Theorem 4.3, mild genericity conditions on the choices of λ have sufficed to guarantee that the metrics g_λ and $g_{\lambda'}$ are not isometric, provided of course that $\lambda'\neq F^*\lambda$ for any isometry F of (N,h).

The dependence of the isometry F_μ on μ is crucial here: If there were an isometry F of (N,h) satisfying the condition above for all μ, then λ and λ' themselves would be intertwined by F and hence give rise to isometric metrics g_λ and $g_{\lambda'}$.

There are several generalizations of this theorem. In its most general version (see Theorem 4.6 below), the torus action is not even required to be free anymore. However, already Theorem 4.3 has many nice applications. In most of them, the key to constructing suitable 1-forms λ,λ' are pairs or families of so-called isospectral j-maps, defined as follows.

Let H be a compact connected semisimple Lie group with Lie algebra $\mathfrak{h}$, and let the Lie algebra $\mathfrak{t}$ of T be endowed with a fixed euclidean inner product.

DEFINITION 4.1.

(i) Two linear maps $j,j':\mathfrak{t}\to\mathfrak{h}$ are called *isospectral* if for each $Z\in\mathfrak{t}$ there is $a_Z\in H$ such that $j'_Z = \operatorname{Ad}_{a_Z}(j_Z)$.
(ii) j and j' are called *equivalent* if there is $\Phi\in\operatorname{Aut}(\mathfrak{h})$ and $C\in\mathrm{O}(\mathfrak{t})$ such that $j'_Z=\Phi(j_{C(Z)})$ for all $Z\in\mathfrak{t}$.

REMARK 4.4. Let $\mathfrak{t}=\mathbb{R}^2$, equipped with the standard metric, and denote by $\mathcal{J}$ the vector space of all linear maps from $\mathfrak{t}$ to $\mathfrak{h}$.

(i) [**36**] If $\mathfrak{h}=\mathfrak{so}(m)$, where m is any positive integer other than $1,2,3,4$, or 6, then there is a Zariski open subset $\mathcal{O}$ of $\mathcal{J}$ such that each $j\in\mathcal{O}$ belongs to a d-parameter family of isospectral, inequivalent elements of $\mathcal{J}$. Here $d\geq m(m-1)/2-[m/2]([m/2]+2)>1$. For $m=6$, there exist at least 1-parameter families in $\mathcal{J}$ with these properties.
(ii) [**53**] If $\mathfrak{h}=\mathfrak{su}(m)$, where $m\geq 3$, then there is a Zariski open subset $\mathcal{O}$ of $\mathcal{J}$ such that each $j\in\mathcal{O}$ belongs to a continuous family of isospectral, inequivalent elements of $\mathcal{J}$.
(iii) [**48**] Recently, Emily Proctor established results analogous to those in (i) (multiparameter families) for $\mathfrak{su}(m\geq 5)$ and $\mathfrak{sp}(m\geq 2)$.

EXAMPLE 4.1. [**29**] Let $(N,h):=S^{m-1\geq 4}$, endowed with the standard metric. Let T be two-dimensional. For each linear map $j:\mathfrak{t}\to\mathfrak{so}(m)$ define a T-valued 1-form λ^j on S^{m-1} by

$$\langle\lambda^j_q(X),Z\rangle := \langle j_Z q, X\rangle$$

for all $X\in T_qS^{m-1}$. If two such maps $j,j':\mathfrak{t}\to\mathfrak{so}(m)$ are isospectral, then the associated forms $\lambda^j,\lambda^{j'}$ satisfy the condition of Theorem 4.3. In fact, if $\mu\in\mathfrak{t}^*$ and $Z\in\mathfrak{t}$ is the dual vector with respect to the inner product on $\mathfrak{t}$, and a_Z is chosen as in Definition 4.1(i), then the isometry $F_\mu := a_Z\in\mathrm{SO}(m)$ of S^{m-1}

satisfies $\mu \circ \lambda^j = F_\mu(\mu \circ \lambda^{j'})$. Theorem 4.3 thus yields pairs of isospectral metrics on $S^{m-1} \times T$.

Actually the construction of these manifolds $(S^{m-1} \times T, g_{\lambda^j})$ in [**29**] had not been done using the approach above of associating metrics to certain 1-forms λ; rather, the manifolds there occurred as submanifolds of certain two-step nilpotent Lie groups with a left invariant metric. These submanifolds, in turn, were the boundaries of certain Dirichlet- and Neumann-isospectral subdomains diffeomorphic to the product $B^m \times T$ of a ball with a torus, which had been given in [**36**]. The latter had been the first examples of *continuous* families of isospectral metrics which were not locally isometric.

The isospectral metrics constructed above on $S^{m-1} \times T$ are in general not locally isometric when j and j' are inequivalent. For example, the metrics can in general be distinguished by the maximum of the associated scalar curvature function on the manifold.

