Lecture CUNY February 3, 2006

A chat on simple closed geodesics

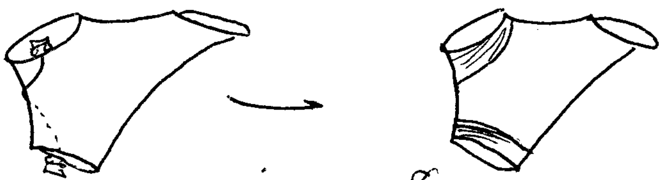
With Hugo Parlier, Klaus-Dieter Semmler

Simple closed geodesics on Riemann surfaces
$K = -1$.

Origin of the problem: Numerical computations lead to possibly bad generators. Alg. Curves $\rightarrow$ hyperbolic metric $\rightarrow$ generators $\rightarrow$ iterate

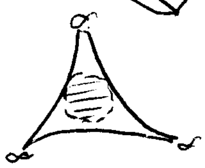
Fact complement of simple closed non intersecting geod has large discs:

Construction

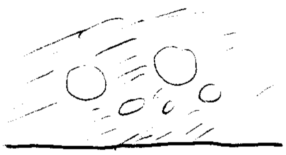


do twice

get two ideal

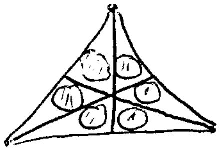


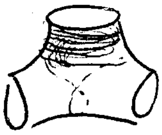
Hence in $\mathbb{H}$



can you see the surface by looking at this?

Funny question: Are there discs in S touched by no simple closed geodesics?

Example  radius $= \operatorname{arcsinh} \frac{1}{\sqrt{23}}$

Example 

Recall: closed geodesics are dense.

Theorem (Birman/Series Topology '85) On any hyperbolic surface, the simple closed geodesics are nowhere dense.

Set c_S = radius of the largest avoided disc.

Theorem There exist $c_g > 0$ such that $c_S \geq c_g$ for all $S \in \mathcal{M}_g$.

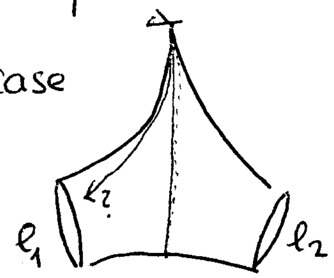
Proof. By compactness (unfortunately)

$\{S_k\} \subset \mathcal{M}_g$ with $c_{S_k} \to$ infimum

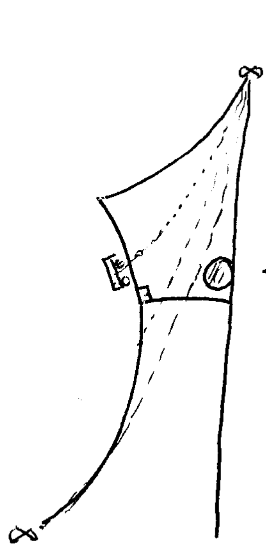
convergent in $\overline{\mathcal{M}_g}$

Case 1 systole $\to 0$.

Limit case



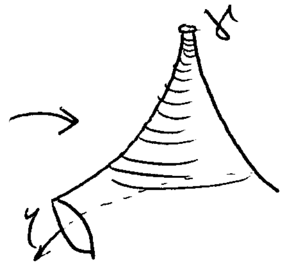
by Bers' theorem, $\ell_1, \ell_2 \leqslant L_g$



$\leftarrow$ radius $\geqslant$ some $f(L_g) > 0$

Continuity?

