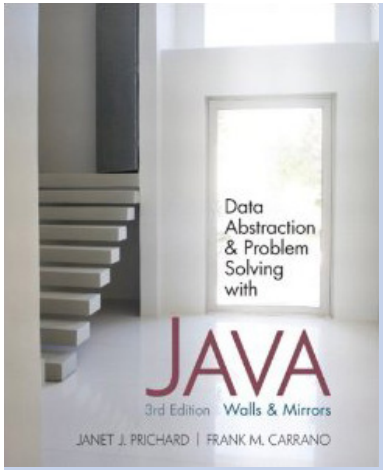


Chapter 14

Graphs

Terminology

- G = {V, E}
- A graph G consists of two sets
  - A set V of vertices, or nodes
  - A set E of edges
- A subgraph
  - Consists of a subset of a graph's vertices and a subset of its edges
- Adjacent vertices
  - Two vertices that are joined by an edge

Terminology

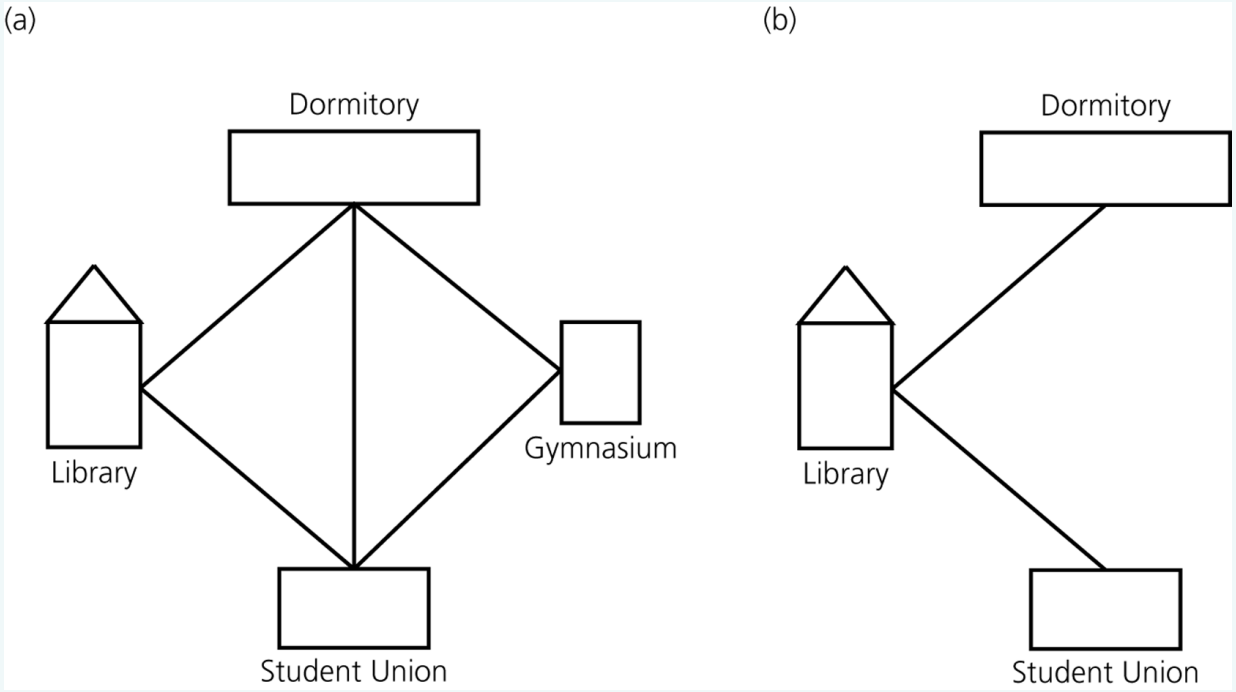


Figure 14-2

a) A campus map as a graph; b) a subgraph

Terminology

- A path between two vertices
  - A sequence of edges that begins at one vertex and ends at another vertex
  - May pass through the same vertex more than once
- A simple path
  - A path that passes through a vertex only once
- A cycle
  - A path that begins and ends at the same vertex
- A simple cycle
  - A cycle that does not pass through a vertex more than once

Terminology

- A connected graph
  - A graph that has a path between each pair of distinct vertices
- A disconnected graph
  - A graph that has at least one pair of vertices without a path between them
- A complete graph
  - A graph that has an edge between each pair of distinct vertices

