The Loop Product and Closed Geodesics

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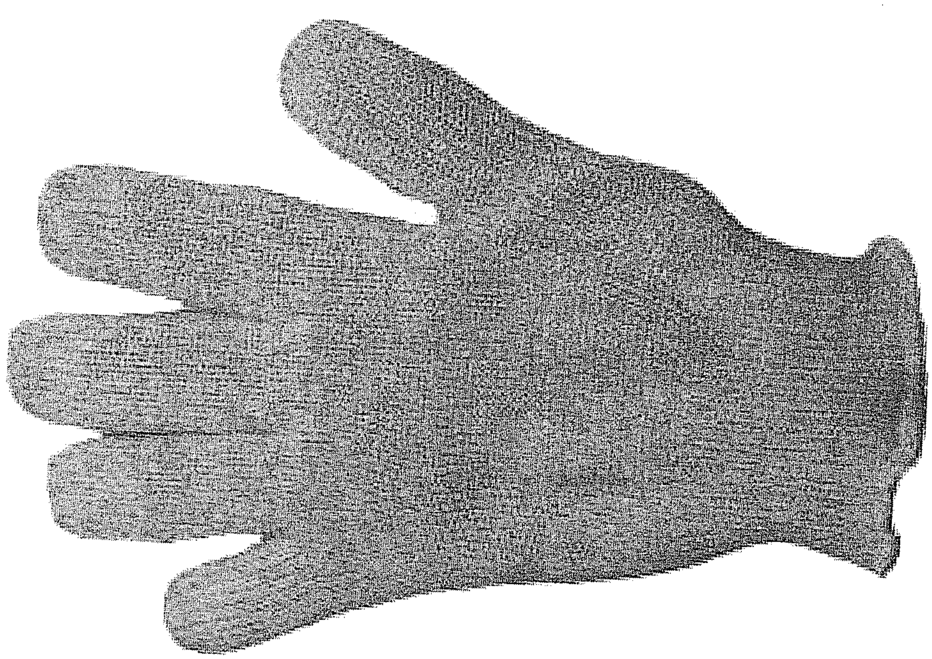


Geometric Interpretation of the Chas-Sullivan Loop Product

"Fits like a glove!"

- Closed geodesic with limiting index growth (slowest possible) is perfect local model for the Loop Product

- Local Nilpotence of Loop Product $\longleftrightarrow$ Theorem (H, 93): A homologically visible closed geodesic @ limiting index growth (slowest poss) $\Rightarrow$ $\exists$ ∞ many closed geodesics

(work in progress)

... on the other hand

<u>Theorem</u> (H, '97) (much harder)

A homologically visible closed geodesic with limiting index growth (<u>fastest</u> possible) $\Rightarrow$ $\exists$ ∞ many closed geodesics

<u>QUESTION:</u>

Where is the other glove?

Geometry suggests the existence of a loop coproduct

$$C^k \otimes C^j \longrightarrow C^{k+j+n-1}$$

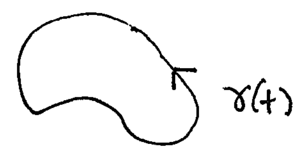
This coproduct does exist at least locally, and gives a new, easier proof of the above theorem.

Closed Geodesics
Background & Examples

M^n compact Riemannian manifold

$\Lambda = \text{Maps } \gamma : S^1 \to M$ the free loop space

$E : \Lambda \to \mathbb{R} \quad E(\gamma) = \int |\dot{\gamma}|^2 dt$



Critical points of E $\equiv$ closed geodesics on M

(Length spectrum $= \{ \sqrt{e} \mid e$ is a critical point for $E \}$)

Morse theory : Topology of $\Lambda \rightsquigarrow$ Critical points of E

Difficulties/ Beautiful Structure

1) $O(2)$ = isometries of S^1 acts on Λ

2) Iteration : If γ is a closed geod., so is $\gamma^m(t) = \gamma(mt)$

$$\Lambda \underset{\gamma \mapsto \gamma^m}{\xrightarrow{\sim}} FPS(\mathbb{Z}_m) \hookrightarrow \Lambda$$

1 honest closed geodesic $\rightsquigarrow$ infinite family of $O(2)$-orbits of critical points of E (all topology independent of the metric)

What can we prove?