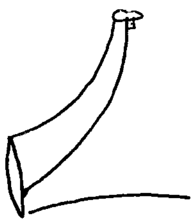
is this possible? $\to$

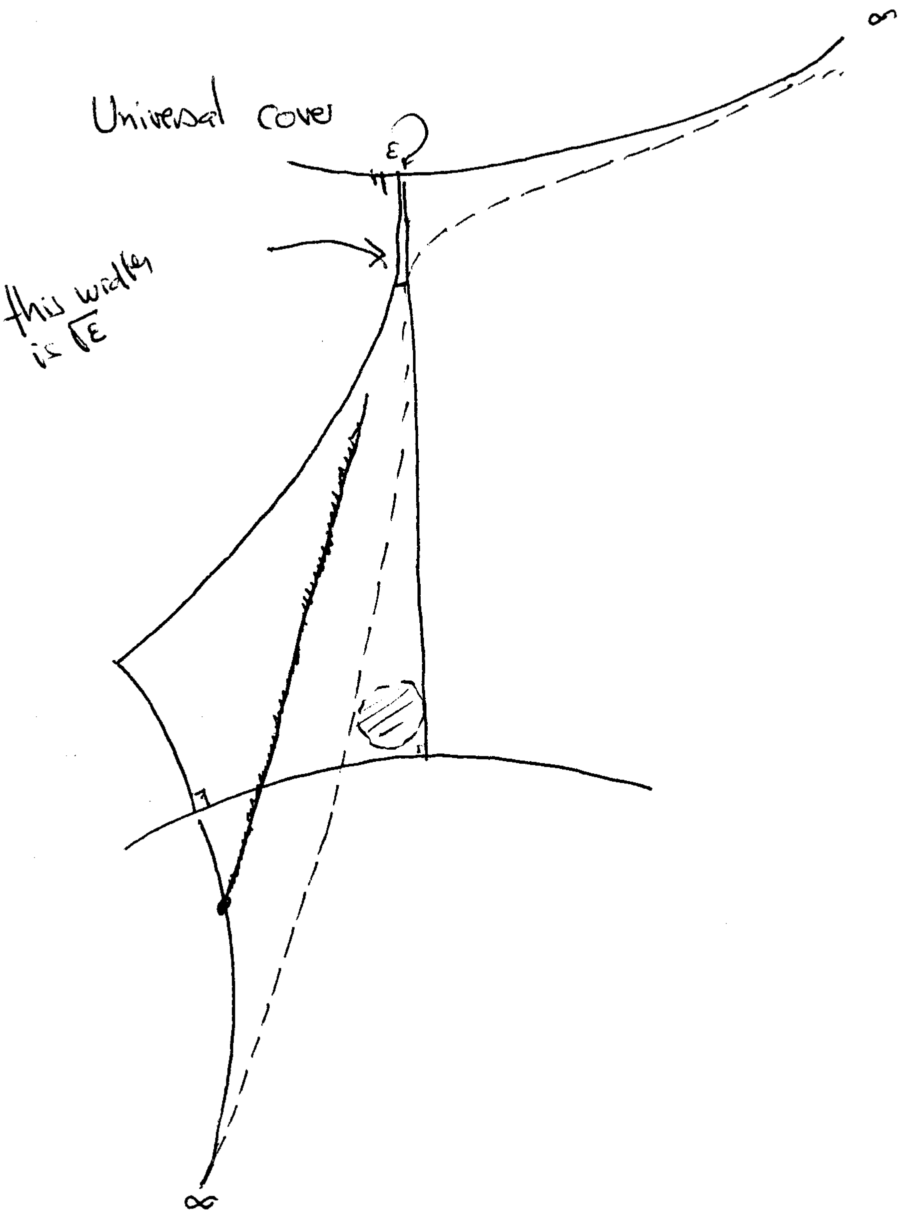


answer: No!

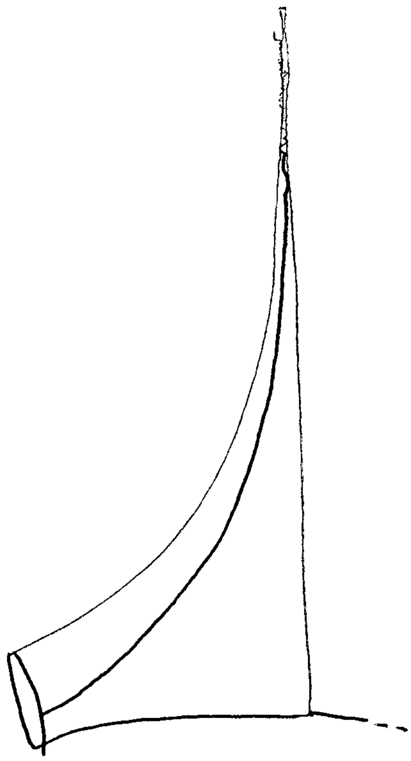
Lemma (False) If $\ell(\gamma) < \varepsilon$ (ε small)

then any ~~closed~~ simple geodesic arc from η to γ is $\perp \gamma$





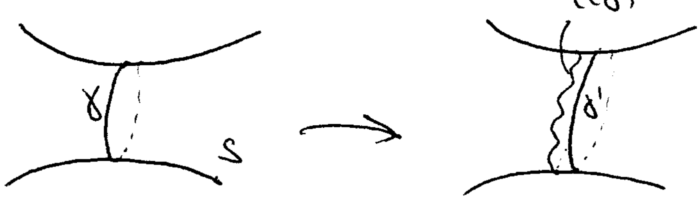
The correct picture:



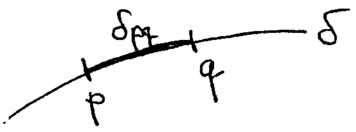
Case 2 $\quad S_k \to S_1 \qquad S \in \mathcal{M}_g$

