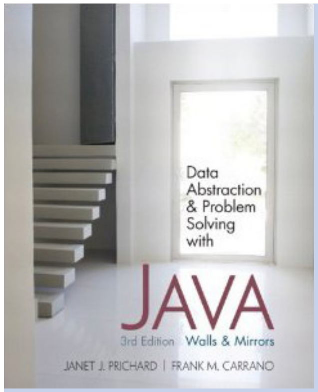


Chapter 3

Recursion: The Mirrors

Recursive Solutions

- Recursion
  - An extremely powerful problem-solving technique
  - Breaks a problem in smaller identical problems
  - An alternative to iteration
    - An iterative solution involves loops

Recursive Solutions

- Sequential search
  - Starts at the beginning of the collection
  - Looks at every item in the collection in order until the item being searched for is found
- Binary search
  - Repeatedly halves the collection and determines which half could contain the item
  - Uses a divide and conquer strategy

$$t = 8$$

$$A = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

```
for (i=0 ; i < A.length ; i++) {
    if (A[i] == t ) {
        return i;
    }
}
return -1;
```

$$M = \frac{L + H}{2}$$

$$A = \{2, 3, 5, 7, 9, 12, 14\}$$

$$\sigma = 4$$

L	H	M
0	6	3
2	2	1
	1	2

t is target

```
LOW = 0;
HIGH = A.length - 1;
if (A[LOW] > t || A[HIGH] < t) {
    return -1;
}
WHILE (LOW <= HIGH) {
    MID = (LOW + HIGH)/2;
    if (t == A[MID]) then {
        return MID;
    } else if (t < A[MID]) then {
        HIGH = MID - 1;
    } else {
        LOW = MID + 1;
    }
}
return -1;
```

t = 12

t = 9

A = {2, 4, 6, 8, 10, 12, 14, 16, 18}

L	H	M
0	8	4
5	8	6
5	5	5

0	8	4
0	3	1
2	3	2
3	3	3
3	2	

```
int bSearch(int[] A, int x, int low, int high) {
    if (low > high) {
            return -1;
    }
    mid = (low + high)/2;
    if (A[mid] == x) {
            return mid;
    } else (if A[mid] > x ) {
Ra      return bSearch(A, x, low, mid-1);
    } else
        return bSearch(A, x, mid+1, high);
          Rb
    }
```

Recursive Solutions

- Facts about a recursive solution
  - A recursive method calls itself
  - Each recursive call solves an identical, but smaller, problem
  - A test for the base case enables the recursive calls to stop
    - Base case: a known case in a recursive definition
  - Eventually, one of the smaller problems must be the base case

Recursive Solutions

- Four questions for construction recursive solutions
  - How can you define the problem in terms of a smaller problem of the same type?
  - How does each recursive call diminish the size of the problem?
  - What instance of the problem can serve as the base case?
  - As the problem size diminishes, will you reach this base case?

A Recursive Valued Method: The Factorial of n

- Problem
  - Compute the factorial of an integer n
- An iterative definition of factorial(n)

factorial(n) = n * (n-1) * (n-2) * … * 1
for any integer n > 0
factorial(0) = 1

A Recursive Valued Method: The Factorial of n

- A recursive definition of factorial(n)

$$\text{factorial}(n) = \begin{cases} 1 & \text{if } n = 0 \\ n * \text{factorial}(n-1) & \text{if } n > 0 \end{cases}$$

- A recurrence relation
  - A mathematical formula that generates the terms in a sequence from previous terms
  - Example

$$\begin{aligned} \text{factorial}(n) &= n * [(n-1) * (n-2) * \ldots * 1] \\ &= n * \text{factorial}(n-1) \end{aligned}$$

A Recursive Valued Method: The Factorial of n

- Box trace
  - A systematic way to trace the actions of a recursive method
  - Each box roughly corresponds to an activation record
  - An activation record
    - Contains a method's local environment at the time of and as a result of the call to the method