Terminology

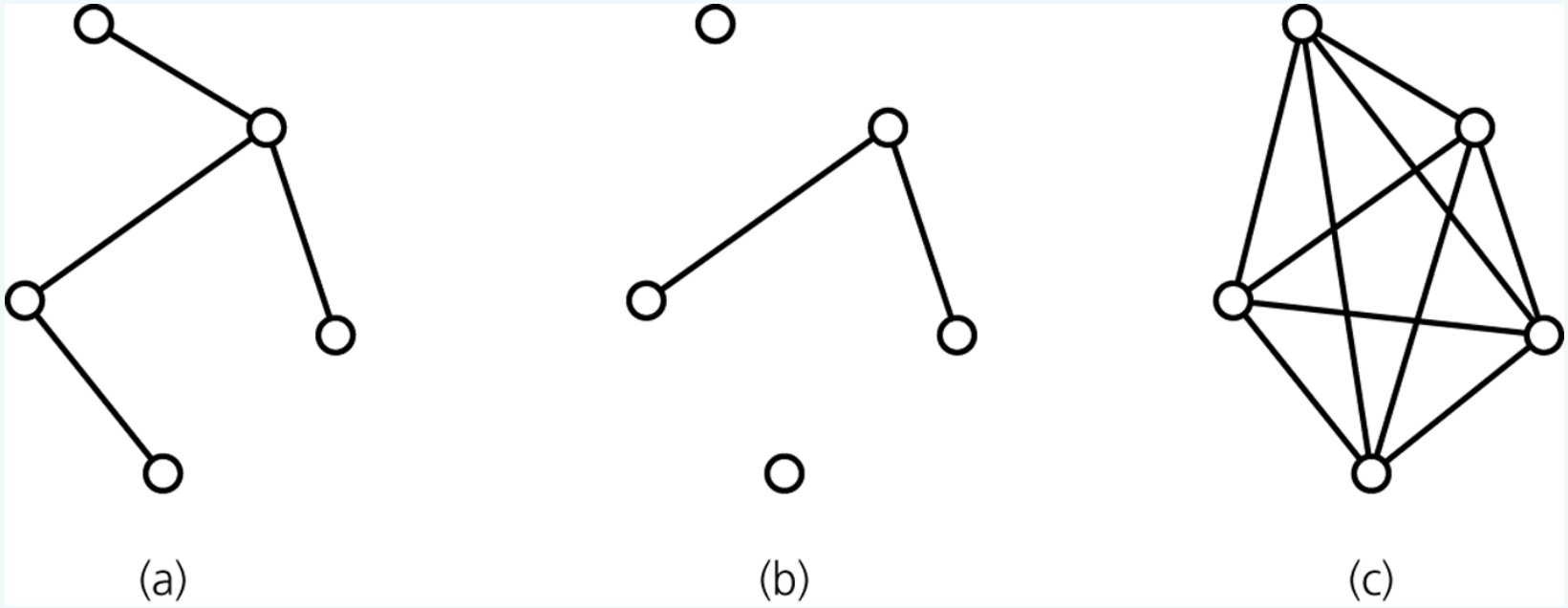


Figure 14-3

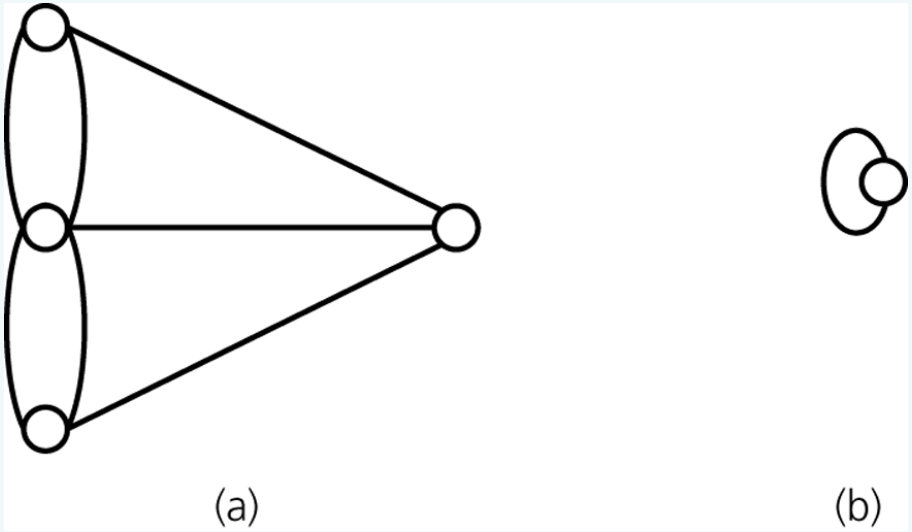
Graphs that are a) connected; b) disconnected; and c) complete