By using multiparameter families of isospectral j-maps, one obtains multiparameter families of isospectral metrics on $S^{m-1} \times T$ which, again, can be shown to be nontrivial in most cases.

Independently of [**29**], Szabo had constructed pairs of isospectral metrics on certain products of spheres (or balls) with tori [**58**]. Excitingly, these examples include a pair of manifolds where one is homogeneous and the other is not even locally so.

Example 4.2. Although in the example above the dimension of the sphere factor was required to be at least four, pairs of $\mathfrak{t} \cong \mathbb{R}^2$-valued 1-forms λ, λ' (not arising from j-maps) which satisfy the condition of Theorem 4.3 nontrivially can be found even on S^2. Using such 1-forms, the third author constructed in [**53**] pairs of isospectral metrics on $S^2 \times T$ (with T two-dimensional) which can be distinguished by the dimension of the locus of the maximal scalar curvature. No examples of locally nonisometric isospectral manifolds in dimension lower than four are known so far.

Example 4.3. [**53**] Let again $\dim(\mathfrak{t}) = 2$, let $\mathfrak{h}$ be any of the Lie algebras from Remark 4.4, and let H be a Lie group with Lie algebra $\mathfrak{h}$, endowed with a bi-invariant metric h. We will let H play the role of the manifold N in the discussion above, so that each $\mathfrak{t}$-valued 1-form λ gives rise to a Riemannian metric g_λ on $H \times T$. If we choose λ to be *left-invariant* (i.,e., λ is defined by a linear map $\mathfrak{h} \to \mathfrak{t}$), then g_λ will be a left-invariant metric on $H \times T$.

For each linear map $j : \mathfrak{t} \to \mathfrak{h}$, define a left invariant $\mathfrak{t}$-valued 1-form λ^j on H by

$$\langle \lambda^j(X), Z \rangle = \langle j_Z, X \rangle$$

for all $Z \in \mathfrak{t}$ and all $X \in \mathfrak{h}$. If $j, j' : \mathfrak{t} \to \mathfrak{h}$ are isospectral, then $\lambda^j, \lambda^{j'}$ again satisfy the conditions of Theorem 4.3: For $\mu \in \mathfrak{t}$, the isometry F_μ of (H, h) which satisfies $\mu \circ \lambda^j = F_\mu^*(\mu \circ \lambda^{j'})$ is now conjugation $I_{a_Z} = L_{a_Z} \circ R_{a_z}^{-1} \in \mathrm{Isom}(H, h)$, where again Z is the vector dual to μ, and $a_Z \in H$ is chosen as in Definition 4.1(i). Theorem 4.3 thus gives us isospectral left-invariant metrics on $H \times T$.

The metrics in this example are homogeneous and can in general be distinguished, as shown in [**53**], by the norm of the associated Ricci tensor.

REMARK 4.5. The construction of the 1-forms $\langle \lambda^j(.), Z\rangle$ on S^{m-1} in Example 4.1 can be interpreted as taking duals to the Killing vectorfields corresponding to the $j_Z \in \mathfrak{so}(m)$, induced on S^{m-1} by the action of $\mathrm{SO}(m)$. Viewed in this way, the construction immediately generalizes to any other base manifold (N, h) admitting an effective isometric H-action, where H is a Lie group whose Lie algebra is one of those from Remark 4.4. Then one canonically obtains isospectral metrics of the type g_{λ^j} on $N \times T$, using pairs or families of isospectral j-maps from $\mathfrak{t}$ to $\mathfrak{h}$.

As an illustration, if we endow $N := H$ with a bi-invariant metric and consider the *left* action of H on itself, then we obtain *right* invariant isospectral metrics g_{λ^j} on $H \times T$. The left invariant isospectral metrics g_{λ^j} from Example 4.3 correspond, in the same sense, to the right action of H on itself (up to the sign of the 1-forms λ^j).

The two cases above, namely, that N is chosen to be either S^{m-1} (in case $H = \mathrm{SO}(m)$) or to be H itself, are in a sense extreme; see [**55**] for a discussion of the case of other homogeneous spaces $N = H/K$.

Below we will, as promised, present the current state of the art—as formulated in Theorem 4.6—concerning the method of torus actions. There had been several intermediate steps:

- The case of nontrivial T-bundles with totally geodesic fibers: [**26**] (certain two-step nilmanifolds, constituting the first examples of locally nonisometric isospectral manifolds), [**52**] (the first examples of isospectral metrics on simply connected manifolds; namely, certain products of spheres); [**53**] (including continuous isospectral families of left invariant metrics on irreducible compact Lie groups);
- the case without the assumption of totally geodesic fibers, but still with a free T-action: [**33**] (including examples of continuous isospectral families of negatively curved manifolds with boundary), [**53**] (including the first examples of pairs of conformally equivalent, locally nonisometric manifolds);
- general T-actions, but still for compact manifolds [**28**] (the first examples of continuous families of isospectral metrics on spheres and balls; namely, continuous families on $B^{n\geq 9}$ and $S^{n-1\geq 8}$), [**54**] (continuous families of isospectral metrics on B^8 and S^7, and pairs on B^6 and S^5); [**32**] (the first examples of isospectral potentials and conformally equivalent isospectral metrics on simply connected manifolds);
- finally, general T-actions on noncompact manifolds [**31**] (isophasal scattering metrics which are compact perturbations of the euclidean metric on $\mathbb{R}^n$), [**47**] (isophasal scattering metrics which are compact perturbations of any rotational metric on $\mathbb{R}^n$).

