RESEARCH SUMMARY – JANUARY 2015

ROB SCHNEIDERMAN

Abstract. This article summarizes my research and provides a guide to reading my papers (the first 14 entries in the bibliography), with emphasis given to more recent results.

My area of research is centered in the geometric topology of 3- and 4-dimensional manifolds. My work focuses on problems involving the search for certain embedded and/or disjoint submanifolds, or the determination of obstructions to their existence. I am also interested in fitting such problems and their (partial) solutions into frameworks that shed light on the bigger picture and in particular provide topological interpretations of algebraic structure.

The main methodology guiding my work is to study low-dimensional topological phenomena by "measuring" as directly as possible the well-known general failure of the Whitney move in dimensions less than or equal to four. A successful Whitney move is shown in Figure 1:

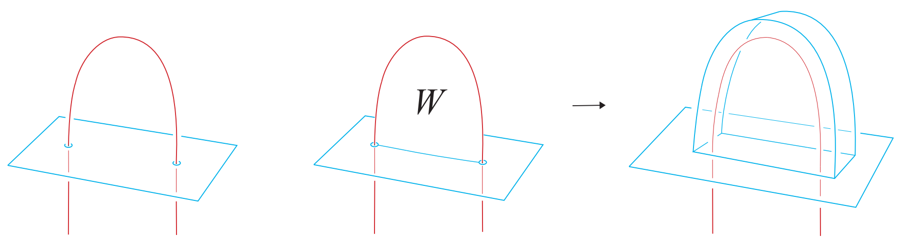
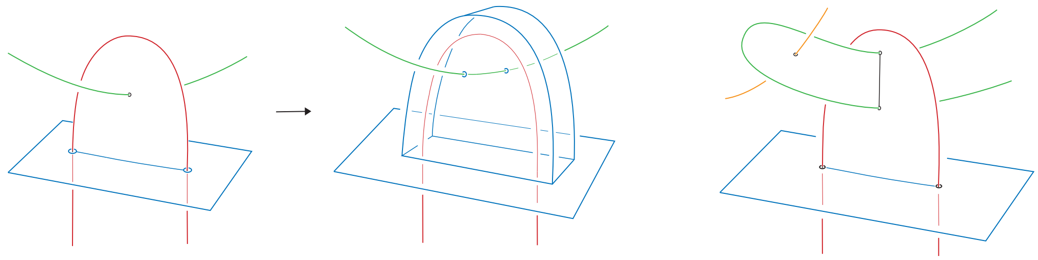


Figure 1. Left: A canceling pair of transverse intersections between two local sheets of surfaces in a 3-dimensional slice of 4–space. The translucent horizontal blue sheet appears entirely in this 3-dimensional 'present', and the red sheet appears as an arc which is assumed to extend into 'past' and 'future'. Middle: A Whitney disk W pairing the intersections. Right: A Whitney move guided by W eliminates the intersection pair, without creating any new intersections.

Via general position arguments available in higher dimensions, this move allows for the extraction of important topological information from algebraic data in many settings (e.g. surgery programs for classifying manifolds). In four dimensions, generic intersections between Whitney disks and surface sheets can obstruct a successful Whitney move: Figure 2(A) shows how such an intersection point leads to an *un*successful Whitney move.

Frequently working in collaboration with J. Conant (UT Knoxville) and P. Teichner (Max-Planck-Institute for Mathematics and UC Berkeley), I have developed a theory of *Whitney towers* which describes this failure in terms of higher-order intersections among iterated layers of Whitney disks in 4–manifolds (Figure 2(B)). Letting the geometric topology guide the construction of invariants has led to interesting algebra and combinatorics, as well as the uncovering of connections between Whitney towers and a variety of topics including Feynman diagrams and the Kontsevich



(A) (B)

FIGURE 2. (A) This Whitney move eliminates the previous pair of intersections but creates a new pair of intersections between the translucent blue sheet and the sheet described by the green arc. (B) Higher-order intersections and Whitney disks. All arcs are assumed to extend into past and future, describing local sheets of surfaces in a 4–ball.

invariant, tree homology and quasi-Lie algebras, Milnor invariants and gropes, bordism groups of homology cylinders, and quadratic forms.

The next section gives a very quick overview of my papers. The subsequent section gives a paper-by-paper outline, presenting enough details along the way to sketch the development of the main results.

1. Brief research summary

In [1] I defined (relative) algebraic linking invariants for (homologically essential) knots and links in 3–manifolds. These concordance invariants characterize cobounding immersed annuli whose intersections can be paired by Whitney disks. The paper [2] with Peter Teichner defines homotopy invariants for immersed 2–spheres in 4–manifolds with vanishing Wall intersection invariants. The invariants provide embedding obstructions which are defined by counting intersections between Whitney disks and spheres. These invariants were later adapted in [7] to classify stable concordance of knots in many 3–manifolds. An interesting aspect of both [2, 7] is that the indeterminacies in the invariants reflect the topology of both the ambient manifold and the homotopy classes of submanifolds under consideration.

A general obstruction theory for *order n Whitney towers* on immersed surfaces in 4–manifolds was presented in [3], motivated by the idea that Whitney towers represent "approximations" of embeddings of the underlying immersed surfaces. It was also shown in [3] that the (reduced) Kontsevich invariant gives obstructions for links in S^3 to bound higher-order Whitney towers in B^4. For knots in S^3, the classical Arf invariant was shown in [4] to be the only obstruction to bounding order n Whitney towers in B^4 for $n \geq 2$.

Intersections among higher-order Whitney disks can represent obstructions to embedding the underlying immersed surfaces, and the Whitney tower obstruction theory is given in terms of an invariant taking values in abelian groups generated by trivalent trees associated to such intersections. A key step in the development of the theory was the geometric realization of a Jacobi identity (IHX relation) for Whitney tower trees which allows the construction of an order $n+1$

Whitney tower after a controlled homotopy, given the vanishing of the order n intersection invariant. The geometric Jacobi identity construction is described in [6], which also relates Whitney towers to finite-type invariants for string links.

In [5] it was shown that order n Whitney towers are essentially equivalent to class $n+1$ gropes (recall that a *grope* is a "geometric commutator" built by gluing punctured surfaces together along symplectic basis curves). A key subtlety here is that Whitney towers are more "flexible," as reflected by the fact that their trees are unrooted, whereas grope-trees are rooted (with root vertex corresponding to the bottom stage surface). This Whitney tower-grope relationship turns out to play a role in the eventual classification of Whitney towers in B^4 and the connection with Milnor invariants.

The recent series of papers [8, 9, 10, 11, 12] (joint with James Conant and Peter Teichner) describes a classification of order n (twisted) Whitney towers in B^4 bounded by links in S^3 modulo order $n+1$ (twisted) Whitney tower concordance. This classification will be surveyed in some detail in the next section, including some elaboration on relevant points from the above mentioned results, as well as connections with other works. A critical step in the classification involved the computation in [9] of the abelian group generated by labeled vertex-oriented trivalent trees modulo IHX and antisymmetry relations. This group was previously understood only with rational coefficients, and the move to integral coefficients unlocked vital combinatorial/topological information, which also has implications for string links and 3-dimensional homology cylinders, as described in [13].

The classification of Whitney towers in B^4 includes the formulation of higher-order Arf invariants which take values in finite-dimensional $\mathbb{Z}_2$-vector spaces and are obstructions to "un-twisting" a twisted Whitney tower. Although the classification relies on algebraic invariants, especially Milnor invariants [10], all the invariants have combinatorial/geometric formulations, and can be extended to immersed 2–spheres in 4–manifolds. Applications of certain *non-repeating* Whitney towers to the problem of representing homotopy classes of 2–spheres by disjoint maps are presented in [14].

2. Summaries of papers

The following summaries are in rough chronological order, with emphasis given to more recent published results. Some details and background material are included, especially regarding Whitney towers.

Statements are given in the smooth oriented category (with discussions of orientations mostly suppressed), even though all results hold in the locally flat topological category by the basic results on topological immersions in Freedman–Quinn [29] (see [11, Rem.2.1]).

[1] "Algebraic linking numbers of knots in 3–manifolds" *Algebraic and Geometric Topology* 3 (2003) 921–968.

Relative self-linking and linking "numbers" for pairs of oriented knots and 2–component links in oriented 3–manifolds are defined in terms of Wall's μ and λ intersection invariants applied to immersed annuli in 3–manifolds crossed with an interval. The resulting concordance invariants generalize the usual homological notion of linking by taking into account the fundamental group of the ambient manifold and often map onto infinitely generated groups. The knot invariants generalize the type 1 invariants of Kirk and Livingston [37, 38] and when taken with respect to certain preferred knots (which depend on the free homotopy class under consideration) are characterized geometrically as the complete obstruction to the existence of a singular concordance which has all singularities paired by Whitney disks. (This paper was shaped by my Ph.D. dissertation.)