Terminology

- Multigraph
  - Not a graph
  - Allows multiple edges between vertices

Figure 14-4

a) A multigraph is not a graph; b) a self edge is not allowed in a graph



Terminology

- Weighted graph
  - A graph whose edges have numeric labels

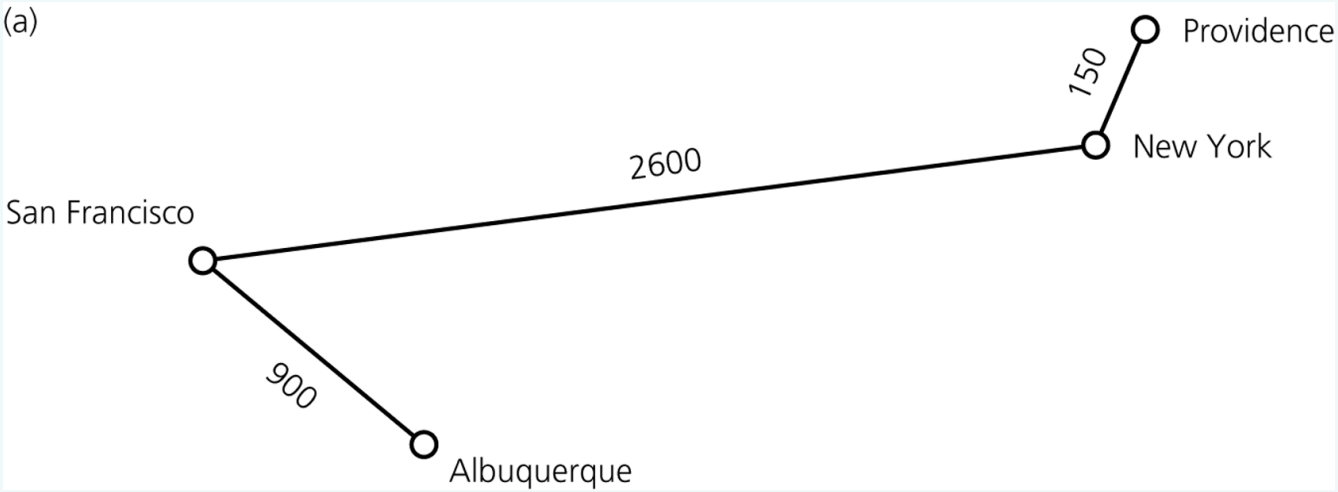


Figure 14-5a

a) A weighted graph

Terminology

- Undirected graph
  - Edges do not indicate a direction
- Directed graph, or diagraph
  - Each edge is a directed edge

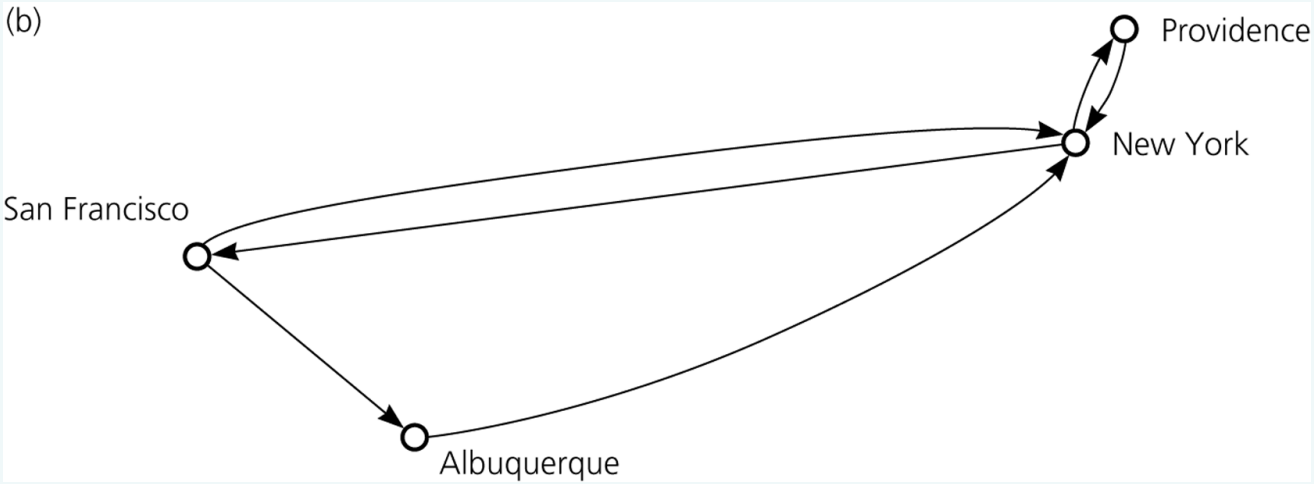


Figure 14-5b

b) A directed graph

Terminology

- Directed graph
  - Can have two edges between a pair of vertices, one in each direction
  - Directed path
    - A sequence of directed edges between two vertices
  - Vertex y is adjacent to vertex x if
    - There is a directed edge from x to y

Graphs As ADTs

- Graphs can be used as abstract data types
- Two options for defining graphs
  - Vertices contain values
  - Vertices do not contain values
- Operations of the ADT graph
  - Create an empty graph
  - Determine whether a graph is empty
  - Determine the number of vertices in a graph
  - Determine the number of edges in a graph

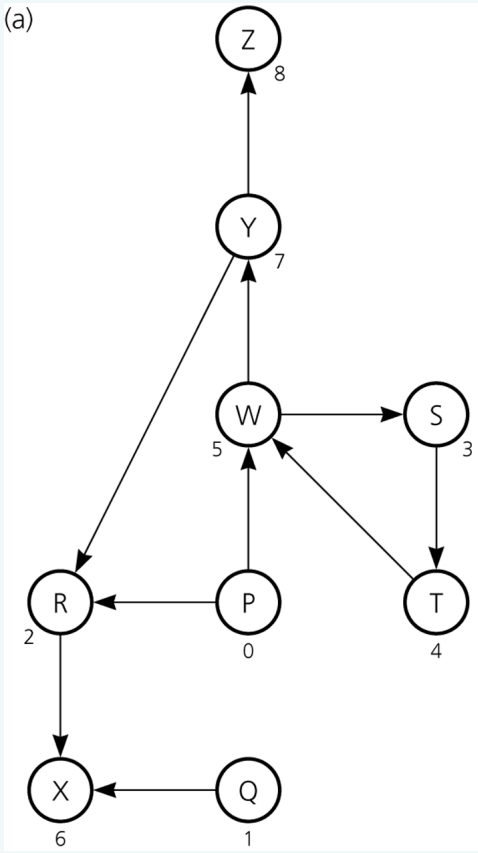
Graphs As ADTs

- Operations of the ADT graph (Continued)
  - Determine whether an edge exists between two given vertices; for weighted graphs, return weight value
  - Insert a vertex in a graph whose vertices have distinct search keys that differ from the new vertex's search key
  - Insert an edge between two given vertices in a graph
  - Delete a particular vertex from a graph and any edges between the vertex and other vertices
  - Delete the edge between two given vertices in a graph
  - Retrieve from a graph the vertex that contains a given search key