We remark that Z. Szabo [**59**] constructed the first examples of pairs of isospectral pairs of metrics on balls and spheres using a different technique involving explicit computations. His construction slightly preceded the construction cited above of continuous families of isospectral metrics on balls and spheres.

Before presenting the method we will use for constructing isophasal metrics, we review basic properties of group actions, in particular, torus actions. Given an action of a compact Lie group G on a manifold M, the *principal orbits* are the orbits with minimal isotropy. The union of the principal orbits is an open dense subset

$\widehat{M}$ of M. There exists a subgroup H of G such the isotropy group of every element of $\widehat{M}$ is conjugate to H. Moreover, the isotropy group of an arbitrary element of M contains a subgroup conjugate to H. In case G is a torus, it follows that the isotropy group of every element contains H itself. In particular, if a torus action is effective, then H is trivial and so the action on the principal orbits is free. Thus $\widehat{M}$ is a principal G-bundle.

THEOREM 4.6. *Let T be a torus which acts effectively on two complete Riemannian manifolds (M, g) and (M', g') by isometries. For each subtorus $W \subset T$ of codimension one, suppose that there exists a T-equivariant diffeomorphism $F_W : M \to M'$ which satisfies $F_W^* \,\mathrm{dvol}_{g'} = \mathrm{dvol}_g$ and induces an isometry $\bar{F}_W$ between the quotient manifolds $(\widehat{M}/W, g^W)$ and $(\widehat{M'}/W, g'^W)$, where $\widehat{M}$ (resp. $\widehat{M'}$) denote the union of the principal orbits in M (resp. M').*

(i) *Suppose (M, g) and (M', g') are compact. Then (M, g) and (M', g') are isospectral; if the manifolds have boundary then they are Dirichlet and Neumann isospectral.*
(ii) *Suppose $M = M'$, M is noncompact, and that g and g' are compact perturbations of a complete T-invariant Riemannian metric g_0 on M with* $\mathrm{dvol}_{g_0} = \mathrm{dvol}_g = \mathrm{dvol}_{g'}$. *Furthermore, assume that the maps F_W can be chosen such that they commute with Δ_{g_0}. Then (Δ_g, Δ_{g_0}) and $(\Delta_{g'}, \Delta_{g_0})$ have the same scattering phase.*

Part (i) of this theorem was first formulated in a slightly different version by the first author in [**28**]. For the proof of (i) (in the version above) and (ii) see [**54**] and [**47**], respectively. In each case, the heart of the proof consists in showing that there exists an L^2-norm preserving isometry from $\mathcal{H}^1(M, g')$ to $\mathcal{H}^1(M, g)$, where $\mathcal{H}^1$ actually means either $\mathcal{H}^1$ (closed case) or $\mathcal{H}^1_D$ resp. $\mathcal{H}^1_N$ (case with boundary) in (i), and $\mathcal{H}^1_0$ in (ii), as explained in section 2.

To construct this isometry, we decompose $\mathcal{H} := \mathcal{H}^1(M, g)$ and $\mathcal{H}' := \mathcal{H}^1(M', g')$ using Fourier decomposition with respect to the T-action and obtain

$$\mathcal{H} = \mathcal{H}_T \oplus \bigoplus\nolimits_W (\mathcal{H}_W \ominus \mathcal{H}_T), \qquad \mathcal{H}' = \mathcal{H}'_T \oplus \bigoplus\nolimits_W (\mathcal{H}'_W \ominus \mathcal{H}'_T),$$

where the sum runs over all subtori $W \subset T$ of codimension one, and $\mathcal{H}_W$ denotes the subspace of W-invariant functions (similarly for T and for $\mathcal{H}'$). Therefore, it suffices to find an L^2-norm preserving isometry from $\mathcal{H}'_W$ to $\mathcal{H}_W$ for each W. It turns out that the pullbacks F_W^* by the maps F_W chosen as in the assumptions do map $\mathcal{H}'_W$ to $\mathcal{H}_W$ isometrically. Preservation of L^2-norms is trivial here by the assumption $F_W^*(\mathrm{dvol}_{g'}) = \mathrm{dvol}_g$; preservation of $\mathcal{H}^1$-norms then follows from preservation of norms of gradients (of smooth W-invariant functions), which in turn follows from the assumption that F_W induces an isometry from $(M/W, g^W)$ to $(M'/W, g'^W)$.