Lemma $\quad C_S$ is continuous

Proof. Let $\varphi : S \to S'$ be a $(1+k)$-quasi-isometry ($0 \leq k <$ small)

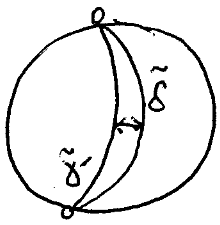


$\delta := \varphi(\gamma)$ is a $(1+k)^2$-quasi geodesic:

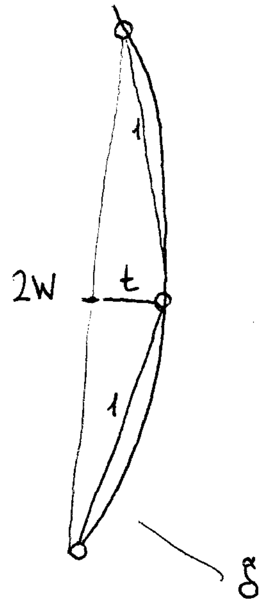


$\ell(\delta_{pq}) \leq (1+k)^2 \operatorname{dist}(p,q)$

In universal cover:



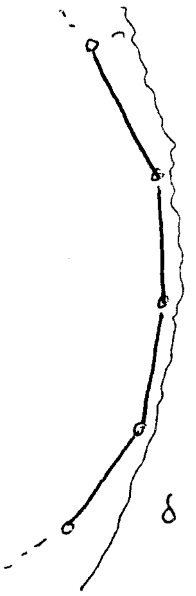
$\longleftrightarrow \; = ?$



$$2W \geq \frac{2}{(1+k)^2}$$

$$\xRightarrow{\text{Compute}} \quad t < \sqrt{4k}$$

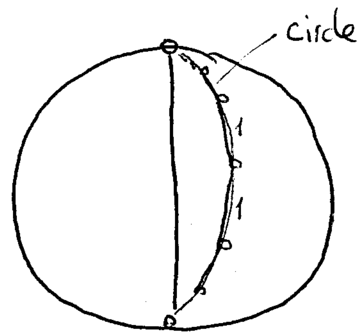
Take



with $t_* = \sqrt{4k}$ then $\tilde{\delta}$ outside

since same endpoints at ∞
one gets estimate:

the distance to $\tilde{\gamma}'$ is smaller than the one for the parallel line with "curvature $\sqrt{k}$"



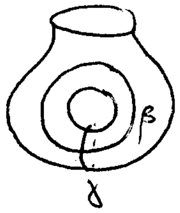
Open problem:
is the extremal case the parallel line?

Figures on slides.

Figure	01	02	03	04	15	16	17	21	22
# geod	8	69	619	619	8	16	204	108	204

Figure	31	32	33	34
	*	*	24	72

Example: one-holed torus



↓ (This part was not covered in the lecture anymore).

Theorem The only simple geodesics are

$\gamma, \beta, \gamma\beta\gamma^{-1}\beta^{-1}$

$\gamma\beta^{N_1}\gamma\beta^{N_2}\cdots\gamma\beta^{N_r}$

$\beta\gamma^{N_1}\beta\gamma^{N_2}\cdots\beta\gamma^{N_r}$

where $N_1, \dots, N_r$ is of small variation

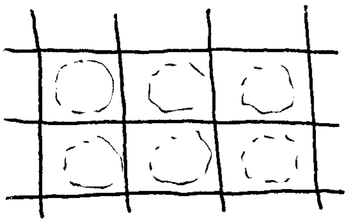
Example

34 34 34 334

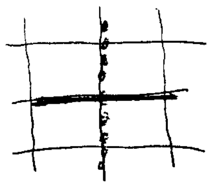
56 56 556 56 56 556 56 56 56 556

*

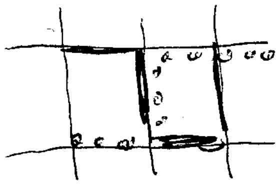
Proof. $\mathbb{Z} \times \mathbb{Z}$ covering



Cases that cause intersections



$\Longrightarrow$ no $\gamma^N \cdots \beta^M$ with $|N|, |M| \geq 2$

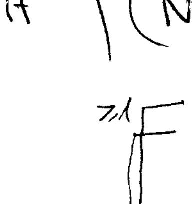


+ others $\Longrightarrow$ no sign changes.

Small variation:

Finally

if $\left| (N_1 + \cdots + N_m) - (N_{l+1} + \cdots + N_{l+m}) \right| \geq 2$



$\square$