Implementing Graphs

- Most common implementations of a graph
  - Adjacency matrix
  - Adjacency list
- Adjacency matrix
  - Adjacency matrix for a graph with n vertices numbered 0, 1, …, n – 1
    - An n by n array matrix such that matrix[i][j] is
      - 1 (or true) if there is an edge from vertex i to vertex j
      - 0 (or false) if there is no edge from vertex i to vertex j

Implementing Graphs



(b)

		0	1	2	3	4	5	6	7	8
		P	Q	R	S	T	W	X	Y	Z
0	P	0	0	1	0	0	1	0	0	0
1	Q	0	0	0	0	0	0	1	0	0
2	R	0	0	0	0	0	0	1	0	0
3	S	0	0	0	0	1	0	0	0	0
4	T	0	0	0	0	0	1	0	0	0
5	W	0	0	0	1	0	0	0	1	0
6	X	0	0	0	0	0	0	0	0	0
7	Y	0	0	1	0	0	0	0	0	1
8	Z	0	0	0	0	0	0	0	0	0

Figure 14-6

a) A directed graph and b) its adjacency matrix

Implementing Graphs

- Adjacency matrix for a weighted graph with n vertices numbered 0, 1, …, n – 1
  - An n by n array matrix such that matrix[i][j] is
    - The weight that labels the edge from vertex i to vertex j if there is an edge from i to j
    - ∞ if there is no edge from vertex i to vertex j