There is a useful specialization of Theorem 4.6 in which $M = M'$, and g, g' arise from g_0 by changing the horizontal distribution on $\widehat{M}$ using a pair λ, λ' of $\mathfrak{t}$-valued 1-forms on M; this specialization is actually a generalization of Theorem 4.3. (See Remark 4.2 to clarify the relationship between Theorem 4.3 and the construction below.)

Begin with a complete Riemannian manifold (M, g_0) on which a torus T acts by isometries. For $Z \in \mathfrak{t}$, the action of T on M gives rise to a vector field Z^* on M given by $Z^*_p = \frac{d}{dt}\big|_{t=0} e^{tZ} \cdot p$. Given a $\mathfrak{t}$-valued 1-form λ on M which is T-invariant and vanishes on vectors tangent to the T-orbits, we define a new metric g_λ on M by

$$g_\lambda(X,Y) := g_0(X + \lambda(X)^*, Y + \lambda(Y)^*)$$

for all $X, Y \in T_pM$, $p \in M$. Note that by the presupposed properties of λ, the metric g_λ is again a T-invariant Riemannian metric, coincides with g_0 on vectors tangent to the T-orbits, and satisfies $g_0^T = g_\lambda^T$. The metrics g_0 and g_λ differ only by the associated horizontal distributions on $\widehat{M}$, which are related by

$$\mathrm{Hor}_p(g_\lambda) = \{X - \lambda(X)^* \mid X \in \mathrm{Hor}_p(g_0)\}$$

where $\mathrm{Hor}_p(g)$ denotes the g-orthogonal complement of $\mathfrak{t}$ in $T_p(M)$. In particular, the volume elements of g_0 and g coincide. In this situation we have:

THEOREM 4.7. [**47**], [**54**] *Let λ, λ' be T-invariant, $\mathfrak{t}$-valued 1-forms on M which vanish on vectors tangent to the T-orbits. Assume that for each $\mu \in \mathfrak{t}^*$ there exists a T-equivariant isometry F_μ of (M, g_0) such that $\mu \circ \lambda = F_\mu^*(\mu \circ \lambda')$. Moreover, assume λ, λ' to be compactly supported* (*in case M is noncompact*). *Then (M, g_λ) and $(M, g_{\lambda'})$ satisfy the conditions of Theorem* 4.6 (*with respect to g_0 in part* (ii)).

A typical situation in which Theorem 4.7 can be applied occurs when (M, g_0) admits an effective isometric $H \times T$-action, where H is a compact Lie group whose Lie algebra $\mathfrak{h}$ is one of the Lie algebras from Remark 4.4, and T is two-dimensional. Again one considers pairs or families of isospectral maps $j : \mathfrak{t} \to \mathfrak{h}$, and for each j defines a $\mathfrak{t}$-valued 1-form λ^j on M by letting $\langle \lambda^j(.), Z \rangle$ be dual to the Killing field corresponding to $j_Z \in \mathfrak{h}$ on M, induced by the action of H (compare Remark 4.5). Here, $\langle\, , \rangle$ denotes some fixed auxiliary scalar product on $\mathfrak{t}$ (not to be confused with the metrics on any of the T-orbits, which now anyway are, in general, no longer isometric to each other).

Since the actions of H and T commute, it is clear that these λ^j will be T-invariant. Also, if j and j' are isospectral, then for each $Z \in \mathfrak{t}$ the 1-form $\langle \lambda^j(.), Z \rangle$ is the pullback of $\langle \lambda^{j'}(.), Z \rangle$ by the element $a_Z \in H$ from the isospectrality condition on j, j'; so these $a_Z \in H$ can serve as the T-equivariant g_0-isometries F_μ required in the assumption of Theorem 4.7. There is one difficulty, namely, the condition that the λ^j vanish on vectors tangent to the T-orbits. This can be ensured by assuming the action of $H \times T$ on M to be such that T-orbits and H-orbits meet g_0-orthogonally in every point. Even if this is not the case, one can achieve this condition by modifying the λ^j a bit: Namely, by first multiplying them with the squared norm of the volume form of the T-orbits, and then projecting the form thus obtained to its horizontal part (which will then still be smooth); see [**55**].

In the case that M is noncompact, the 1-forms constructed above will have noncompact support. This, however, can easily be mended using an idea from [**54**]: If one already has *some* pair of $\mathfrak{t}$-valued 1-forms λ, λ' on M which satisfy the conditions of the theorem, then so do $\psi\lambda, \psi\lambda'$, where ψ is any smooth function on M which is invariant under T and under all the F_μ from the assumption (more precisely, with the property that the F_μ can be *chosen* such that they preserve ψ). In our case, where the F_μ are actually elements of the compact group H, we can choose ψ to be any nontrivial smooth $H \times T$-invariant function on M; e.g., with support in an invariant neighborhood of any of the $H \times T$-orbits. This idea can be used in the compact case, too: It shows that the 1-forms can be chosen to have support in arbitrary small subsets of (M, g_0), outside which the associated metrics will be equal to g_0.