[2] **"Higher order intersection numbers of 2–spheres in 4–manifolds" (with P. Teichner)** ***Algebraic and Geometric Topology* 1 (2001) 1–29.**

A homotopy invariant $\tau(f)$ is defined for a map $f : S^2 \to X$ of a 2–sphere in a 4–manifold X with vanishing Wall self-intersection number $\mu(f)$ by counting intersections between Whitney disks and the sphere in a quotient of the group ring $\mathbb{Z}[\pi_1 X \times \pi_1 X]$ modulo an $\mathcal{S}_3$-symmetry. (Note that $\mu(f)$ takes values in $\mathbb{Z}[\pi_1 X]$ modulo an $\mathcal{S}_2$-symmetry.) The invariant τ is an embedding obstruction which generalizes to the non-simply connected setting the Kervaire-Milnor invariant defined in [29] and [54].

Necessary and sufficient conditions are given for homotoping three maps $f_1, f_2, f_3 : S^2 \to X$ to a position in which they have *disjoint* images. The obstruction $\lambda(f_1, f_2, f_3)$ generalizes Wall's intersection number $\lambda(f_1, f_2)$ which answers the same question for a pair of spheres but is not sufficient (in dimension 4) for a triple. In the same way as intersection numbers correspond to linking numbers in 3–space, this new invariant corresponds to the Milnor invariant $\mu(123)$, generalizing the Matsumoto triple [45] to the non simply-connected setting.

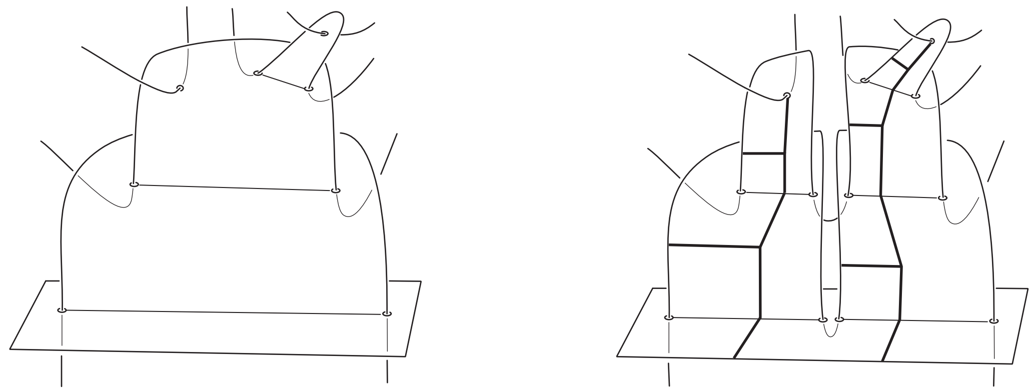
The algebraic properties of these new cubic forms on $\pi_2 X$ are generalizations of the properties of quadratic forms as defined by Wall [57, §5]. For instance, $\lambda(f, f, f) = \sum_{\sigma \in \mathcal{S}_3} \tau(f)^\sigma$ generalizes the well known fact that Wall's invariants satisfy $\lambda(f, f) = \mu(f) + \overline{\mu(f)} = \sum_{\sigma \in \mathcal{S}_2} \mu(f)^\sigma$ for an immersion f with trivial normal bundle.

At this point it was known that the vanishing of τ and λ implied the existence of another "layer" of "higher-order" Whitney disks, but a clear notion of Whitney towers was not yet formulated. The invariant τ would turn out to be the case $n = 1$ of the order n intersection invariants τ_n associated to order n Whitney towers.

[3] **"Whitney towers and the Kontsevich integral" (with P. Teichner)** ***Proceedings of a conference in honor of Andrew Casson*, UT Austin 2003, Geometry and Topology Monograph Series, Vol. 7 (2004) 101–134.**

This paper introduces order n Whitney towers in 4–manifolds, including the intersection/obstruction theory which associates to an order n Whitney tower $\mathcal{W}$ built on a collection A of immersed surfaces in a 4–manifold X an intersection invariant $\tau_n(\mathcal{W}) \in \mathcal{T}_n$, where the abelian group $\mathcal{T}_n$ is generated by labelled vertex-oriented trivalent trees modulo the IHX (Jacobi) and antisymmetry relations well-known from the 3–dimensional theory of finite type invariants. (Figure 3 and Definitions 1–4 below.)

A Whitney tower is constructed recursively starting with A (which by definition has order 0, since there are no Whitney disks), by adjoining Whitney disks pairing up intersections among previously-added Whitney disks and A. Any unpaired intersections determine trivalent trees which bifurcate down through the Whitney tower, with each trivalent vertex contained in a Whitney disk, and each edge a sheet-changing arc joining vertices in adjacent Whitney disks (with univalent vertices lying in the components of A). A Whitney tower is order n if all its associated trees have at least n trivalent vertices. Univalent vertices are labeled by the components of A, and trivalent vertices inherit a cyclic ordering of the adjacent edges from orientations of the Whitney disks.



(A) (B)

Figure 3. (A) A local picture of part of a Whitney tower $\mathcal{W}$. (B) The unpaired intersections determine trivalent trees, and $\mathcal{W}$ can be 'split' so that all singularities are contained in neighborhoods of embeddings of these trees.

These trees which define the invariants are actually "spines" of the Whitney towers, and it can be arranged that all singularities are contained in thickenings of the trees (Figure 3(B)). The relations in the target can be realized by geometric constructions and the main result [3, Thm.2] is that if $\tau_n(\mathcal{W}) = 0 \in \mathcal{T}_n$, then (after a controlled homotopy) the A support a Whitney tower of order $n+1$ (see Theorem 5 below).

The fundamental problem for Whitney towers is to determine exactly the geometric relations needed in the target groups to promote the sufficiency of the vanishing of $\tau_n(\mathcal{W})$ to a necessary condition for the existence of an order $n+1$ Whitney tower on the underlying order 0 immersed surfaces. Taking τ_n in the resulting quotient will then give a homotopy invariant of the order 0 surfaces (which is independent of the choice of Whitney tower). These indeterminacies will in general depend on both the ambient 4–manifold and the order 0 surfaces.

The finite type theory [51] is used to show that, after tensoring with $\mathbb{Q}$, τ_n agrees with (the leading term of the tree part of) the Kontsevich invariant in the case of Whitney towers on immersed disks in the 4–ball bounded by links in the 3–sphere [3, Thm.4]. By work of Habegger–Masbaum [33], this also implies that τ_n rationally computes the first non-vanishing Milnor invariants of links, but this story would not be clarified until later [11].

In this summary, details, notation and terminology are given for Whitney towers in simply connected 4–manifolds only.

Whitney towers.

Definition 1. A *surface of order 0* in an oriented 4–manifold X is a connected oriented surface in X with boundary embedded in the boundary and interior immersed in the interior of X. A *Whitney tower of order 0* is a collection of order 0 surfaces. The *order of a (transverse) intersection point* between a surface of order n and a surface of order m is $n+m$. The *order of a Whitney disk* is $(n+1)$ if it pairs intersection points of order n. For $n \geq 1$, a *Whitney tower of order n* is a Whitney tower $\mathcal{W}$ of order $(n-1)$ together with (immersed) Whitney disks pairing all order $(n-1)$ intersection points of $\mathcal{W}$.

The Whitney disks in a Whitney tower may self-intersect and intersect each other as well as lower order surfaces but the boundaries of all Whitney disks are required to be disjointly embedded. In addition, all Whitney disks are required to be *framed* (see e.g. [11, Sec.2.2]).

Definition 2. All trees are unitrivalent, and *oriented* by cyclic orderings of the edges at all trivalent vertices, with univalent vertices labeled from an index set $\{1, 2, 3, \ldots, m\}$. A *rooted* tree has one unlabeled univalent vertex designated as the *root*. Such rooted trees correspond to formal non-associative bracketings of elements from the index set. The *rooted product* (I, J) of rooted trees I and J is the rooted tree gotten by identifying the root vertices of I and J to a single vertex v and sprouting a new rooted edge at v. This operation corresponds to the formal bracket, and we identify rooted trees with formal brackets. The *inner product* $\langle I, J\rangle$ of rooted trees I and J is the unrooted tree gotten by identifying the roots of I and J to a single non-vertex point. Note that all the univalent vertices of $\langle I, J\rangle$ are labeled.

The *order* of a tree, rooted or unrooted, is defined to be the number of trivalent vertices.

The following associations of trees to Whitney disks and intersection points respects the notion of order given in Definition 1.

To each order zero surface A_i is associated the order zero rooted tree consisting of an edge with one vertex labeled by i, and to each transverse intersection $p \in A_i \cap A_j$ is associated the order zero tree $t_p := \langle i, j\rangle$ consisting of an edge with vertices labeled by i and j. The order 1 rooted Y-tree (i, j), with a single trivalent vertex and two univalent labels i and j, is associated to any Whitney disk $W_{(i,j)}$ pairing intersections between A_i and A_j. This rooted tree can be thought of as being embedded in X, with its trivalent vertex and rooted edge sitting in $W_{(i,j)}$, and its two other edges descending into A_i and A_j as sheet-changing paths. Orientations of trivalent vertices and Whitney disks are related by a convention described in [3, Sec. 3.4].