Highlights:

Birkhoff (27): $\exists$ at least one closed geodesic for <u>any</u> metric on S^n

Lusternik-Fet (51): $\exists$ at least one closed geodesic for any metric on M.

Lusternik-Schnirelmann (29) + Grayson (89): $\exists$ at least 3 simple closed geodesics for any metric on S^2.

$\vdots$

not much more without

<u>Bott formula</u> for the index of the iterates (56):

completely explicit formula in terms of geometry

$$\Rightarrow \left|\operatorname{index}(\gamma^m) - m\cdot\operatorname{index}(\gamma)\right| \leq (m-1)(n-1)$$

Betti #'s of Λ unbounded $\xRightarrow[\text{(69)}]{\text{Gromoll-Meyer}}$ $\exists\ \infty$ many closed geodesics on M for any metric.

$\Big\Uparrow$ Sullivan-Vigué-Poirrier (76)

$$H^*(M,\mathbb{Q}) \neq \mathbb{Q}[x]/x^d$$

Moser (77) + H (84) + Rademacher (89) $\Rightarrow \exists\ \infty$ many for a generic metric on M.

Birkhoff (27) + Bangert (80) + Lusternik-Schnirelmann + Grayson (29, 89) + $\left\{\begin{array}{l}\text{Franks (92)} \\ \text{H (93)}\end{array}\right\} \Rightarrow$ infinitely many for any metric on S^2

Examples

- S^2, standard metric. All geodesics closed, index $2m-1$

Nondegenerate critical manifold of dim 3.

Geodesics of length $2m\pi$ contribute homology in dimension $2m-1$ thru $2m+2$

$m=1$	1	4
$m=2$	3	6
$m=3$	5	8
	⋮	⋮

- Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, $a<b<c$, $a\approx b\approx c\approx 1$

$\exists$ 3 simple closed geodesics:

α^m index 1, 3, 5, ... (ev. drops)

β^m index 2, 4, 6, ...

γ^m index 3, 5, 7, ... (ev. rises)

Morse: $\exists$ ∞ many closed geodesics, but length of 4th shortest $\to\infty$ as $a,b,c\to 1$.

- Katok examples: Nonsymmetric Finsler metric on S^n, projective space @ $\approx n$ closed geodesics.
- H + Bangert-Long (preprint) $\Rightarrow$ $\exists$ at least 2 on any Finsler S^2

How to keep track of iterates?

$$\text{Algebraic Operation} \overset{?}{\longleftrightarrow} \text{Iteration}$$

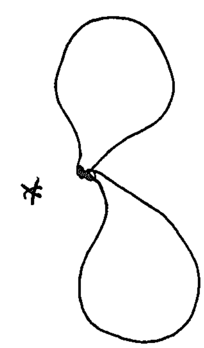
possible candidate:

Pontrjagin Product

$$C_j(\Omega) \otimes C_k(\Omega) \to C_{j+k}(\Omega)$$

need to fix a base point

$$\Omega = \{ \gamma \mid S^1 \to M \mid \gamma(0) = * \}$$



models "hyperbolic" growth rate

$$x \in H_k(\Omega) \Rightarrow x^m \in H_{mk}(\Omega)$$

<u>Thm</u> H $\begin{cases} 93 & \text{Easy} \\ 97 & \text{Much harder} \end{cases}$

M^n compact, γ a closed geodesic of length L.