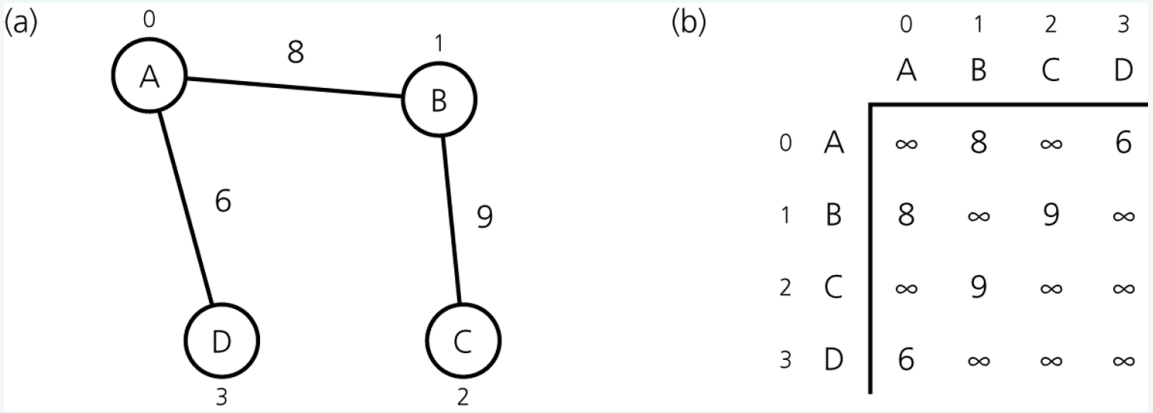


Figure 14-7

a) A weighted undirected graph and b) its adjacency matrix

Implementing Graphs

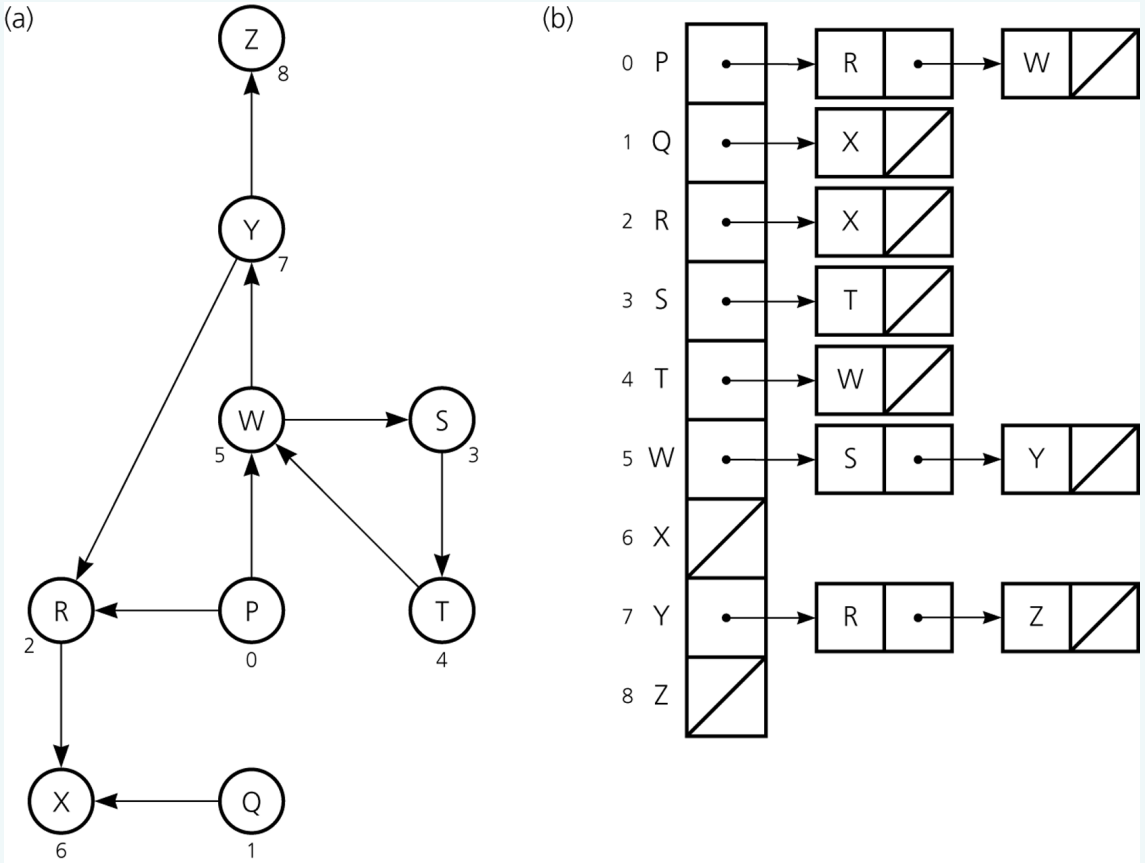
- Adjacency list
  - An adjacency list for a graph with n vertices numbered 0, 1, …, n – 1
    - Consists of n linked lists
    - The i^{th} linked list has a node for vertex j if and only if the graph contains an edge from vertex i to vertex j
      - This node can contain either
        - » Vertex j's value, if any
        - » An indication of vertex j's identity

Implementing Graphs

Figure 14-8

a) A directed graph and

b) its adjacency list



Implementing Graphs

- Adjacency list for an undirected graph
  - Treats each edge as if it were two directed edges in opposite directions

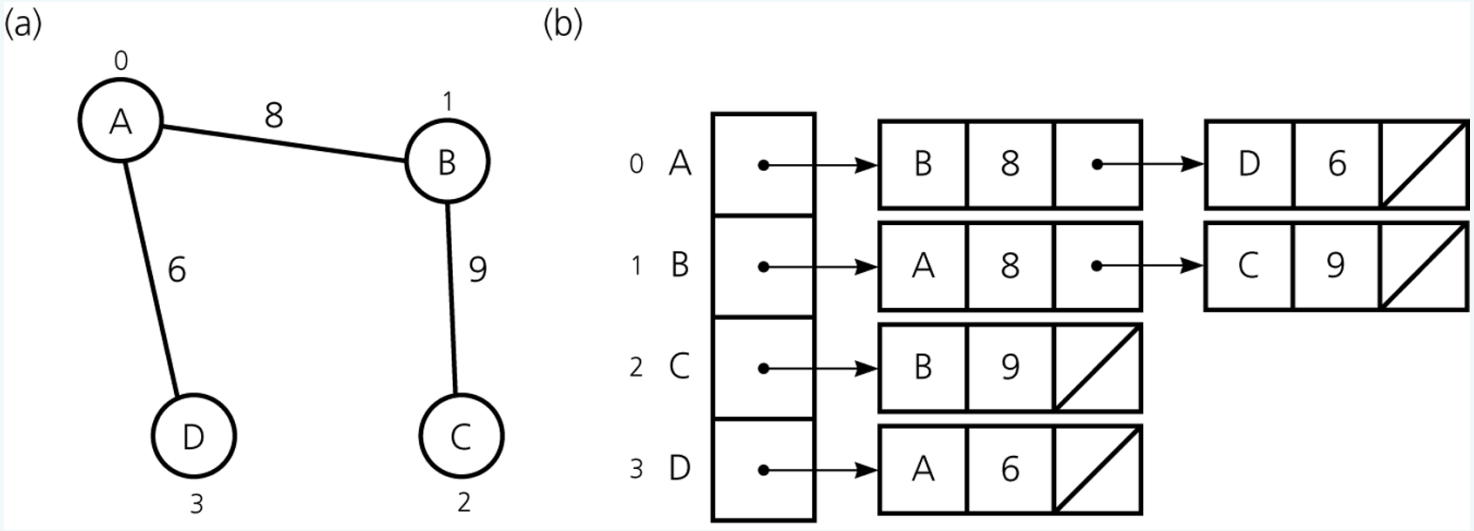


Figure 14-9

a) A weighted undirected graph and b) its adjacency list

Implementing Graphs

- Adjacency matrix compared with adjacency list
  - Two common operations on graphs
    - Determine whether there is an edge from vertex i to vertex j
    - Find all vertices adjacent to a given vertex i
  - Adjacency matrix
    - Supports operation 1 more efficiently
  - Adjacency list
    - Supports operation 2 more efficiently
    - Often requires less space than an adjacency matrix

Implementing a Graph Class Using the JCF

- ADT graph not part of JCF
- Can implement a graph using an adjacency list consisting of a vector of maps
- Implementation presented uses `TreeSet` class