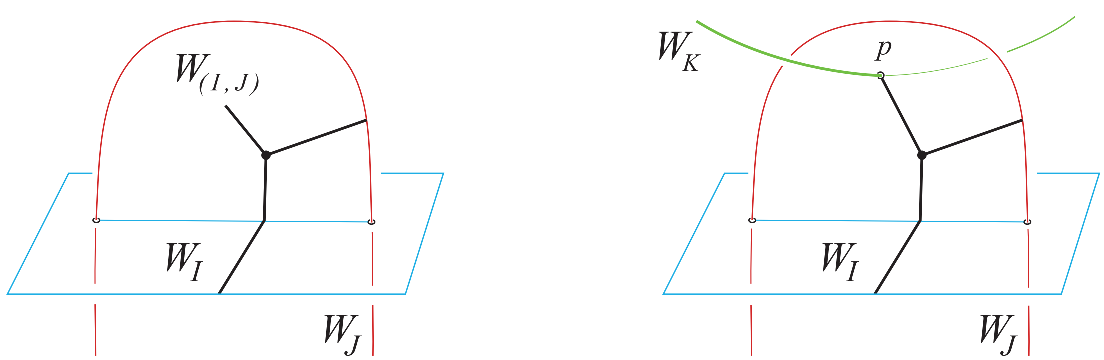
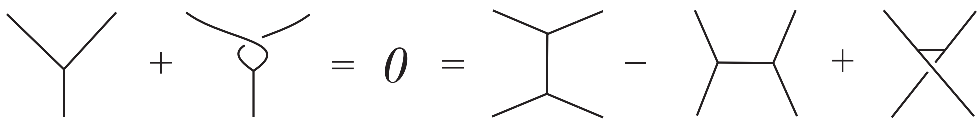


FIGURE 4

Recursively, the rooted tree (I, J) is associated to any Whitney disk $W_{(I,J)}$ pairing intersections between W_I and W_J (see left-hand side of Figure 4); with the understanding that if, say, I is just a singleton i, then W_I denotes the order zero surface A_i. To any transverse intersection $p \in W_{(I,J)} \cap W_K$ between $W_{(I,J)}$ and any W_K is associated the un-rooted tree $t_p := \langle (I, J), K\rangle$ (see right-hand side of Figure 4). Figure 6 shows an explicit example of a Whitney tower in B^4 bounded by a link in S^3.

Definition 3. The group $\mathcal{T}_n$ (for each $n = 0, 1, 2 \ldots$) is the free abelian group on (unitrivalent labeled vertex-oriented) order n trees, modulo the AS (antisymmetry) and IHX (Jacobi) local relations:



The obstruction theory works as follows:

Definition 4. The *order n intersection invariant* $\tau_n(\mathcal{W})$ of an order n Whitney tower $\mathcal{W}$ is defined to be

$$\tau_n(\mathcal{W}) := \sum \epsilon_p \cdot t_p \in \mathcal{T}_n$$

where the sum is over all order n intersections p, with $\epsilon_p = \pm 1$ the usual sign of a transverse intersection point.

(The invariant τ_n was actually called the order n intersection "tree" in [3]; the more recent papers use the more appropriate "invariant".)

All relations in $\mathcal{T}_n$ can be realized by controlled manipulations of Whitney towers, and further maneuvers allow algebraically canceling pairs of trees to be converted into intersection-point pairs admitting Whitney disks. As a result, we get the following partial recovery of the "algebraic cancellation implies geometric cancellation" principle available in higher dimensions:

Theorem 5. *If a collection A of properly immersed surfaces in a simply connected 4–manifold supports an order n Whitney tower $\mathcal{W}$ with $\tau_n(\mathcal{W}) = 0 \in \mathcal{T}_n$, then A is regularly homotopic (rel ∂) to A' which supports an order $n+1$ Whitney tower.*

The analogous result without the assumption that X is simply connected is Theorem 2 of [3]. In the general setting, tree edges are also decorated with elements of $\pi_1 X$, and there are additional relations in the target group. These relations reduce to the above AS and IHX relations for $\pi_1 X$ trivial.

[4] **"Simple Whitney towers, half-gropes and the Arf invariant of a knot"**
***Pacific Journal of Mathematics* Vol. 222, No. 1, Nov (2005) 169–184.**

This paper gives a geometric characterization of the classical Arf invariant of a knot in the 3–sphere in terms of bordism by certain *simple Whitney towers* and *half-gropes*, which correspond to right- or left-normed iterated commutators (called *simple commutators* in [46]). It is shown constructively (by geometrically manipulating framing obstructions on higher-order Whitney disks) that the Arf invariant is exactly the obstruction to cobording pairs of knots by half-gropes and simple Whitney towers in $S^3 \times I$ of arbitrarily high class and order, respectively. In particular, a knot $K \subset S^3$ bounds an order n Whitney tower or a class n grope in B^4 for all n if and only if K has vanishing classical Arf invariant.

This illustrates geometrically how, in the setting of knot *concordance*, the Vassiliev (isotopy) invariants (which are known to correspond to 3-dimensional grope-cobordism [24, 25]) "collapse" to the Arf invariant. Since the classical Arf invariant is the mod 2 reduction of the simplest non-trivial Vassiliev invariant, this integer-valued isotopy invariant can interpreted as the obstruction to "pushing this construction down into 3–space".

On the other hand, there is a highly non-trivial filtration of knot concordance by *symmetric* Whitney towers (graded by *height*) which are closely related to the notion of n-solvability introduced in [21, 22, 23] (see also "**Comparisons with other iterated disk constructions**" below). Thus, the signature invariants which obstruct n-solvability of knots can interpreted as obstructions to "symmetrizing" the construction of this paper.

[5] "Whitney towers and Gropes in 4–manifolds" *Transactions of the American Mathematical Society* 358 (2006) 4251–4278.

This paper describes a precise correspondence between order n Whitney towers and class $n+1$ embedded gropes in 4–manifolds, in particular showing how one can be locally converted into the other, and vice versa. The "flexibility" of Whitney towers is used to demonstrate some geometric consequences for knot and link concordance connected to n-solvability [21] ("embedded height n grope implies height n Whitney tower" – the converse is not known), k-cobordism [35] ("class $2k$ grope concordance implies k-cobordism") and grope concordance ("half-gropes generate grope concordance"). A key observation is that the unitrivalent trees associated to gropes and Whitney towers can can be preserved during the surgeries and Whitney moves which convert one to the other. In particular, the conversion of a Whitney tower to a grope only involves a choice of preferred *root* univalent vertex on each tree, giving a geometric interpretation of a well-known map from trees to commutators that is used in the classification of Whitney towers in the 4–ball (compare the η'- and η-maps of [9, 10] described below).

[6] "Jacobi identities in low-dimensional topology" (with J. Conant and P. Teichner) *Compositio Mathematica* 143 Part 3 (2007) 780–810.

This paper exposes the underlying topological unity between the 3- and 4-dimensional IHX-relations, deriving from a picture, Figure 5, of the Borromean rings embedded on the boundary of an unknotted genus 3 handlebody in 3–space. Interpreted as sitting in a 3-dimensional slice of 4–space, this picture leads to the construction of the three trees of an IHX relator (Definition 3 above) in a Whitney tower on a quadruple of 2–spheres in 4–space. By tubing such 2–spheres into Whitney disks in a Whitney tower this allows for the controlled geometric realization of any IHX relation, a key step in the obstruction theory order-raising theorem of [3] Theorem 5). The analogous relation for knot, string link and 3–manifold invariants is described via grope cobordisms and claspers. (This 3-dimensional direction is pursued further in [13].)

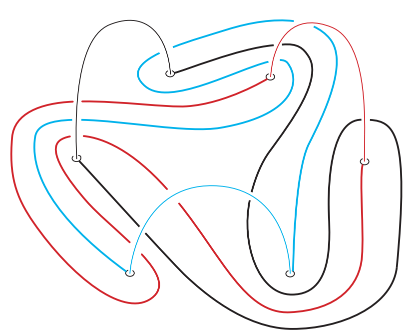


Figure 5. The geometric origin of the 4-dimensional Jacobi identity.

[7] **"Stable concordance of knots in 3–manifolds"** ***Algebraic and Geometric Topology*** **10 (2010) 373–432.**

Building on the notions of [1] and [2], this paper applies a variation of the order 1 invariant τ_1 to define concordance invariants of knots and links in 3–manifolds which generalize the Arf invariant, the mod 2 Sato–Levine invariants, and Milnor's triple linking numbers. Besides fitting into the general theory of Whitney towers, these (relative) invariants provide obstructions to the existence of a singular concordance in the product $M \times I$ of a 3–manifold M with an interval which can be homotoped to an embedding after stabilization by connected sums with $S^2 \times S^2$. Results include classifications of stably slice links in orientable 3–manifolds, stable knot concordance in products of an orientable surface with the circle, and stable link concordance for many links of null-homotopic knots in orientable 3–manifolds. An interesting aspect here is that the indeterminacies in the invariants generally depend on both the order 0 invariants and the topology of the 3-manifold, especially the existence of non-orientable base surfaces and singular fibers in Seifert fibered characteristic submanifolds.