A Recursive Valued Method: The Factorial of n

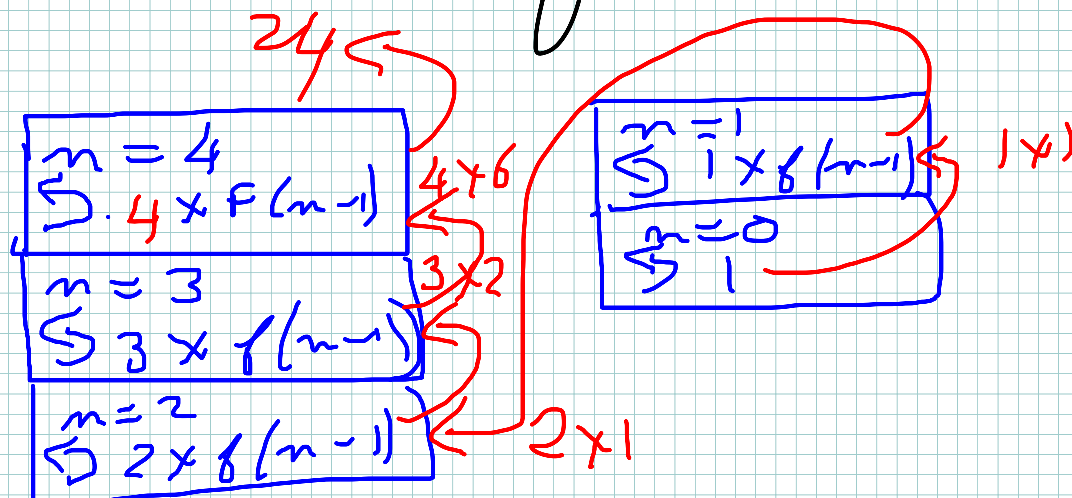
- A method's local environment includes:
  - The method's local variables
  - A copy of the actual value arguments
  - A return address in the calling routine
  - The value of the method itself

```
n = 3
A: fact(n-1) = ?
return ?
```

Figure 3-3

A box

```
int factorial(int n){
    if(n==0){
        return 1;
    }
    return n * factorial(n-1);
}
```



A Recursive `void` Method: Writing a String Backward

- Problem
  - Given a string of characters, write it in reverse order
- Recursive solution
  - Each recursive step of the solution diminishes by 1 the length of the string to be written backward
  - Base case
    - Write the empty string backward