If γ is homologically visible in dimension k, and if

$$\begin{cases} \text{index} + \text{nullity}\,(\gamma^m) \leq mk - (m-1)(n-1) & (\text{slowest possible}) \\ \text{index}\,(\gamma^m) \geq mk + (m-1)(n-1) & (\text{fastest possible}) \end{cases}$$

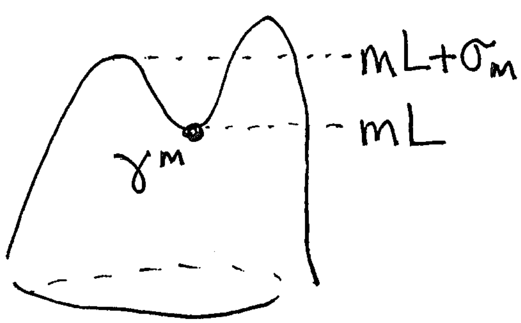
then $\exists\, m_0 \in \mathbb{Z}^+$, $\{\sigma_m\} \downarrow 0$ so that $m \geq m_0 \Rightarrow$

M has a closed geodesic with length

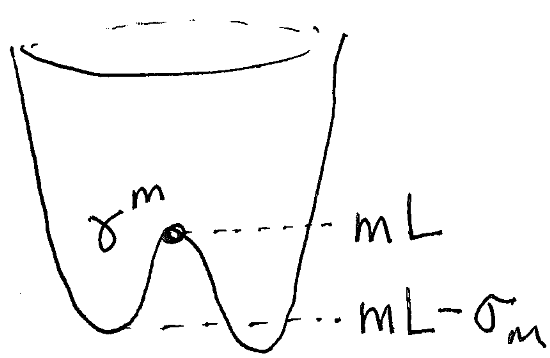
$$\ell \in \begin{cases} (mL, mL+\sigma_m) \\ (mL-\sigma_m, mL) \end{cases}.$$

This $\Rightarrow$ M has infinitely many closed geodesics.

Picture:



new critical value just <u>above</u> mL (homology)



new critical value just below mL (cohomology)

Chas-Sullivan Loop Product

(rough idea)

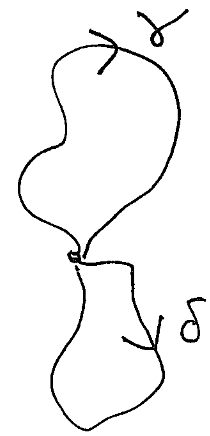
$K, \tau \in C_*(\Lambda)$ $\qquad$ $e : \Lambda \to M$

$e(\gamma) = \gamma(0)$

Supp. $K = \{\gamma\}$ $\qquad$ Assume $\{e(\gamma)\}$ and

$\tau = \{\delta\}$ $\qquad$ $\{e(\sigma)\}$ intersect $\pitchfork$.

$$K * \tau = \{\gamma \cdot \delta \mid e(\gamma) = e(\delta)\}$$



$$\boxed{C_k \otimes C_j \to C_{k+j-n}}$$

Bring in geometry. $\Lambda^a = E^{-1}[0, a^2]$

$\approx \ell^{-1}[0, a]$

Loop product: $C_k(\Lambda^b, \Lambda^a) \otimes C_j(\Lambda^d, \Lambda^c)$

$$\longrightarrow C_{k+j-n}\left(\Lambda^{b+d}, \Lambda^{\max(a+d,\, b+c)}\right)$$