Graph Traversals

- A graph-traversal algorithm
  - Visits all the vertices that it can reach
  - Visits all vertices of the graph if and only if the graph is connected
    - A connected component
      - The subset of vertices visited during a traversal that begins at a given vertex
  - Can loop indefinitely if a graph contains a loop
    - To prevent this, the algorithm must
      - Mark each vertex during a visit, and
      - Never visit a vertex more than once

Graph Traversals

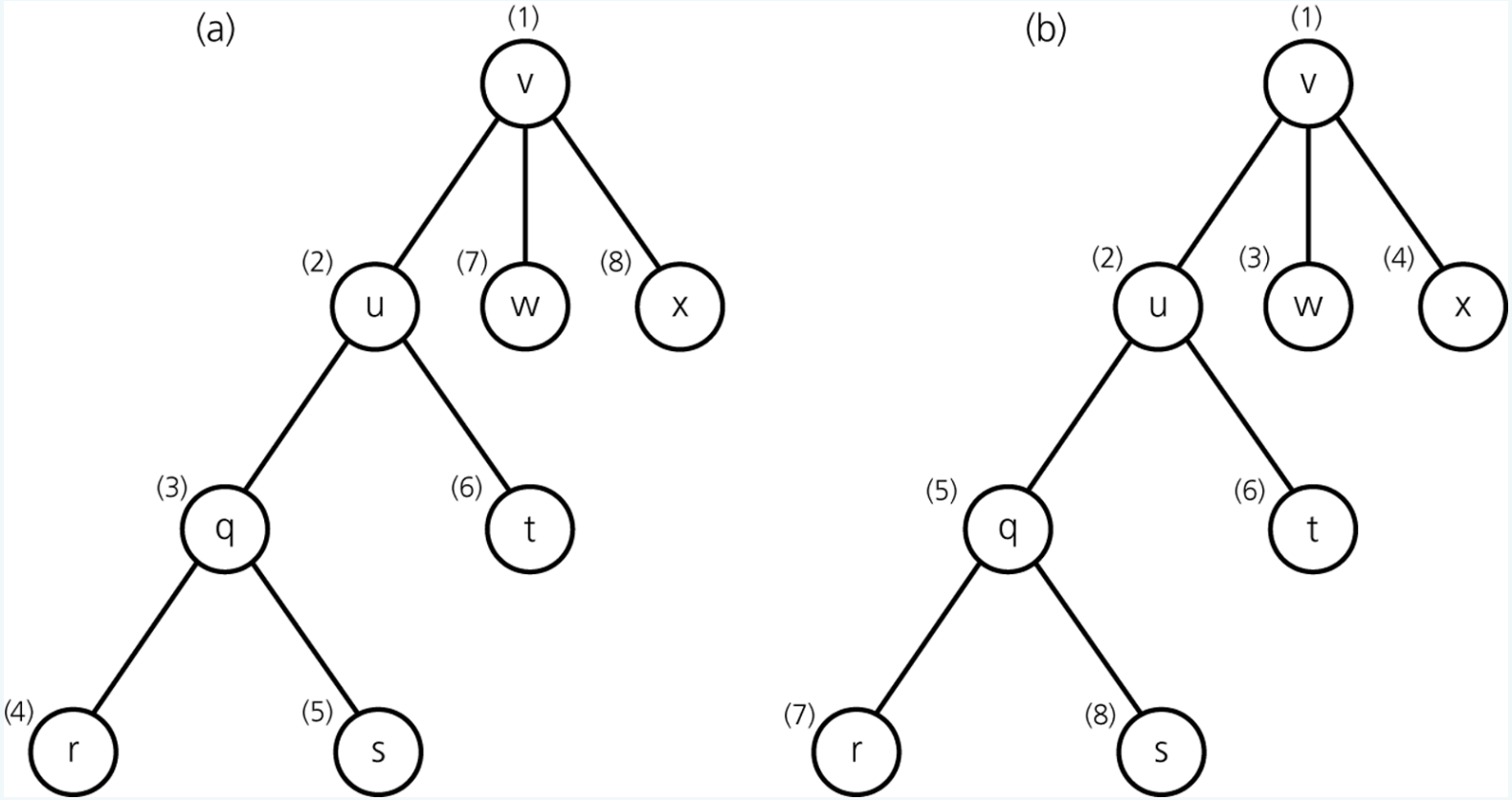


Figure 14-10

Visitation order for a) a depth-first search; b) a breadth-first search

Depth-First Search

- Depth-first search (DFS) traversal
  - Proceeds along a path from *v* as deeply into the graph as possible before backing up
  - Does not completely specify the order in which it should visit the vertices adjacent to *v*
  - A last visited, first explored strategy

Breadth-First Search

- Breadth-first search (BFS) traversal
  - Visits every vertex adjacent to a vertex v that it can before visiting any other vertex
  - A first visited, first explored strategy
  - An iterative form uses a queue
  - A recursive form is possible, but not simple

Implementing a BFS Iterator Class Using the JCF

- `BFSIterator` class uses the `ListIterator` class
  - As a queue to keep track of the order the vertices should be processed
- `BFSIterator` constructor
  - Initiates methods used to determine BFS order of vertices for the graph
- Graph is searched by processing vertices from each vertex's adjacency list
  - In the order that they were pushed onto the queue