A Recursive `void` Method: Writing a String Backward

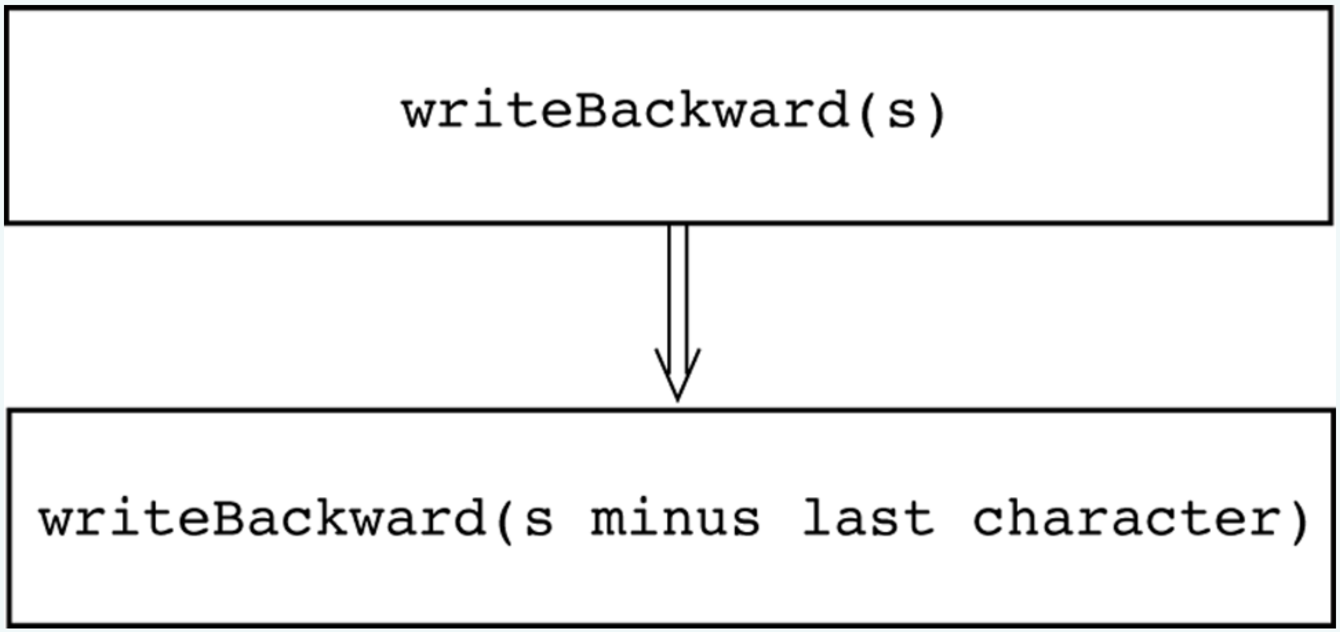


Figure 3-6

A recursive solution

A Recursive `void` Method: Writing a String Backward

- Execution of `writeBackward` can be traced using the box trace
- Temporary `System.out.println` statements can be used to debug a recursive method

Counting Things

- Next three problems
  - Require you to count certain events or combinations of events or things
  - Contain more than one base cases
  - Are good examples of inefficient recursive solutions

Multiplying Rabbits (The Fibonacci Sequence)

- “Facts” about rabbits
  - Rabbits never die
  - A rabbit reaches sexual maturity exactly two months after birth, that is, at the beginning of its third month of life
  - Rabbits are always born in male-female pairs
    - At the beginning of every month, each sexually mature male-female pair gives birth to exactly one male-female pair

Multiplying Rabbits (The Fibonacci Sequence)

- Problem
  - How many pairs of rabbits are alive in month n?
- Recurrence relation

  rabbit(n) = rabbit(n-1) + rabbit(n-2)

Multiplying Rabbits (The Fibonacci Sequence)