[8] **"Higher-order intersections in low-dimensional topology"** **(with J. Conant and P. Teichner)** ***Proceedings of the National Academy of Sciences USA*** **2011 108 (20) 8081–8084.**

This paper surveys the recent classification of Whitney towers in the 4–ball as detailed in [9, 10, 11, 12] (summarized below) and touches on the related results for homology cylinders in [13] (see also below). It had become clear from the above summarized papers (as well as preliminary work on [14] below) that further progress on the general theory of Whitney towers would depend on first understanding the setting of Whitney towers on immersed disks in the 4–ball bounded by links in the 3–sphere. This classification represents several years of work, and is complete modulo computation of the image (within known bounds) of certain *higher-order Arf invariants*, which are conjectured to be new concordance invariants (that can also be formulated for 2–spheres in 4–manifolds). As discussed next, the main developments include the computation of the target groups $\mathcal{T}_n$ using discrete Morse theory for chain complexes with torsion [9], the geometric interpretation of Milnor invariants in terms of twisted Whitney towers [10], the extension of the obstruction theory to twisted Whitney towers and the formulation of the higher-order Sato-Levine and Arf invariants [11], and the interpretation of the target for the twisted Whitney tower invariants as a quadratic refinement of the intersection pairing for framed Whitney disks [12].

[9] **"Tree homology and a conjecture of Levine"** **(with J. Conant and P. Teichner)** ***Geometry and Topology*** **16 (2012) 55–600.**

As an important first step towards the 4–ball Whitney tower classification, this paper computes the groups $\mathcal{T}_n$ for all n, confirming a conjecture of J. Levine which was formulated during his study of 3-dimensional homology cylinders.

Definition 6. Let $\mathsf{L} = \mathsf{L}(m)$ denote the free Lie algebra (over the ground ring $\mathbb{Z}$) on generators $\{X_1, X_2, \ldots, X_m\}$. It is $\mathbb{N}$-graded, $\mathsf{L} = \oplus_n \mathsf{L}_n$, where the degree n part L_n is the additive abelian group of length n brackets, modulo Jacobi identities and the self-annihilation relations $[X, X] = 0$. The free *quasi-Lie algebra* L' is gotten from L by replacing the self-annihilation relations with the weaker anti-symmetry relations $[X, Y] = -[Y, X]$. Note that L' can be identified with the abelian group on *rooted* trees (unitrivalent, oriented and labeled as in Definition 2) modulo IHX and antisymmetry relations.

The bracketing map $\mathsf{L}_1 \otimes \mathsf{L}_{n+1} \to \mathsf{L}_{n+2}$, has a nontrivial kernel, denoted D_n. The analogous bracketing map on the free quasi-Lie algebra is denoted D'_n.

Levine studied a natural map $\eta'_n : \mathcal{T}_n \to \mathsf{D}'_n$ defined as follows. For v a univalent vertex of an order n tree t, denote by $B'_v(t) \in \mathsf{L}'_{n+1}$ the quasi-Lie bracket of generators $X_1, X_2, \ldots, X_m$ determined by the formal bracketing of indices which is gotten by considering v to be a root of t.

Definition 7. Denoting the label of a univalent vertex v by $\ell(v) \in \{1, 2, \ldots, m\}$, the map $\eta'_n : \mathcal{T}_n \to \mathsf{L}'_1 \otimes \mathsf{L}'_{n+1}$ is defined on generators by

$$\eta'_n(t) := \sum_{v \in t} X_{\ell(v)} \otimes B'_v(t)$$

where the sum is over all univalent vertices v of t.

The Lie bracket map kernel D_n is relevant to a variety of topological settings (see e.g. the introduction to [9]) and was known to be isomorphic to $\mathcal{T}_n$ after tensoring with $\mathbb{Q}$ when Levine's study of the cobordism groups of 3-dimensional homology cylinders [42, 43] led him to conjecture that $\mathcal{T}_n$ is in fact isomorphic to the quasi-Lie bracket map kernel D'_n, via the map η'_n. Levine made progress in [43, 44], and in theorems 1.1 and 1.4 of this paper we affirm his conjecture:

Theorem 8. *$\eta'_n : \mathcal{T}_n \to \mathsf{D}'_n$ is an isomorphism for all n.*

The proof of Theorem 8 uses techniques from discrete Morse theory on chain complexes [26, 40], including an extension of the theory to complexes containing torsion. A key idea involves defining combinatorial vector fields that are inspired by the Hall basis algorithm for free Lie algebras and its generalization by Levine to quasi-Lie algebras.

Via Levine's description of the structure of L'_n and D'_n from [44], Theorem 8 gives the following useful corollary:

Corollary 9 ([9] Cor 1.2)**.** *The groups $\mathcal{T}_{2k}$ are free abelian (of known rank) and the torsion in $\mathcal{T}_{2k+1}$ is generated by symmetric trees of the form $i \!-\!\!<^{J}_{J}$ where J has order k.*

As described below, this result will play an essential role in both the classification of Whitney towers in B^4 [11] and the extension [13] of Levine's study of homology bordism groups of 3-dimensional homology cylinders.

[10] "Milnor Invariants and Twisted Whitney Towers" (with J. Conant and P. Teichner) *Journal of Topology* 7 no. 1 (2014) 187–224. http://arxiv.org/abs/1102.0758.

The main result of this paper describes a precise correspondence between the Milnor invariants of links in S^3 and the intersection invariants of certain *twisted* Whitney towers in B^4. The (first non-vanishing) Milnor μ-invariants [49] inductively measure the link longitudes as iterated commutators in the lower central quotients of the link group. Given that gropes are geometric embodiments of commutators, and that Whitney towers and gropes are essentially equivalent, one might expect a close correspondence between Milnor invariants and the Whitney tower obstruction theory. However, the classical Arf invariant of a knot shows that Milnor invariants will not provide complete obstructions to the existence of Whitney towers (by [5] and the fact that Milnor invariants vanish on knots). It turns out that appropriately weakening the framing requirement on certain Whitney disks in a Whitney tower does indeed capture the geometry of both the Milnor and Arf invariants:

Definition 10. A *twisted Whitney tower of order* 0 is a collection of properly immersed surfaces in a 4–manifold (without any framing requirement).

For $k > 0$, a *twisted Whitney tower of order* $2k - 1$ is just a (framed) Whitney tower of order $2k - 1$ as in Definition 1 above.

For $k > 0$, a *twisted Whitney tower of order* $2k$ is a Whitney tower having all intersections of order less than $2k$ paired by Whitney disks, with all Whitney disks of order less than k required to be framed, but Whitney disks of order at least k allowed to be twisted.

Here *twisted* Whitney disks are just Whitney disks without the framing requirement [11, Sec.2.2]. Special "twisted" trees are assigned to the twisted Whitney disks in a twisted Whitney tower as follows. If W_J is a twisted Whitney disk with associated rooted tree J (Definition 2), then the twisted ∞*-tree* denoted by J^∞ associated to W_J is gotten from J by labeling the root with the twist-symbol "∞":

$$J^\infty := \boldsymbol{\infty}\!-\!\!-J$$

The obstruction theory of Theorem 5 is extended to twisted Whitney towers in [11] by including such trees into a quadratic refinement $\mathcal{T}_n^\infty$ of the untwisted tree groups [12], and defining an intersection invariant $\tau_n^\infty \in \mathcal{T}_n^\infty$ which sums over all order n (untwisted) trees and (if n is even) all twisted trees of order $n/2$.

The connection to Milnor invariants is described using a variation of the η'-map in Definition 7:

Theorem 11. [10, Thm.5] *If L bounds a twisted Whitney tower $\mathcal{W}$ of order n, then the order k Milnor invariants $\mu_k(L)$ vanish for $k < n$ and*

$$\mu_n(L) = \eta_n \circ \tau_n^\infty(\mathcal{W}) \in \mathsf{D}_n$$

Here the map $\eta_n : \mathcal{T}_n \to \mathsf{D}_n$ is defined on (untwisted) trees analogously to the sum-over-all-choices-of-root η'_n-map above, and extended to ∞-trees via $\eta_n(J^\infty) := \frac{1}{2}\eta_n(\langle J, J\rangle)$ which lies in $\mathsf{L}_1 \otimes \mathsf{L}_{n+1}$ because the coefficient of $\eta_n(\langle J, J\rangle)$ is even. It turns out that η_n maps $\mathcal{T}_n^\infty$ onto D_n. The order n Milnor invariant $\mu_n(L)$ corresponds to all the *length* $n + 2$ Milnor invariants of L in the traditional indexing, and the group D_n is free abelian of known rank equal to the number of independent first non-vanishing length $n + 2$ Milnor invariants [52].