In the following examples, the isospectral (resp. isophasal) metrics either turned out to be, or were constructed to be, of the type just described; that is, they are associated with $\mathfrak{t}$-valued 1-forms on M which are of the above form λ^j—possibly modified as mentioned to ensure horizontality and/or compact support. Here we only list examples with a *nonfree* T-action:

- [**28**] Continuous multiparameter families of isospectral metrics on $S^{m+3\geq 8}$ and $B^{m+4\geq 9}$, associated with the standard action of $\mathrm{SO}(m)\times T \subset \mathrm{SO}(m)\times \mathrm{SO}(4) \subset \mathrm{SO}(m+4)$ on $\mathbb{R}^{m+4}$;
- [**54**] Continuous families of isospectral metrics on S^7 and B^8 associated with the action of $\mathrm{SU}(3) \times T \subset \mathrm{U}(3) \times \mathrm{U}(1) \subset \mathrm{U}(4)$ on $\mathbb{C}^4 \cong \mathbb{R}^8$;
- [**31**] Continuous multiparameter families of isophasal scattering metrics on $\mathbb{R}^{m+4\geq 9}$ which are compact perturbations of the Euclidean metric;
- [**47**] Continuous multiparameter families of isophasal scattering metrics on $\mathbb{R}^{m+4\geq 9}$ which are compact perturbations of *any* rotational (that is, $\mathrm{O}(m+4)$-invariant) metric on $\mathbb{R}^{m+4}$; for example, compact perturbations of the hyperbolic metric.

The same $\mathrm{SO}(m) \times T$-action on $\mathbb{R}^{m+4}$ as in the first item is used in the third and fourth one. Using the $\mathrm{SU}(3) \times T$-action from the second item, one also obtains isophasal scattering metrics on $\mathbb{R}^8$.

Finally, we mention that using certain suitable pairs of $\mathfrak{t} \cong \mathbb{R}^2$-valued 1-forms λ, λ' related to those used in Example 4.2, it is possible to apply Theorem 4.7 to obtain pairs of isospectral, resp. isophasal, metrics also on $S^2 \times S^3$ [**2**], S^5, B^6 [**54**] and $\mathbb{R}^6$.

5. Summary and Open Problems

5.1. Structure of isospectral or isopolar sets of metrics. We list below, in various contexts, the lowest dimensions in which examples of isospectral or isopolar manifolds are known.

- Pairs of isospectral manifolds with different global, though the same local, geometry: dimension 2 [**63**]. (As discussed in Section 3, there are large families of isospectral Riemann surfaces constructed by Sunada's technique [**10**].)
- Continuous isospectral deformations with different global, though the same local, geometry: dimension 5 [**35**].
- Pairs of isospectral metrics with different local geometry: dimension 4. (Metrics on $S^2 \times T^2$ [**53**]; see Section 4.)
- Continuous isospectral deformations with different local geometry: dimension 6. (Metrics on $S^4 \times T^2$ [**29**]; see Section 4.)
- Pairs of isospectral simply-connected manifolds: dimension 5. (Metrics on $S^2 \times S^3$ [**2**] and on S^5 [**54**]; see Section 4.)
- Continuous isospectral deformations of metrics on a simply-connected manifold: dimension 7. (Metrics on S^7 [**54**], see section 4.)
- Pairs of isophasal manifolds with same global, different local geometry: dimension 2. (Riemann surfaces [**8**], [**38**]; see Section 3.)
- Pairs of isophasal metrics with different local geometry: dimension 6. (Compactly supported perturbations of the Euclidean metric on $\mathbf{R}^6$; see Section 4.)

- Continuous isophasal deformations of metrics with different local geometry: dimension 8. (Compactly supported perturbations of the Euclidean metric on $\mathbf{R}^8$; see Section 4.)

PROBLEM 5.1. *In all cases above for which the lowest known dimension is greater than two, it remains open whether lower-dimensional examples exist. What is the critical dimension in each case?*

Given a closed Riemannian manifold (M, g), the set of all Riemannian metrics on M isospectral to g will be called the *isospectral set* of g. A more general problem than the one above is to describe the structure of the isospectral set of a metric. One can define a $\mathcal{C}^\infty$ topology on the set of all isometry classes of metrics on M.

PROBLEM 5.2. *Is the isospectral set of a metric always compact in the $\mathcal{C}^\infty$ topology?*

In dimension 2, the answer is yes, as proven by Osgood, Phillips, and Sarnak [**46**]. In higher dimensions, the problem remains open, although a number of partial results are known (in conformal classes on 3-manifolds [**16**], [**21**], [**22**]; on negatively curved 3-manifolds [**1**], [**13**]; for 3-manifolds with metrics in a spectrally determined neighborhood of a constant curvature metric [**15**]).