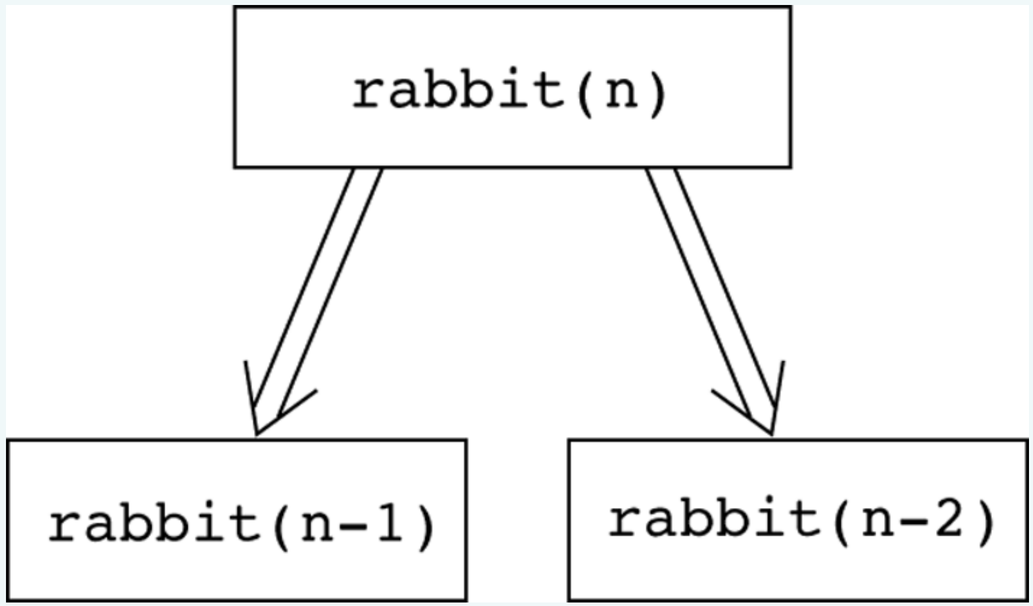


Figure 3-10

Recursive solution to the rabbit problem

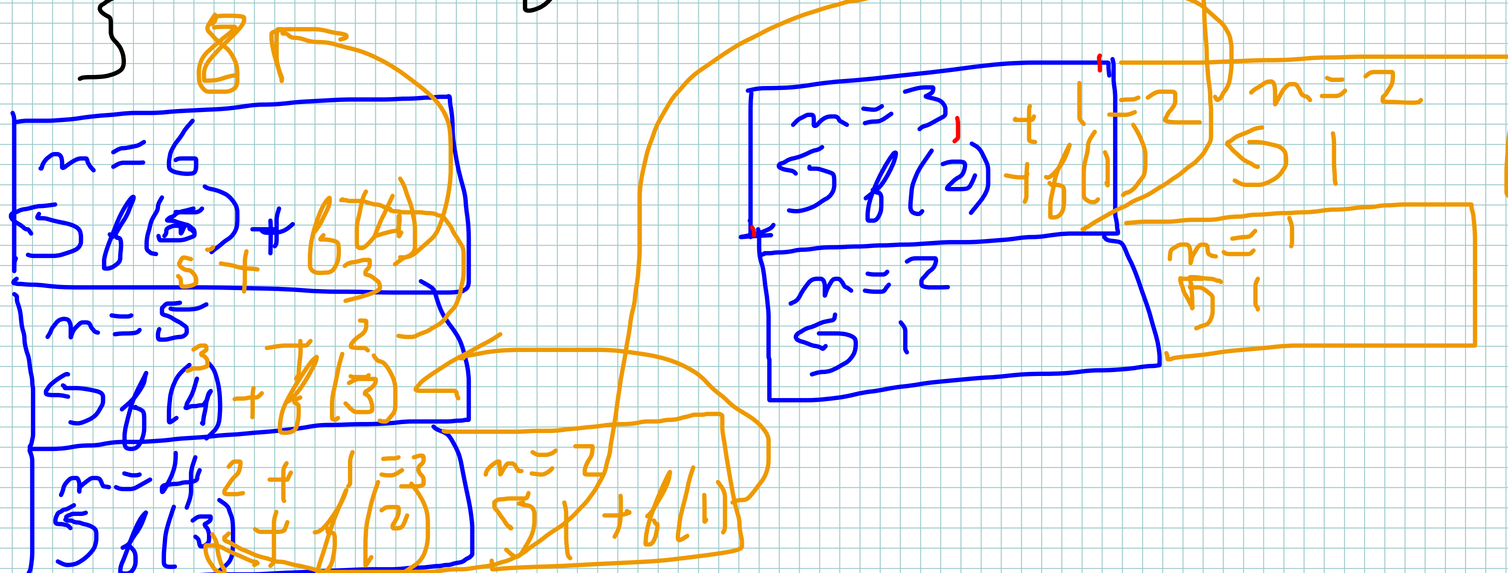
Multiplying Rabbits (The Fibonacci Sequence)

- Base cases
  - rabbit(2), rabbit(1)
- Recursive definition

$$\text{rabbit}(n) = \begin{cases} 1 & \text{if } n \text{ is 1 or 2} \\ \text{rabbit}(n-1) + \text{rabbit}(n-2) & \text{if } n > 2 \end{cases}$$

- Fibonacci sequence
  - The series of numbers rabbit(1), rabbit(2), rabbit(3), and so on

```
int fib(int n) {
    if (n==1 || n==2) {
        return 1;
    }
    return fib(n-1) + fib(n-2);
}
```



Organizing a Parade

- Rules about organizing a parade
  - The parade will consist of bands and floats in a single line
  - One band cannot be placed immediately after another
- Problem
  - How many ways can you organize a parade of length n?

Organizing a Parade

- Let:
  - $P(n)$ be the number of ways to organize a parade of length n
  - $F(n)$ be the number of parades of length n that end with a float
  - $B(n)$ be the number of parades of length n that end with a band
- Then
  - $P(n) = F(n) + B(n)$

Organizing a Parade

- Number of acceptable parades of length n that end with a float

  $F(n) = P(n-1)$

- Number of acceptable parades of length n that end with a band

  $B(n) = F(n-1)$

- Number of acceptable parades of length n
  - $P(n) = P(n-1) + P(n-2)$

Organizing a Parade

- Base cases

  $P(1) = 2$ (The parades of length 1 are float and band.)

  $P(2) = 3$ (The parades of length 2 are float-float, band-float, and float-band.)