In [3] the above result was shown for framed Whitney towers, using a translation into claspers together with the Habegger-Masbaum identification of the Milnor invariants with the tree part of the Kontsevich invariant [33]. This roundabout argument is replaced here by a very direct geometric one, using the notion of *grope duality* from [41] and the resolution of a Whitney tower to a grope described in [4]. It shows clearly the relationship between higher-order intersections and the iterated commutators determined by the link longitudes, as expressed algebraically by the map η, and also works for twisted Whitney towers. The proof explains why twisting is allowed in half-order Whitney disks and sheds light on the geometry behind Habegger and Masbaum's computation of the image of the first non-vanishing Milnor invariants as a lattice in the tree-subspace of Feynman diagrams [33, Sec.8]. In particular, the coefficients of 1/2 on certain symmetric trees in the image lattice correspond to the effect of "reflecting" iterated commutators which is provided by twisted Whitney disks of order $n/2$ in an order n twisted Whitney tower.

The twisted Whitney tower-Milnor invariant correspondence plays a role the classification of Whitney towers in the 4–ball [11], as well giving some new geometric characterizations of Milnor invariants [10, Sec.1.6]. We note here one geometric characterization in the setting of k-slice links:

k-slice links: Recall (e.g. from [56]) that a *grope* of *class* k is defined recursively as follows: A grope of class 1 is a circle. A grope of class 2 is an orientable surface Σ with one boundary component. A grope of class k is formed by attaching to every dual pair of curves in a symplectic basis for Σ a pair of gropes whose classes add to k.

Gropes are "geometric embodiments" of iterated commutators in the sense that a loop in a topological space represents a k-fold commutator in the fundamental group if and only if it extends to a continuous map of a grope of class k. Since Milnor invariants measure how deeply the link longitudes extend into the lower central series of the link group, Milnor invariants obstruct bounding *immersed* gropes essentially by definition. On the other hand, extracting information on bounding *embedded* gropes from the vanishing of Milnor invariants is much more difficult. Embedded framed gropes have usefully served as "approximations" to embedded disks in many topological settings (see e.g. [56]).

Perhaps the most notable previously known geometric "if and only if" characterization of Milnor invariants is the *k-slice Theorem*, due to K. Igusa and K. Orr: Expressed in the language of gropes, a link $L \subset S^3$ is said to be *k-slice* if the link components L_i bound disjointly embedded (oriented) surfaces $\Sigma_i \subset B^4$ such that a symplectic basis of curves on each Σ_i bound class k gropes immersed in the complement of $\Sigma := \cup_i \Sigma_i$. Via a very careful analysis of the third homology of the nilpotent quotients F/F_k of the (rank m) free group F, Igusa and Orr [35] proved the following difficult result.

Theorem 12 ([35])**.** *A link L is k-slice if and only if $\mu_i(L) = 0$ for all $i \leq 2k-2$ (equivalently, all Milnor invariants of length $\leq 2k$ vanish).*

The k-slice condition says that the link components bound certain immersed gropes in B^4 whose embedded bottom stage surfaces are "algebraic approximations" of slice disks modulo the kth term of the lower central series of the link group.

Via results in [9, 11] we have the following geometric improvement:

Theorem 13 ([10])**.** *A link $L = \cup_i L_i$ has $\mu_i(L) = 0$ for all $i \leq 2k-2$ if and only if the link components L_i bound disjointly embedded surfaces Σ_i in the 4–ball, with each surface a connected sum of two surfaces Σ_i' and Σ_i'' such that*

(i) *a symplectic basis of curves on Σ_i' bound disjointly embedded framed gropes $G_{i,j}$ of class k in the complement of $\Sigma := \cup_i \Sigma_i$, and*
(ii) *a symplectic basis of curves on Σ_i'' bound immersed disks in the complement of $\Sigma \cup G$, where G is the union of all $G_{i,j}$.*

Theorem 13 is a considerable strengthening of the above Igusa-Orr k-slice Theorem: Since the geometric conditions in both theorems are equivalent to the vanishing of Milnor's invariants through order $2k-2$ (length $2k$), one can read this result as saying that the *immersed gropes* of class k found by Igusa and Orr can be cleaned up to immersed *disks* (these are immersed gropes of arbitrarily high class) or *embedded* gropes of class k. As explained next, certain *higher-order Arf invariants* are exactly the obstructions to eliminating the need for the Σ_i'' and these immersed disks.

[11] "Whitney tower concordance of classical links" (with J. Conant and P. Teichner) *Geometry and Topology* 16 (2012) 1419–1479.

The main goal of this paper is to provide an answer to the following question for any given n: "Which links in the 3–sphere bound an order n Whitney tower in the 4–ball?" The answer to this question is roughly summarized by the following theorem (compare Corollary 27):

Theorem 14. *A link bounds a Whitney tower of order n if and only if its Milnor invariants, higher-order Sato-Levine invariants and higher-order Arf invariants vanish up to order n.*

These higher-order Sato-Levine and Arf invariants turn out to be exactly the obstructions to converting twisted Whitney towers bounded by links to framed Whitney towers, as will be sketched below (closely following the introduction of [11]).

To explain this result, start by defining the *Whitney tower filtration*:

$$\cdots \subseteq \mathbb{W}_3 \subseteq \mathbb{W}_2 \subseteq \mathbb{W}_1 \subseteq \mathbb{W}_0 \subseteq \mathbb{L}$$

on the set $\mathbb{L} = \mathbb{L}(m)$ of m-component framed links in S^3. Here $\mathbb{W}_n = \mathbb{W}_n(m)$ is the subset of those framed links that bound immersed disks supporting order n (framed) Whitney towers in B^4.

This filtration factors through link concordance, and the intersection of all $\mathbb{W}_n$ contains all slice links since a properly embedded 2–disk is a Whitney tower of arbitrarily large order.

Whitney towers built on immersed annuli connecting link components in $S^3 \times I$ induce equivalence relations of *Whitney tower concordance* on links. The quotient W_n of $\mathbb{W}_n$ modulo the equivalence relation of Whitney tower concordance of order $n+1$ is the *associated graded* of the filtration in the sense that $L \in \mathbb{W}_{n+1}$ if and only if $L \in \mathbb{W}_n$ and $[L] = 0 \in \mathsf{W}_n$.

The Whitney tower obstruction theory leads to:

Theorem 15 ([11] Thm.1.3)**.** *The sets W_n are finitely generated abelian groups under the (well-defined) operation of band sum, and there are* realization *epimorphisms $R_n : \mathcal{T}_n \twoheadrightarrow \mathsf{W}_n$.*

These realization maps R_n are defined similarly to T. Cochran's iterated Bing-doubling construction for realizing Milnor invariants [18, 19], and are equivalent to "simple clasper surgery along trees" in the sense of Goussarov [31] and Habiro [34] (see Figure 6 for an example).

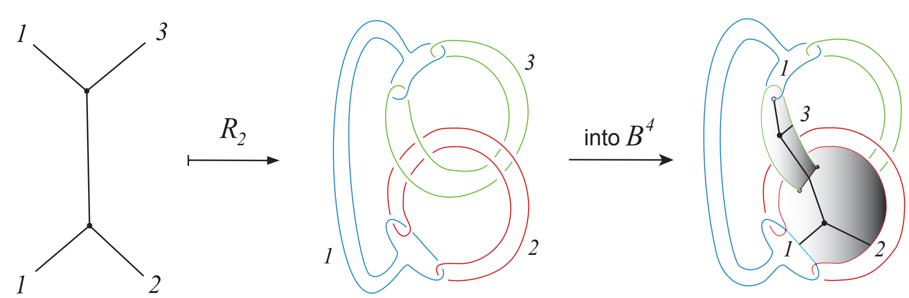


FIGURE 6. The realization map R_2 sends the tree t on the left to the link $L \subset S^3$ shown in the middle. The trace of a null-homotopy of L described by a pair of crossing-changes between the blue component 1 and the red component 2 supports an order 2 Whitney tower $\mathcal{W} \subset B^4$ bounded by L, with $\tau_2(\mathcal{W}) = t$, as shown on the right. (Pushing further into B^4 would show a 3-component unlink bounding disjointly embedded disks).

The following result follows from the Milnor invariant-Whitney tower relationship and the affirmation of the Levine Conjecture:

Theorem 16 ([11] Thm.1.4)**.** *In all even orders, the realization maps $R_{2k} : \mathcal{T}_{2k} \to \mathsf{W}_{2k}$ are isomorphisms and W_{2k} are free abelian groups of known rank, detected by Milnor invariants.*

The affirmation of Levine's conjecture also implies that the torsion in $\mathcal{T}_{2k-1}$ is generated by symmetric trees of the form $i -\!\!<^{J}_{J}$, where J is a subtree of order $k-1$, and i is a univalent vertex label (Corollary 9). These trees are actually 2-torsion by the antisymmetry relation and hence all torsion in $\mathcal{T}$ is 2-torsion. The next result shows that a large part of this 2-torsion actually maps trivially to W_{2k-1}.

