A problem in number theory

by Professor Isaac

I am interested in probability and related areas. Here is a problem I have been working on. This is a problem in number theory which I have looked at from a probability point of view.

Every real number x between 0 and 1 has a binary expansion, that is, x has the representation $x = .x_1x_2\cdots$ where the x_i are 0 or 1. This is the base 2 analog of our usual base 10 decimal expansion. This expansion is unique except for the situations where the expansion ends in all 0's or all 1's, e.g., $.1001000\cdots$ ending in all 0's has the equivalent expansion $.1000111\cdots$ ending in all 1's.

A famous theorem of number theory says that "almost all" real numbers x between 0 and 1 have the property (called simple normality to the base 2) that if you take the first n digits of the expansion of x and n is large, then approximately half of the digits will be 0 and half will be 1. This statement needs some explanation. The term "almost all" can be made precise using probability. Intuitively speaking, it means that if you randomly choose a point x of the unit interval, the probability that x is simply normal (to the base 2) is equal to 1. So the simply normal numbers are all over the place even though lots of numbers you can think of won't have this property—the expansions ending in all 0's for instance, or $x = .100100100\cdots$.

The property of simple normality is really a statement about limits. It says that the average of the first n digits of the expansion of a simply normal number converges to 1/2 as n tends to infinity. In mathematical notation this is written

$$\lim_{n\to\infty} \frac{x_1 + x_2 + \cdots + x_n}{n} = 1/2.$$

Now here is something very interesting. As I have mentioned, it is known that there are loads of these simply normal numbers around. You would therefore think that it would be very easy to identify lots of particular numbers as simply normal. But it isn't! To this date the only ones found have been artificially constructed to have the property. It is not known whether common numbers like π or $\sqrt{2}$ are simply normal (although I may have a proof of it for $\sqrt{2}$). This is a kind of paradox that arises from time to time in mathematics: it is proved that there exist a lot of objects with a specified property, but it is hard to determine whether any particular object has that property.

What is the interest in such a problem? Aside from the insight we get into the properties of numbers, identification of simply normal numbers may have applications to computers, in particular, to the generation of random numbers.

Richard Isaac