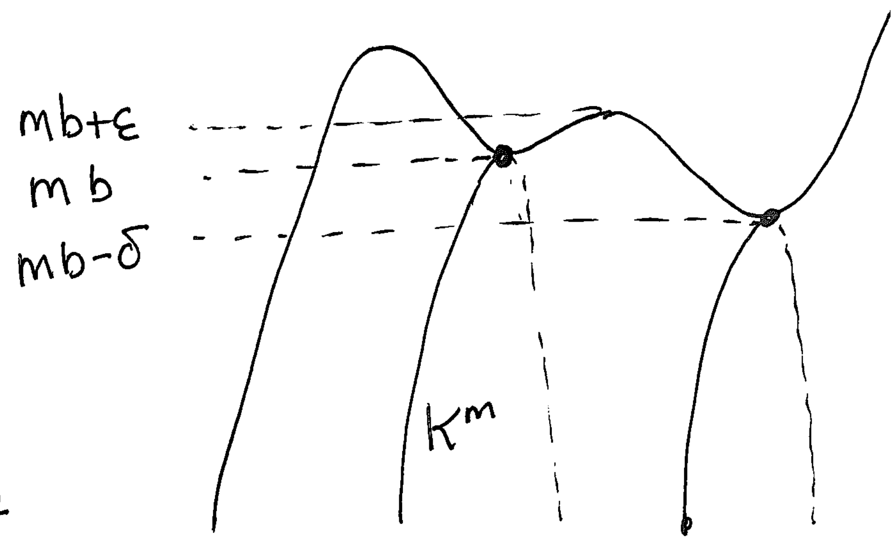
Geometry of the Loop Product

$[K] \in H_*(\Lambda)$ is <u>nilpotent</u> if $[K^m] = 0$ for some m.

$[K] \in H_*(\Lambda^b, \Lambda^a)$ is <u>locally nilpotent</u> if given $\varepsilon > 0$ $\exists$ $m \in \mathbb{Z}$ so that $[K^m] = 0$ in $H_*(\Lambda^{mb+\varepsilon}, \Lambda^{mb-\delta})$ for some $\delta > 0$.

Roughly, some power of $[K]$ can be pushed down (always true unless $[K]$ is pushed down)

(1) On S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$ or CaP^2, there is an element in $H_*(\Lambda, \Lambda^0)$ that is not nilpotent. In the standard metric this element is not locally nilpotent.



(2) In nondegenerate case, Bott $\Rightarrow$ every element is locally nilpotent.

(3) Thm (H) If all closed geodesics are isolated, every element is locally nilpotent.

Remark. If $[\kappa] \in H_k(\Lambda^b)$, then

$$[\kappa^m] \in H_{mk-(m-1)n}(\Lambda^{mb}). \qquad (*)$$

(Example: On S^2 if $[\kappa] \in H_4(\Lambda)$, then
$[\kappa^m] \in H_{2m+2}(\Lambda)$: 4, 6, 8, ...)

If $[\kappa^m] \neq 0 \ \forall m$, there must be a closed geodesic of length b with limiting index growth (slowest possible). This index growth forces the local geometry into a very rigid mold.

(*) is the only degree that can have this geometric significance.

Coproduct

Should be a map

$$H^k(\Lambda^b, \Lambda^a) \otimes H^j(\Lambda^d, \Lambda^c)$$
$$\to H^{k+j+n-1}(\Lambda^{\min(a+d,\, b+c)}, \Lambda^{a+c})$$

(1) On S^n, $\mathbb{C}P^n$, $\mathbb{H}P^n$ or CaP^2, there is an element in $H^*(\Lambda, \Lambda^0)$ that is not nilpotent.

In the standard metric, this element is not locally nilpotent.

(2) In the nondegenerate case, Bott $\Rightarrow$ every element is locally nilpotent.

(3) Theorem (H) If all closed geodesics are isolated, every element is locally nilpotent.

Claim: The coproduct exists at least locally, and has the above properties.

the length function induces a filtration on the cochain complex of Λ

$C^*(\Lambda^b, \Lambda^a)$ = Cohains vanishing on chains at length $< a$, mod cochains vanishing on chains at length $< b$.

Locally nilpotent

$\mu \in H^*(\Lambda^b, \Lambda^a)$ is locally nilpotent if given $\varepsilon > 0$ $\exists$ $m \in \mathbb{Z}$ so that $[\mu^m] = 0$ in $H^*(\Lambda^{ma+\delta}, \Lambda^{ma-\varepsilon})$ for some $\delta > 0$.

Roughly, some power of μ rises above the level ma.