Theorem 17 ([11] Thm.1.5)**.** *The realization maps R_{2k-1} factor through a quotient $\widetilde{\mathcal{T}}_{2k-1}$ of $\mathcal{T}_{2k-1}$.*

The Whitney tower obstruction theory also descends to these *reduced groups* $\widetilde{\mathcal{T}}_{2k-1}$:

Definition 18. Let $\widetilde{\mathcal{T}}_{2k-1} := \mathcal{T}_{2k-1}/\operatorname{Im}\Delta_{2k-1}$, where $\Delta_{2k-1} \colon \mathcal{T}_{k-1} \to \mathcal{T}_{2k-1}$ is defined on generators t of order $k-1$ as follows. For any univalent vertex v of t, denote by $\ell(v)$ the label of v, and write $t = \ell(v) \text{ --- } T_v(t)$. Then we get a 2-torsion element of $\mathcal{T}_{2k-1}$ defined by

$$\Delta_{2k-1}(t) := \sum_v \ell(v) -\!\!<^{T_v(t)}_{T_v(t)}$$

where the sum is over all univalent vertices v of t.

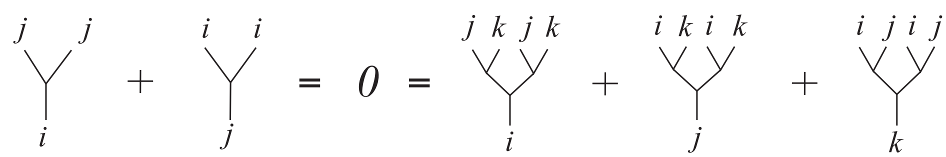


FIGURE 7. The framing relations in orders 1 and 3.

The relations $\Delta_{2k-1}(t) = 0$ are called *framing relations* because they correspond to the image of *twisted* IHX relations in a *twisted Whitney tower* via a conversion to a framed Whitney tower [11, Sec.4.4].

Conjecturally, all odd order *reduced realization maps* $\widetilde{R}_{2k-1} : \widetilde{\mathcal{T}}_{2k-1} \to \mathsf{W}_{2k-1}$ are isomorphisms, and the following theorem confirms this in half of the cases:

Theorem 19 ([11] Thm.1.7)**.** *The reduced realization maps $\widetilde{R}_{4k-1}$ are isomorphisms and the torsion of W_{4k-1} is a $\mathbb{Z}_2$-vector space of known dimension, detected by higher order Sato-Levine invariants.*

The *higher-order Sato-Levine invariants* are certain projections of Milnor invariants, shifted down one order. They represent obstructions to framing a twisted Whitney tower, as explained in [11, Sec.5]. In order to sketch the proof of Theorem 19, we next introduce the *twisted Whitney tower filtration*, and explain how *higher-order Arf invariants* play a role in completing the classifications of both the twisted and framed filtrations.

The twisted Whitney tower filtration. Denote by $\mathbb{W}^{\infty}_n = \mathbb{W}^{\infty}_n(m)$ the set of framed m-component links that bound immersed disks supporting order n *twisted* Whitney towers (Definition 10), and by W^{∞}_n the associated graded, defined as the quotient by order $n+1$ twisted Whitney tower concordance. This gives the *twisted Whitney tower filtration*:

$$\cdots \subseteq \mathbb{W}^{\infty}_3 \subseteq \mathbb{W}^{\infty}_2 \subseteq \mathbb{W}^{\infty}_1 \subseteq \mathbb{W}^{\infty}_0 = \mathbb{L}$$

As stated in general by Theorem 2.10 of [11], the order-raising obstruction theory (Theorem 5 above) also holds for the *twisted intersection invariant* $\tau^{\infty}_n(\mathcal{W}) \in \mathcal{T}^{\infty}_n$. Briefly, the odd order groups $\mathcal{T}^{\infty}_{2k-1}$ are defined as quotients of $\mathcal{T}_{2k-1}$ by the torsion subgroups, generated by trees of the form $i \,-\!\!<^{J}_{J}$; where J is a subtree of order $k-1$, and i is a univalent vertex label. These *boundary-twist relations* correspond to the intersections created by performing a boundary-twist on an order k Whitney disk. In even orders, the twisted trees $J^{\infty} = \infty\!-\!\!-J$ in $\mathcal{T}^{\infty}_{2k}$ which represent framing obstructions on order k Whitney disks are involved in new *symmetry*, *twisted IHX*, and *interior twist* relations (see summary of [12] below), all of which have geometric interpretations [11, Def.2.8].

As a consequence of the twisted obstruction theory [11, Thm.1.9] and an extension of the realization maps to twisted trees we have:

Theorem 20 ([11] Thm.1.8)**.** *The sets W^{∞}_n are finitely generated abelian groups under the (well-defined) operation of connected sum $\#$ and there are epimorphisms $R^{\infty}_n : \mathcal{T}^{\infty}_n \twoheadrightarrow \mathsf{W}^{\infty}_n$.*

From the main result of [10] we get the following commutative triangle:

Corollary 21 ([11] Cor.1.12)**.** *There is a commutative diagram of epimorphisms*

$$\begin{tikzcd} \mathcal{T}^{\infty}_n \arrow[r, two heads, "R^{\infty}_n"] \arrow[dr, two heads, "\eta_n"'] & \mathsf{W}^{\infty}_n \arrow[d, two heads, "\mu_n"] \\ & \mathsf{D}_n \end{tikzcd}$$

The affirmation of the Levine Conjecture [9] implies that $\eta_n : \mathcal{T}^{\infty}_n \to \mathsf{D}_n$ is an *isomorphism* except when $n \equiv 2 \mod 4$, so the computation of W^{∞}_n in three quarters of the cases is complete (in terms of the known group D_n):

Theorem 22 ([11] Thm.1.13)**.** *If $n \not\equiv 2 \mod 4$, the maps R^{∞}_n and μ_n give rise to isomorphisms*

$$\mathcal{T}^{\infty}_n \cong \mathsf{W}^{\infty}_n \cong \mathsf{D}_n$$

The main result from [9] also gives a complete understanding of the kernel of the combinatorial side of the above triangle of maps for all $n \equiv 2 \mod 4$:

Proposition 23 ([11] Prop.1.14)**.** *The map sending $1 \otimes J$ to $\infty\!-\!\!<^{J}_{J} \in \mathcal{T}^{\infty}_{4k-2}$ for rooted trees J of order $k-1$ defines an isomorphism $\mathbb{Z}_2 \otimes \mathsf{L}_k \cong \operatorname{Ker}(\eta_{4k-2} : \mathcal{T}^{\infty}_{4k-2} \to \mathsf{D}_{4k-2})$.*

It follows from Corollary 21 that $\mathbb{Z}_2 \otimes \mathsf{L}_k$ is also an upper bound on the kernels of the epimorphisms R^{∞}_{4k-2} and μ_{4k-2}, and the calculation of $\mathsf{W}^{\infty}_{4k-2}$ will be completed by invariants defined on the kernel of μ_{4k-2} which are concordance invariants generalizing the classical Arf invariant, as described next.

Higher-order Arf invariants. Let $\mathsf{K}^{\infty}_{4k-2}$ denote the kernel of $\mu_{4k-2} : \mathsf{W}^{\infty}_{4k-2} \twoheadrightarrow \mathsf{D}_{4k-2}$. It follows from Corollary 21 and Proposition 23 above that mapping $1 \otimes J$ to $R^{\infty}_{4k-2}(\infty\!-\!\!\!<^{J}_{J})$ induces a surjection $\alpha^{\infty}_{k} : \mathbb{Z}_2 \otimes \mathsf{L}_k \twoheadrightarrow \mathsf{K}^{\infty}_{4k-2}$, for all $k \geq 1$. Denote by $\overline{\alpha^{\infty}_{k}}$ the induced isomorphism on $(\mathbb{Z}_2 \otimes \mathsf{L}_k)/\operatorname{Ker}\alpha^{\infty}_{k}$.

Definition 24 ([11] Def.1.15)**.** The *higher-order Arf invariants* are defined by

$$\operatorname{Arf}_k := (\overline{\alpha^{\infty}_{k}})^{-1} : \mathsf{K}^{\infty}_{4k-2} \to (\mathbb{Z}_2 \otimes \mathsf{L}_k)/\operatorname{Ker}\alpha^{\infty}_{k}$$

From Corollary 21, Theorem 22, Proposition 23 and Definition 24 we see that the groups W^{∞}_{n} are computed by the Milnor and higher-order Arf invariants:

Corollary 25 ([11] Cor.1.16)**.** *The groups* W^{∞}_{n} *are classified by Milnor invariants* μ_n *and, in addition, higher-order Arf invariants* Arf_k *for* $n = 4k-2$*.*

In particular, it follows that a link bounds an order $n+1$ twisted Whitney tower if and only if its Milnor invariants and higher-order Arf invariants vanish up to order n.