Applications of Graphs: Topological Sorting

- Topological order
  - A list of vertices in a directed graph without cycles such that vertex x precedes vertex y if there is a directed edge from x to y in the graph
  - There may be several topological orders in a given graph
- Topological sorting
  - Arranging the vertices into a topological order

Topological Sorting

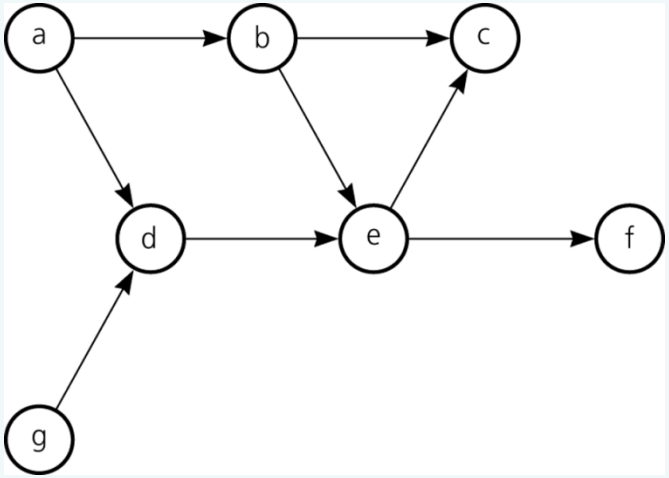


Figure 14-14

A directed graph without cycles

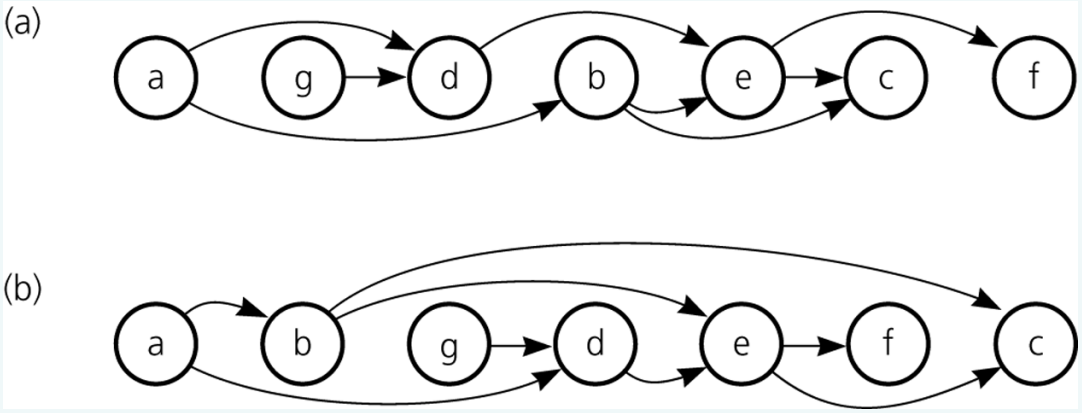


Figure 14-15

The graph in Figure 14-14 arranged according to the topological orders a) *a, g, d, b, e, c, f* and b) *a, b, g, d, e, f, c*

Topological Sorting

- Simple algorithms for finding a topological order
  - topSort1
    - Find a vertex that has no successor
    - Remove from the graph that vertex and all edges that lead to it, and add the vertex to the beginning of a list of vertices
    - Add each subsequent vertex that has no successor to the beginning of the list
    - When the graph is empty, the list of vertices will be in topological order

Topological Sorting

- Simple algorithms for finding a topological order (Continued)
  - topSort2
    - A modification of the iterative DFS algorithm
    - Strategy
      - Push all vertices that have no predecessor onto a stack
      - Each time you pop a vertex from the stack, add it to the beginning of a list of vertices
      - When the traversal ends, the list of vertices will be in topological order

Spanning Trees

- A tree
  - An undirected connected graph without cycles
- A spanning tree of a connected undirected graph G
  - A subgraph of G that contains all of G's vertices and enough of its edges to form a tree
- To obtain a spanning tree from a connected undirected graph with cycles
  - Remove edges until there are no cycles

Spanning Trees

- You can determine whether a connected graph contains a cycle by counting its vertices and edges
  - A connected undirected graph that has n vertices must have at least n – 1 edges
  - A connected undirected graph that has n vertices and exactly n – 1 edges cannot contain a cycle
  - A connected undirected graph that has n vertices and more than n – 1 edges must contain at least one cycle