Negatively curved metrics on closed manifolds are spectrally rigid; i.e., they do not admit nontrivial continuous isospectral deformations; this fact was proven by Guillemin and Kazhdan [**37**] in dimension 2 and by Croke and Sharafutdinov [**24**], [**25**] in arbitrary dimension. In two dimensions, this local rigidity along with the compactness result cited above leads one to ask the following question:

PROBLEM 5.3. *Is the isospectral set of a negatively curved metric on a surface always finite?*

For Riemann surfaces with the hyperbolic metric, McKean proved that isospectral sets are always finite and P. Buser [**19**] obtained an explicit, though huge, upper bound depending only on the genus. However, the problem remains open for surfaces of variable negative curvature.

5.2. Geometry of the Spectrum. The various examples of isospectral manifolds reveal a number of geometric and topological properties that are not spectrally determined. In the following list, we indicate the first example which revealed that the specified property is not spectrally determined. (This list is not complete.)

(a) Fundamental group [**62**].
(b) Diameter [**18**].
(c) Maximal scalar curvature [**29**].
(d) $\int_M \mathrm{Scal}_g^2\, dv_g$, $\int_M \|\mathrm{Ric}_g\|^2 dv_g$, and $\int_M \|R_g\|^2 dv_g$. [**52**], [**53**]. (Note that a specific linear combination of these three integrals gives the second heat invariant, a spectrally determined quantity.)
(e) Whether a closed manifold is homogeneous and whether it is locally homogeneous [**58**], [**59**].
(f) Whether a closed manifold has constant scalar curvature; whether a manifold with boundary is Einstein and whether its curvature tensor is parallel [**33**].

Of course there is still a large gap between the known spectral invariants and the geometric invariants that the counterexamples tell us are not spectrally determined. Some questions suggested by the list above are:

Problem 5.4. *The heat invariants are integrals of (in general very complicated) polynomial expressions in the curvature and its covariants. Are the heat invariants the only spectral invariants of that form?* (*See* (d) *above.*)

Problem 5.5. *Does the spectrum of a closed Riemannian manifold determine whether the manifold is Einstein and whether it is locally symmetric?* (*See* (f) *above.*)

5.3. Obstacle Scattering. The most interesting open problem in inverse scattering involves obstacle scattering. Let $\mathcal{O}$ be a compact connected subset of $\mathbb{R}^n$ with smooth boundary, let H_0 be the Euclidean Laplacian on $\mathbb{R}^n$, let $\Omega = \mathbb{R}^n \setminus \mathcal{O}$, and let H be the Laplacian on Ω with Dirichlet boundary conditions on $\partial\mathcal{O}$. Although the pair (H, H_0) act on different spaces, one can define relative scattering theory and a spectral shift for the pair (H, H_0). Given the scattering phase for an obstacle $\mathcal{O}$, its *isophasal set* is the set of all obstacles in $\mathbb{R}^n$ modulo isometries of $\mathbb{R}^n$ which have the same scattering phase. Hassell and Zelditch [**39**] showed that the isophasal set of a given obstacle in $\mathbb{R}^2$ is compact in a suitable topology on obstacles. Hassell and Zworski [**40**] showed that the sphere S^2 is uniquely determined by its scattering poles (when viewed as the boundary of an obstacle $\mathcal{O} \subset \mathbb{R}^3$).

Problem 5.6. *Does the isophasal set of an obstacle contain more than one element?*

Problem 5.7. *What is the 'critical dimension' for polar and phasal rigidity for obstacle scattering?*

References

1. M. T. Anderson, *Remarks on the compactness of isospectral sets in low dimensions*, Duke Math. J. **63** (1991), no. 3, 699–711.
2. W. Ballmann, *On the construction of isospectral manifolds*, preprint (2001).
3. P. Bérard, *Transplantation et isospectralité.* I, Math. Ann. **292** (1992), no. 3, 547–559.
4. ———, *Transplantation et isospectralité.* II, J. London Math. Soc. (2) **48** (1993), no. 3, 565–576.
5. P. Bérard and D. L. Webb, *One cannot hear the orientability of surfaces*, C. R. Acad. Sci. Paris **320**, no. 1 (1995), 533–536.
6. R. Brooks, *Constructing isospectral manifolds*, Amer. Math. Monthly **95** (1988), no. 9, 823–839.
7. ———, *The Sunada method*, Tel Aviv Topology Conference: Rothenberg Festschrift (1998), Contemp. Math., **231**, Amer. Math. Soc., Providence, RI, 1999, pp. 25–35.
8. R. Brooks and O. Davidovich, *Isoscattering on surfaces*, J. Geom. Anal. **13** (2003), no. 1, 39–53.
9. R. Brooks and C. Gordon, *Isospectral families of conformally equivalent Riemannian metrics*, Bull. Amer. Math. Soc. (N.S.) **23** (1990), no. 2, 433–436.
10. R. Brooks, R. Gornet, and W. H. Gustafson, *Mutually isospectral Riemann surfaces*, Adv. Math. **138** (1998), no. 2, 306–322.
11. R. Brooks, R. Gornet, and P. Perry, *Isoscattering Schottky manifolds*, Geom. Funct. Anal. **10** (2000), no. 2, 307–326.
12. R. Brooks and P. A. Perry, *Isophasal scattering manifolds in two dimensions*, Comm. Math. Phys. **223** (2001), no. 3, 465–474.
13. R. Brooks, Peter A. Perry, and P. Petersen, *Compactness and finiteness theorems for isospectral manifolds*, J. Reine Angew. Math. **426** (1992), 67–89.
14. ———, *Finiteness of diffeomorphism types of isospectral manifolds*, Differential Geometry: Riemannian Geometry (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., **54**, Part 3, Amer. Math. Soc., Providence, RI, 1993, pp. 89–94.
15. ———, *Spectral geometry in dimension* 3, Acta Math. **173** (1994), no. 2, 283–305.