- Solution

  $P(1) = 2$

  $P(2) = 3$

  $P(n) = P(n-1) + P(n-2)$ for $n > 2$

Mr. Spock's Dilemma (Choosing k out of n Things)

- Problem
  - How many different choices are possible for exploring k planets out of n planets in a solar system?
- Let
  - $c(n, k)$ be the number of groups of k planets chosen from n

Mr. Spock's Dilemma (Choosing k out of n Things)

- In terms of Planet X:

  c(n, k) = (the number of groups of k planets that include Planet X)

  +

  (the number of groups of k planets that do not include Planet X)

Mr. Spock's Dilemma (Choosing k out of n Things)

- The number of ways to choose k out of n things is the sum of
  - The number of ways to choose k-1 out of n-1 things and
  - The number of ways to choose k out of n-1 things

$$c(n, k) = c(n-1, k-1) + c(n-1, k)$$

Mr. Spock's Dilemma (Choosing k out of n Things)

- Base cases
  - There is one group of everything
    $c(k, k) = 1$
  - There is one group of nothing
    $c(n, 0) = 1$
  - $c(n, k) = 0 \quad \text{if } k > n$

Mr. Spock's Dilemma (Choosing k out of n Things)

- Recursive solution

$$c(n,k) = \begin{cases} 1 & \text{if } k = 0 \\ 1 & \text{if } k = n \\ 0 & \text{if } k > n \\ c(n-1, k-1) + c(n-1, k) & \text{if } 0 < k < n \end{cases}$$

Searching an Array: Finding the Largest Item in an Array

- A recursive solution

```
if (anArray has only one item) {
 maxArray(anArray) is the item in anArray
}
else if (anArray has more than one item) {
 maxArray(anArray) is the maximum of
   maxArray(left half of anArray) and
   maxArray(right half of anArray)
}  // end if
```

Finding the Largest Item in an Array

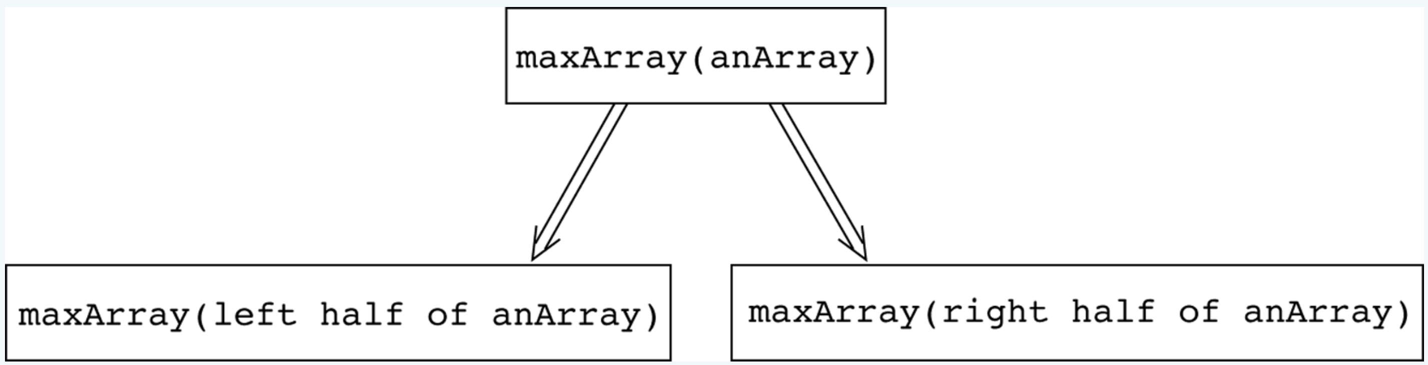


Figure 3-13

Recursive solution to the largest-item problem

Binary Search

- A high-level binary search

```
if (anArray is of size 1) {
 Determine if anArray's item is equal to value
}
else {
 Find the midpoint of anArray
 Determine which half of anArray contains value
 if (value is in the first half of anArray) {
   binarySearch (first half of anArray, value)
 }
 else {
   binarySearch(second half of anArray, value)
 } // end if
} // end if
```

Binary Search

- Implementation issues:
  - How will you pass "half of `anArray`" to the recursive calls to `binarySearch`?
  - How do you determine which half of the array contains value?
  - What should the base case(s) be?
  - How will `binarySearch` indicate the result of the search?