Spanning Trees

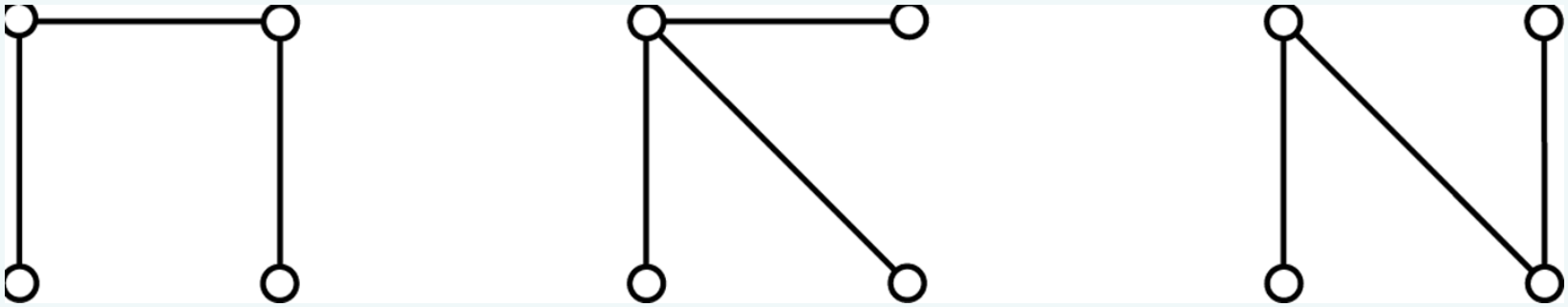


Figure 14-19

Connected graphs that each have four vertices and three edges

The DFS Spanning Tree

- To create a depth-first search (DFS) spanning tree
  - Traverse the graph using a depth-first search and mark the edges that you follow
  - After the traversal is complete, the graph's vertices and marked edges form the spanning tree

The BFS Spanning Tree

- To create a breath-first search (BFS) spanning tree
  - Traverse the graph using a bread-first search and mark the edges that you follow
  - When the traversal is complete, the graph's vertices and marked edges form the spanning tree

Minimum Spanning Trees

- Minimum spanning tree
  - A spanning tree for which the sum of its edge weights is minimal
- Prim's algorithm
  - Finds a minimal spanning tree that begins at any vertex
  - Strategy
    - Find the least-cost edge (v, u) from a visited vertex v to some unvisited vertex u
    - Mark u as visited
    - Add the vertex u and the edge (v, u) to the minimum spanning tree
    - Repeat the above steps until there are no more unvisited vertices

Shortest Paths

- Shortest path between two vertices in a weighted graph
  - The path that has the smallest sum of its edge weights
- Dijkstra's shortest-path algorithm
  - Determines the shortest paths between a given origin and all other vertices
  - Uses
    - A set vertexSet of selected vertices
    - An array weight, where weight[v] is the weight of the shortest (cheapest) path from vertex 0 to vertex v that passes through vertices in vertexSet

Circuits

- A circuit
  - A special cycle that passes through every vertex (or edge) in a graph exactly once
- Euler circuit
  - A circuit that begins at a vertex v, passes through every edge exactly once, and terminates at v
  - Exists if and only if each vertex touches an even number of edges

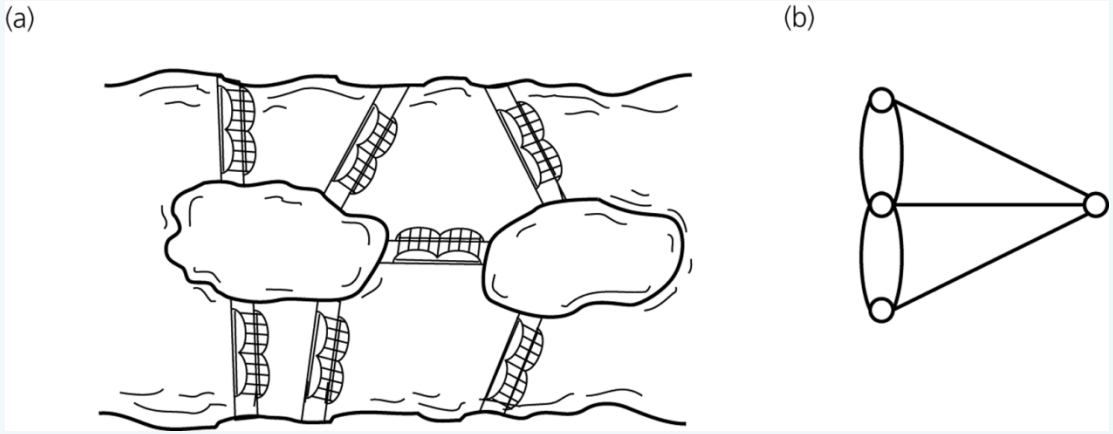


Figure 14-27

a) Euler's bridge problem and b) its multigraph representation

Some Difficult Problems

- Three applications of graphs
  - The traveling salesperson problem
  - The three utilities problem
  - The four-color problem
- A Hamilton circuit
  - Begins at a vertex v, passes through every vertex exactly once, and terminates at v

Summary

- The two most common implementations of a graph are the adjacency matrix and the adjacency list
- Graph searching
  - Depth-first search goes as deep into the graph as it can before backtracking
  - Bread-first search visits all possible adjacent vertices before traversing further into the graph
- Topological sorting produces a linear order of the vertices in a directed graph without cycles

Summary

- Trees are connected undirected graphs without cycles
  - A spanning tree of a connected undirected graph is a subgraph that contains all the graph's vertices and enough of its edges to form a tree
- A minimum spanning tree for a weighted undirected graph is a spanning tree whose edge-weight sum is minimal
- The shortest path between two vertices in a weighted directed graph is the path that has the smallest sum of its edge weights

Summary

- An Euler circuit in an undirected graph is a cycle that begins at vertex v, passes through every edge in the graph exactly once, and terminates at v
- A Hamilton circuit in an undirected graph is a cycle that begins at vertex v, passes through every vertex in the graph exactly once, and terminates at v