16. R. Brooks, P. Perry, and P. Yang, *Isospectral sets of conformally equivalent metrics*, Duke Math. J. **58** (1989), no. 1, 131–150.
17. R. Brooks and R. Tse, *Isospectral surfaces of small genus*, Nagoya Math. J. **107** (1987), 13–24.
18. P. Buser, *Isospectral Riemann surfaces*, Ann. Inst. Fourier **36** (1986), 167–192.
19. ———, *Geometry and Spectra of Compact Riemann Surfaces*, Birkhäuser, Boston 1992.
20. G. Carron, *Déterminant relatif et la fonction Xi.* Amer. J. Math. **124** (2002), no. 2, 307–352.
21. S.-Y. A. Chang and P. C. Yang, *Compactness of isospectral conformal metrics on S^3*, Comment. Math. Helv. **64** (1989), no. 3, 363–374.
22. ———, *Isospectral conformal metrics on* 3*-manifolds* J. Amer. Math. Soc. **3** (1990), no. 1, 117–145.
23. T. Christiansen and M. Zworski, *Resonance wave expansions: two hyperbolic examples*, Comm. Math. Phys. **212** (2000), no. 2, 323–336.
24. C. Croke and V. A. Sharafutdinov, *Spectral rigidity of a compact negatively curved manifold*, Topology **37** (1998), no. 6, 1265–1273.
25. ———, *Spectral rigidity of a compact negatively curved manifold*, Topology **37** (1998), no. 6, 1265–1273.
26. C. S. Gordon, *Isospectral closed Riemannian manifolds which are not locally isometric.* II, Geometry of the Spectrum (Seattle, WA, 1993), Contemp. Math., **173**, Amer. Math. Soc., Providence, RI, 1994, pp. 121–131.
27. ———, *Survey of isospectral manifolds*, Handbook of Differential Geometry, Vol. I, North-Holland, Amsterdam, 2000, pp. 747–778.
28. ———, *Isospectral deformations of metrics on spheres*, Invent. Math. **145** (2001), no. 2, 317–331.
29. C. S. Gordon, R. Gornet, D. Schueth, D. L. Webb, and E. N. Wilson, *Isospectral deformations of closed Riemannian manifolds with different scalar curvature*, Ann. Inst. Fourier (Grenoble) **48** (1998), no. 2, 593–607.
30. C. S. Gordon, E. Makover, and D. L. Webb, *Transplantation and Jacobians of isospectral Riemann surfaces*, Preprint (2003).
31. C. S. Gordon and P. A. Perry, *Continuous families of isophasal scattering manifolds*, submitted to Duke Math. J.
32. C. S. Gordon and D. Schueth, *Isospectral potentials and conformally equivalent isospectral metrics on spheres, balls and Lie groups*, J. Geom. Anal. **13** (2003), no. 2, 279–306.
33. C. S. Gordon and Z. I. Szabo, *Isospectral deformations of negatively curved Riemannian manifolds with boundary which are not locally isometric*, Duke Math. J. **113** (2002), no. 2, 355–383.
34. C. S. Gordon, D. L. Webb, and S. Wolpert, *Isospectral plane domains and surfaces via Riemannian orbifolds*, Invent. Math. **110** (1992), no. 1, 1–22.
35. C. S. Gordon and E. N. Wilson, *Isospectral deformations of compact solvmanifolds*, J. Differential. Geom. **19** (1984), 241–256.
36. ———, *Continuous families of isospectral Riemannian metrics which are not locally isometric*, J. Differential Geom. **47** (1997), no. 3, 504–529.
37. V. Guillemin and D. Kazhdan, *Some inverse spectral results for negatively curved* 2*-manifolds*, Topology **19** (1980), no. 3, 301–312.
38. L. Guillopé and M. Zworski, *Scattering asymptotics for Riemann surfaces*, Ann. Math. **145** (1997), 597–660.
39. A. Hassell and S. Zelditch, *Determinants of Laplacians in exterior domains*, Internat. Math. Res. Notices **1999**, no. 18, 971–1004.
40. A. Hassell and M. Zworski, *Resonant rigidity of S^2*, J. Funct. Anal. **169** (1999), no. 2, 604–609.
41. M. Joshi and A. Sá Barreto, *Inverse scattering on asymptotically hyperbolic manifolds*, Acta Math. **184** (2000), 41–86.
42. P. D. Lax and R. S. Phillips, *Scattering theory*, second edition. With appendices by C. S. Morawetz and G. Schmidt. Pure and Appl. Math., **26**. Academic Press, Inc., Boston, MA, 1989.
43. R. Miatello and J. P. Rossetti, *Flat manifolds isospectral on p-forms*, J. Geom. Anal. **11** (2001), 649–667.
44. ———, *Comparison of twisted p-form spectra for flat manifolds with diagonal holonomy*, Ann. Global Anal. Geom. **21** (2002), no. 4, 341–376.