Finding the k^{th} Smallest Item in an Array

- The recursive solution proceeds by:
  1. Selecting a pivot item in the array
  2. Cleverly arranging, or partitioning, the items in the array about this pivot item
  3. Recursively applying the strategy to one of the partitions

Finding the k^{th} Smallest Item in an Array

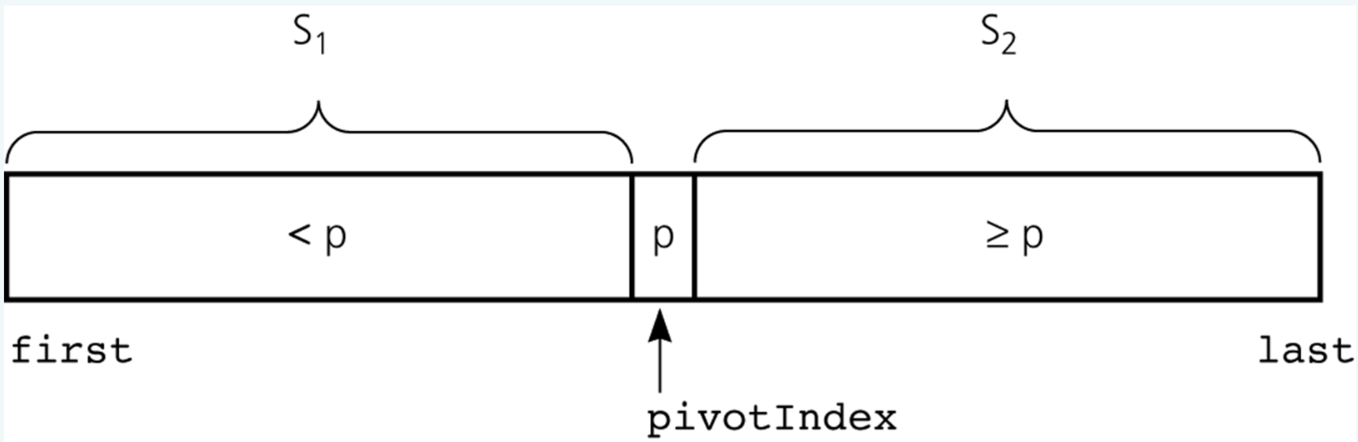


Figure 3-18

A partition about a pivot

Finding the k^{th} Smallest Item in an Array

- Let:

```
kSmall(k, anArray, first, last) =
        kth smallest item in anArray[first..last]
```

- Solution:

```
kSmall(k, anArray, first, last)
      kSmall(k, anArray, first, pivotIndex-1)
                          if k < pivotIndex – first + 1
=                         if k = pivotIndex – first + 1
      p
      kSmall(k-(pivotIndex-first+1), anArray,
             pivotIndex+1, last)
                          if k > pivotIndex – first + 1
```