We conjecture that the α^{∞}_{k} are isomorphisms, which would mean that the Arf_k are very interesting new concordance invariants:

Conjecture 26 ([11] Conj.1.17)**.** $\operatorname{Arf}_k : \mathsf{K}^{\infty}_{4k-2} \to \mathbb{Z}_2 \otimes \mathsf{L}_k$ *are isomorphisms for all* k*.*

Conjecture 26 would imply that $\mathsf{W}^{\infty}_{4k-2} \cong \mathcal{T}^{\infty}_{4k-2} \cong (\mathbb{Z}_2 \otimes \mathsf{L}_k) \oplus \mathsf{D}_{4k-2}$ where the second isomorphism (is non-canonical and) already follows from Proposition 23. Conjecture 26 is true for $k = 1$, with Arf_1 given by the classical Arf invariants of the link components [10, Lem.9]. It remains an open problem whether Arf_k is non-trivial for any $k > 1$. The links $R^{\infty}_{4k-2}(\infty\!-\!\!\!<^{J}_{J})$ realizing the image of Arf_k can all be constructed as internal band sums of iterated Bing doubles of knots having non-trivial classical Arf invariant [10, Lem.12]. Such links are known not to be slice by work of J.C. Cha [17], providing evidence in support of Conjecture 26.

In combination with Theorem 22, Conjecture 26 can be succinctly expressed in terms of the twisted Whitney tower filtration classification as the statement: "the twisted realization maps $R^{\infty}_{n} : \mathcal{T}^{\infty}_{n} \to \mathsf{W}^{\infty}_{n}$ are isomorphisms for all n."

A table of the groups $\mathsf{W}^{\infty}_{n}(m)$ for low values of n, m is given in Figure 8, where the higher-order Arf invariant Arf_2 appears in order 6. The currently unknown ranks of Arf_2 are represented by the ranges of possible ranks of the 2-torsion subgroups of the groups $\mathsf{W}^{\infty}_{6}(m)$.

For $n = 0$, the groups are freely generated by the image under R^{∞}_{0} of trees $i \,—\, j$, with $i \neq j$, and twisted trees $\infty \,—\, j$. The resulting links are detected by linking numbers and framings, respectively. For order $n = 1$, the generators come (via R^{∞}_{1}) from trees $i\!-\!\!\!<^{j}_{k}$ where all indices are distinct (otherwise the tree is zero in $\mathcal{T}^{\infty}_{1}$ by the boundary-twist relations). They are detected by Milnor's triple invariants $\mu(ijk)$.

In order $n = 2$, generators include (R^{∞}_{2} of) ∞-trees $\infty\!-\!\!\!<^{i}_{j}$ (recall that these indeed lie in $\mathcal{T}^{\infty}_{2}$ even though the tree has only one trivalent vertex). If $i \neq j$ these are of infinite order, detected by Milnor's $\mu(ijij)$, but for $i = j$ they have order 2 and are detected by the classical Arf invariant of the ith component. This shows how the groups $\mathcal{T}^{\infty}_{4k-2}$ combine Milnor and Arf invariants in one new formalism.

Framing twisted Whitney towers. As explained in Section 5 of [11], the translation of the classification of the twisted Whitney tower filtration back into the framed setting is accomplished using a new interpretation of certain first non-vanishing Milnor invariants as obstructions to

	1	2	3	4	5
0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^6$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{15}$
1	0	0	$\mathbb{Z}$	$\mathbb{Z}^4$	$\mathbb{Z}^{10}$
2	$\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}_2^2$	$\mathbb{Z}^6\oplus\mathbb{Z}_2^3$	$\mathbb{Z}^{20}\oplus\mathbb{Z}_2^4$	$\mathbb{Z}^{50}\oplus\mathbb{Z}_2^5$
3	0	0	$\mathbb{Z}^6$	$\mathbb{Z}^{36}$	$\mathbb{Z}^{126}$
4	0	$\mathbb{Z}^3$	$\mathbb{Z}^{28}$	$\mathbb{Z}^{146}$	$\mathbb{Z}^{540}$
5	0	0	$\mathbb{Z}^{36}$	$\mathbb{Z}^{340}$	$\mathbb{Z}^{1740}$
6	0	$\mathbb{Z}^6\oplus\mathbb{Z}_2^{e_2}$	$\mathbb{Z}^{126}\oplus\mathbb{Z}_2^{e_3}$	$\mathbb{Z}^{1200}\oplus\mathbb{Z}_2^{e_4}$	$\mathbb{Z}^{7050}\oplus\mathbb{Z}_2^{e_5}$

FIGURE 8. A table of the groups $\mathsf{W}^\infty_n(m)$, where m runs horizontally and n runs vertically. The possible ranges of the torsion exponents in order 6 depend on the currently unknown ranks of Arf_2: $0 \leq e_2 \leq 1$, $0 \leq e_3 \leq 3$, $0 \leq e_4 \leq 6$, $0 \leq e_5 \leq 10$.

framing a twisted Whitney tower. These are the higher-order Sato-Levine invariants which are defined in all odd orders of the framed Whitney tower filtration. The higher-order Arf invariants also appear as framing obstructions, however they are shifted down one order, due to the fact that a twisted Whitney tower of order $2k$ can always be converted into a framed Whitney tower of order $2k-1$ by twisting and IHX constructions. These geometric constructions explain the origin of the *framing relations* introduced above in Definition 18.

Setting $\widetilde{\mathcal{T}}_{2k} := \mathcal{T}_{2k}$ in even orders, the reduced realization maps $\widetilde{R}_n : \widetilde{\mathcal{T}}_n \to \mathsf{W}_n$ for the framed filtration turn out to be isomorphisms in three quarters of the cases, in close analogy with Theorem 22 above. Then the higher-order Arf invariants again appear in the other quarter of cases, and Conjecture 26 has an analogous expression in terms of the framed Whitney tower filtration classification as the statement: "the realization maps $\widetilde{R}_n : \widetilde{\mathcal{T}}_n \to \mathsf{W}_n$ are isomorphisms for all n".

However, the analogy with Theorem 22 does *not* hold for the Milnor invariants μ_n in the framed filtration, leading to the appearance of the higher-order Sato-Levine invariants in the classification of the framed filtration described in the following Corollary 27. This subtle interaction between Milnor invariants and framing obstructions is the reason why the framed classification is trickier to describe.

Corollary 27 ([11] Cor.5.11)**.** *The groups W_n are classified by Milnor invariants μ_n and in addition, Sato-Levine invariants SL_n if n is odd, and finally, Arf invariants Arf_k for $n = 4k-3$.*

In particular, a link bounds an order n Whitney tower if and only it has all vanishing Milnor, Sato-Levine and Arf invariants up to order n (Compare Theorem 14).

A table of the framed filtration groups $\mathsf{W}_n(m)$ for low values of n, m is given in Figure 9, where the higher-order Arf invariant Arf_2 appears in order 5. The higher-order Sato-Levine invariants correspond to 2-torsion in all odd orders (for $m > 1$), and the ranges of possible ranks of the 2-torsion subgroups of the groups $\mathsf{W}_5(m)$ correspond to the possible ranks of Arf_2 (as in Figure 8).

For $n = 0$, the groups come from trees $i \frac{\quad}{} j$, and are detected by linking numbers for $i \neq j$ and framings for $i = j$. For order $n = 1$, the generators come (via R_1) from trees $i\!-\!\!<^{j}_{k}$. If all indices are distinct then they are detected by Milnor's triple invariants $\mu(ijk)$. However, in $\widetilde{\mathcal{T}}_1$ repeating indices also give nontrivial elements of order 2. If $i = j = k$, these are detected by the

	1	2	3	4	5
0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^6$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{15}$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}\oplus\mathbb{Z}_2^6$	$\mathbb{Z}^4\oplus\mathbb{Z}_2^{10}$	$\mathbb{Z}^{10}\oplus\mathbb{Z}_2^{15}$
2	0	$\mathbb{Z}$	$\mathbb{Z}^6$	$\mathbb{Z}^{20}$	$\mathbb{Z}^{50}$
3	0	$\mathbb{Z}_2^2$	$\mathbb{Z}^6\oplus\mathbb{Z}_2^8$	$\mathbb{Z}^{36}\oplus\mathbb{Z}_2^{20}$	$\mathbb{Z}^{126}\oplus\mathbb{Z}_2^{40}$
4	0	$\mathbb{Z}^3$	$\mathbb{Z}^{28}$	$\mathbb{Z}^{146}$	$\mathbb{Z}^{540}$
5	0	$\mathbb{Z}_2^{e_2}$	$\mathbb{Z}^{36}\oplus\mathbb{Z}_2^{e_3}$	$\mathbb{Z}^{340}\oplus\mathbb{Z}_2^{e_4}$	$\mathbb{Z}^{1740}\oplus\mathbb{Z}_2^{e_5}$
6	0	$\mathbb{Z}^6$	$\mathbb{Z}^{126}$	$\mathbb{Z}^{1200}$	$\mathbb{Z}^{7050}$

FIGURE 9. A table of the groups $\mathsf{W}_n(m)$, where m runs horizontally and n runs vertically. The possible ranges of the torsion exponents in order 5 depend on the currently unknown ranks of Arf_2: $3 \leq e_2 \leq 4$, $18 \leq e_3 \leq 21$, $60 \leq e_4 \leq 66$, $150 \leq e_5 \leq 160$.

classical Arf invariant of the ith component. In the case where exactly two indices are equal, one needs the classical Sato-Levine invariant (but has to note the framing relations from Figure 7).