45. ______, *Length spectra and p-spectra of compact flat manifolds*, J. Geom. Anal. **13** (2003), no. 4, 631–657.
46. B. Osgood, R. Phillips, and P. Sarnak, *Compact isospectral sets of metrics*, J. Funct. Anal. **l80** (1988), 212-234.
47. P. A. Perry and D. Schueth, *Continuous families of isophasal scattering manifolds which are asymptotically hyperbolic*, preprint (2004).
48. E. Proctor, *Isospectral metrics on classical compact simple Lie groups*, Ph.D. thesis, Dartmouth College, 2003.
49. M. Reed and B. Simon, *Methods of Modern Mathematical Physics, I. Functional Analysis*, New York, Academic Press, 1972.
50. ______, *Methods of Modern Mathematical Physics, III. Scattering Theory*, New York, Academic Press, 1979.
51. ______, *Methods of Modern Mathematical Physics, IV. Analysis of Operators*, New York, Academic Press, 1978.
52. D. Schueth, *Continuous families of isospectral metrics on simply connected manifolds*, Ann. of Math. (2) **149** (1999), no. 1, 287–308.
53. ______, *Isospectral manifolds with different local geometries*, J. reine angew. Math. **534** (2001), 41–94.
54. ______, *Isospectral metrics on five-dimensional spheres*, J. Differential Geom. **58** (2001), no. 1, 87–111.
55. ______, *Constructing isospectral metrics via principal connections*, Geometric Analysis and Nonlinear Partial Differential Equations (S. Hildebrandt, H. Karcher eds.), Springer, 2003, pp. 69-79.
56. T. Sunada, *Riemannian coverings and isospectral manifolds*, Ann. of Math. (2) **121** (1985), no. 1, 169–186.
57. C. J. Sutton, *Isospectral simply-connected homogeneous spaces and the spectral rigidity of group actions*, Comment. Math. Helv. **77** (2002), no. 4, 701–717.
58. Z. I. Szabo, *Locally non-isometric yet super isospectral spaces*, Geom. Funct. Anal. **9** (1999), no. 1, 185–214.
59. ______, *Isospectral pairs of metrics on balls, spheres, and other manifolds with different local geometries*, Ann. of Math. (2) **154** (2001), no. 2, 437–475.
60. S.-H. Tang and M. Zworski, *Resonance expansions of scattered waves*, Comm. Pure Appl. Math. **53** (2000), no. 10, 1305–1334.
61. B. R. Vainberg, *Exterior elliptic problems that depend polynomially on the spectral parameter, and the asymptotic behavior for large values of the time of the solutions of nonstationary problems* (Russian), Mat. Sb. (N.S.) **92** (134) (1973), 224–241, 343.
62. M.-F. Vignéras, *Exemples de sous-groupes discrets non conjugués de $PSL(2, R)$ qui ont même fonction zéta de Selberg*, C. R. Acad. Sci. Paris Sér. A-B **287** (1978), no. 2, A47–A49.
63. ______, *Variétés riemanniennes isospectrales et non isométriques*, Ann. of Math. (2) **112** (1980), no. 1, 21–32.
64. D. R. Yafaev, *Mathematical Scattering Theory. General Theory*, translated from the Russian by J. R. Schulenberger. Translations of Mathematical Monographs, **105**. American Mathematical Society, Providence, RI, 1992.
65. S. Zelditch, *Isospectrality in the FIO category*, J. Differential Geom. **35** (1992), no. 3, 689–710.
66. ______, *Kuznecov sum formulae and Szegö limit formulae on manifolds*, Comm. Partial Differential Equations **17** (1992), no. 1-2, 221–260.
67. M. Zworski, *Counting scattering poles*, Spectral and scattering theory (Sanda, 1992), 301–331, Lecture Notes in Pure and Appl. Math., **161**, Dekker, New York, 1994.
68. ______, *Resonances in physics and geometry*, Notices Amer. Math. Soc. **46** (1999), no. 3, 319–328.

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