Organizing Data: The Towers of Hanoi

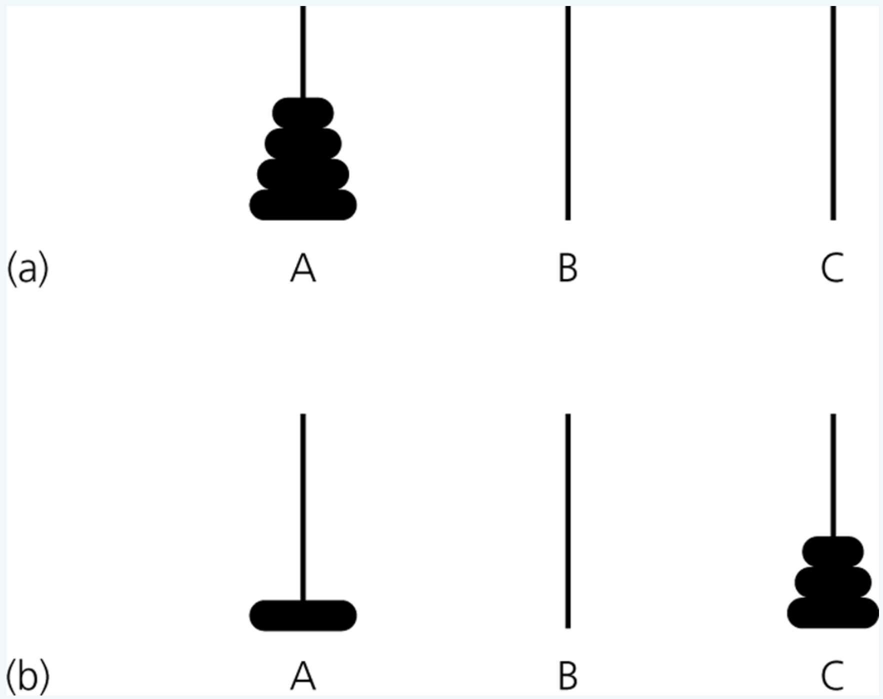


Figure 3-19a and b

a) The initial state; b) move *n* - 1 disks from *A* to *C*

The Towers of Hanoi

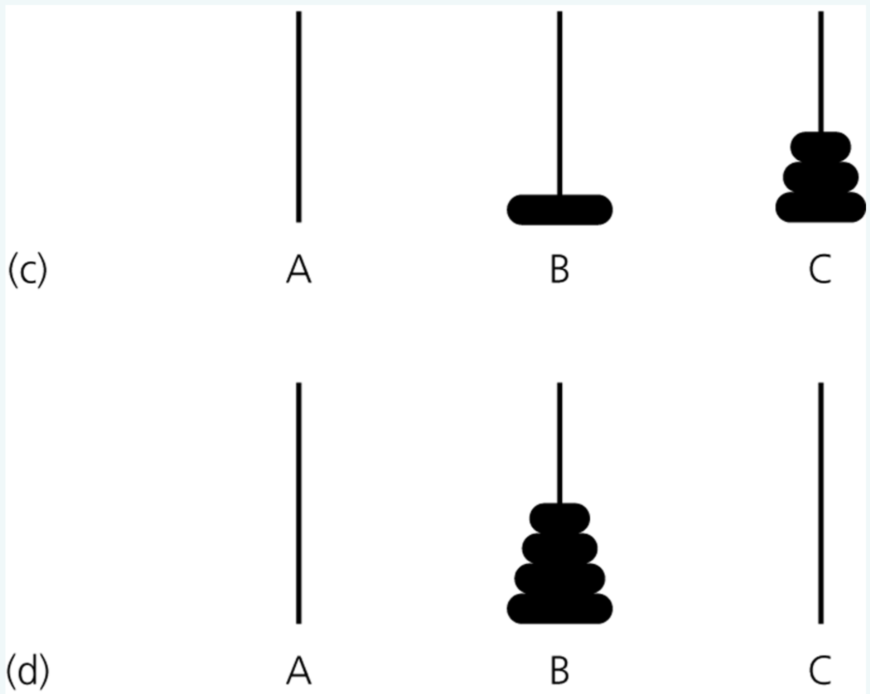


Figure 3-19c and d

c) move one disk from *A* to *B*; d) move *n* - 1 disks from *C* to *B*

The Towers of Hanoi

- Pseudocode solution

```
solveTowers(count, source, destination, spare)
 if (count is 1) {
   Move a disk directly from source to destination
 }
 else {
   solveTowers(count-1, source, spare, destination)
   solveTowers(1, source, destination, spare)
   solveTowers(count-1, spare, destination, source)
 } //end if
```

Recursion and Efficiency

- Some recursive solutions are so inefficient that they should not be used
- Factors that contribute to the inefficiency of some recursive solutions
  - Overhead associated with method calls
  - Inherent inefficiency of some recursive algorithms

Summary

- Recursion solves a problem by solving a smaller problem of the same type
- Four questions to keep in mind when constructing a recursive solution
  – How can you define the problem in terms of a smaller problem of the same type?
  – How does each recursive call diminish the size of the problem?
  – What instance of the problem can serve as the base case?
  – As the problem size diminishes, will you reach this base case?

Summary

- A recursive call's postcondition can be assumed to be true if its precondition is true
- The box trace can be used to trace the actions of a recursive method
- Recursion can be used to solve problems whose iterative solutions are difficult to conceptualize

Summary

- Some recursive solutions are much less efficient than a corresponding iterative solution due to their inherently inefficient algorithms and the overhead of method calls
- If you can easily, clearly, and efficiently solve a problem by using iteration, you should do so