The main tool for deriving the framed classification from the twisted one is the following surprisingly simple relation between the twisted and framed Whitney tower filtrations. Recall that in even orders the reduced groups $\widetilde{\mathcal{T}}_{2k}$ and realization maps $\widetilde{R}_{2k}$ are by definition equal to $\mathcal{T}_{2k}$ and R_{2k}.

Theorem 28. [11, Thm.5.1] *There are commutative diagrams of exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{\mathcal{T}}_{2k} \arrow[r] \arrow[d, two heads, "\widetilde{R}_{2k}"] & \mathcal{T}^\infty_{2k} \arrow[r] \arrow[d, two heads, "R^\infty_{2k}"] & \widetilde{\mathcal{T}}_{2k-1} \arrow[r] \arrow[d, two heads, "\widetilde{R}_{2k-1}"] & \mathcal{T}^\infty_{2k-1} \arrow[r] \arrow[d, two heads, "R^\infty_{2k-1}"] & 0 \\ 0 \arrow[r] & \mathsf{W}_{2k} \arrow[r] & \mathsf{W}^\infty_{2k} \arrow[r] & \mathsf{W}_{2k-1} \arrow[r] & \mathsf{W}^\infty_{2k-1} \arrow[r] & 0 \end{tikzcd}$$

where all maps in the bottom row are induced by the identity on the set of links. Moreover, there are isomorphisms

$$\mathrm{Cok}(\mathcal{T}_{2k} \to \mathcal{T}^\infty_{2k}) \cong \mathbb{Z}_2 \otimes \mathsf{L}'_{k+1} \cong \mathrm{Ker}(\widetilde{\mathcal{T}}_{2k-1} \to \mathcal{T}^\infty_{2k-1})$$

As a consequence of the Levine Conjecture (Theorem 8), all the relevant tree-groups are completely computed. So together with some additional geometric and algebraic arguments, the graded groups associated to the framed filtration can be computed in terms of those of the twisted filtration.

In Section 6 of [11], the diagram of Theorem 28 relating the $\mathcal{T}$- and W-groups is extended by the relevant η- and μ-maps to include exact sequences of D-groups, giving a bird's eye view of the classifications. The resulting pair of *master diagrams* gives a succinct summary of the overall algebraic structure connecting the $\mathcal{T}$-, W-, and D-groups.

Comparisons with other iterated disk constructions. Andrew Casson was the first who tried to recover the Whitney move in dimension four by an iterated disk construction. He started with a simply connected 4–manifold M with a knot K in its boundary. He looked for conditions so that K would bound an embedded disk in M. His starting point was an *algebraically transverse sphere* for a (singular) disk in M bounding K, an assumption that is satisfied in the setting of the s-cobordism theorem or the surgery exact sequence (but not for $M = B^4$). He then showed that

K bounds a *Casson tower* of arbitrary height in M. In such a tower, one attaches an immersed disk to the accessory circles of every intersection point in a previous stage (and requires that the new disk does not intersect previous stages).

Mike Freedman [27] realized that one can actually re-embed one Casson tower into another and that one can obtain enough geometric control to prove his breakthrough result: Any Casson tower of height greater than 3 contains in its neighborhood a topologically-flat embedded disk with boundary K. This implies Freedman's classification result for simply connected closed 4–manifolds and leads to many stunning applications.

However, there can be no obstruction theory for finding Casson towers of larger and larger height, not even in $M = B^4$ (where a transverse sphere cannot exist): Any knot K bounds a Casson tower of height 1 (which is just a singular disk) and if K bounds a Casson tower of height 4 then it is topologically slice (and hence bounds a Casson tower of arbitrary height).

The ground-breaking work of Cochran, Orr and Teichner in the setting of knot concordance [21, 22] includes the study of *symmetric* Whitney towers of *height* n. Here one inductively attaches Whitney disks to previous stages but only allows these new Whitney disks to intersect each other (and not the previously constructed stages). It follows that a symmetric Whitney tower of height h is a (particularly nice) Whitney tower of order 2^h, see [5].

Such symmetric Whitney towers have an extremely rich theory, even in the case of knots (see [23] for the fact that the filtration is nontrivial for all heights). All the iterated graded groups are in fact infinitely generated [20], one reason being the existence of higher-order von Neumann signatures that take values in the reals $\mathbb{R}$ (infinitely generated as abelian group).

There are currently no known algebraic criteria for raising the height of a symmetric Whitney tower, and hence not too much hope for a complete classification of the symmetric Whitney tower filtration of links, or even knots. This motivated the study of the Whitney tower filtrations by the more basic grading by order, and the classification as expressed in Corollary 27 is the first instance of a complete computation of a filtration defined via an iterated disk construction. These Whitney tower filtrations have analogues for immersed 2–spheres in 4–manifolds, including a formulation of the proposed higher-order Arf invariants. The order 1 theory goes back to [28] (see also [2, 45, 54, 59], and 10.8A and 10.8B of [29] where the relation to the Kirby–Siebenmann invariant is explained), but the higher-order theory is not generally understood for closed 4–manifolds.

[12] **"Universal quadratic forms and Whitney tower intersection invariants" (with J. Conant and P. Teichner)**
***Proceedings of the Freedman Fest, Geometry and Topology Monographs*, 18 (2012) 35–60. http://arxiv.org/abs/1207.0109.**

An important step in the above-described computation of the Whitney tower filtration involved determining the role played by framing obstructions (twistings) on Whitney disks. It was particularly satisfying to discover that the target groups $\mathcal{T}^{\infty}_{2n}$ for the twisted Whitney tower obstruction theory can be considered as (universal) quadratic refinements of the groups $\mathcal{T}_{2n}$ for the framed setting. This is made precise by this paper, which develops a general theory of quadratic forms, specializing from the non-commutative to the commutative to finally, the symmetric settings. These notions generalize those introduced by H. Baues [15] and [16, §8], and A. Ranicki [53, p.246].

To indicate some of the results which are directly relevant to the Whitney tower filtration, start by observing that the inner product (Definition 2) on the free abelian group on rooted trees associated to Whitney disks extends uniquely to a bilinear, symmetric, invariant pairing on the free quasi-Lie algebra L'

$$\langle\ ,\ \rangle:\ \mathsf{L}'(m)\times\mathsf{L}'(m)\longrightarrow\mathcal{T}(m).$$

This follows since the AS and IHX relations hold on both sides and are preserved by the inner product, with invariance corresponding to $\langle\, I,(J,K)\,\rangle=\langle\,(I,J),K)\,\rangle$ by the rotation of the tree in the plane [12, Fig.4]. This inner product is in fact *universal* by Lemma 10 of [12]. Note that L' is denoted by $\mathcal{L}$ in [12] (where it is the only type of Lie algebra considered).

Recall from above that the group $\mathcal{T}^{\infty}_{2n}(m)$ is gotten from $\mathcal{T}_{2n}(m)$ by including order n ∞-trees J^{∞} as new generators, where the notation indicates that the order n rooted tree J has its root vertex labeled by the twist symbol ∞. In addition to the usual IHX- and AS-relations on unrooted order $2n$ trees, the order n ∞-trees are involved in the following new *symmetry*, *interior twist* and *twisted IHX* relations:

$$J^{\infty}=(-J)^{\infty}\qquad 2\cdot J^{\infty}=\langle J,J\rangle\qquad I^{\infty}=H^{\infty}+X^{\infty}-\langle H,X\rangle$$

As their names suggest, these new relations arose from geometric considerations for twisted Whitney towers in [11].

As a specialization of the general theory, the universal symmetric pairing $\langle\ ,\ \rangle$ is shown to admit a universal quadratic refinement $q:\mathsf{L}'_n(m)\to\mathcal{T}^{\infty}_{2n}(m)$ defined by $q(J):=J^{\infty}$. In particular, with the right algebraic notion of *quadratic refinement*, the group $\mathcal{T}^{\infty}_{2n}(m)$ is completely determined by the pairing $\langle\ ,\ \rangle$. Most of this paper is dedicated to developing this general theory, which we do not attempt to summarize here (but compare the above relations with those of Wall's intersection form: $\mu(f)=\mu(-f)$, $2\mu(f)=\lambda(f,f)$, $\mu(f+g)=\mu(f)+\mu(g)+\lambda(f,g)$, for f,g in the subgroup of $\pi_{2n}X$ represented by immersed n-spheres with vanishing normal Euler number, X a $4n$-dimensional simply connected manifold.)

The following consequence of general properties of universal symmetric quadratic refinements has direct implications for the classification of Whitney towers in the 4–ball:

Theorem 29 ([12] Thm.9)**.** *For all m,n, the maps $t\mapsto t$ respectively $J^{\infty}\mapsto 1\otimes J$ give an exact sequence:*

$$0\longrightarrow\mathcal{T}_{2k}\longrightarrow\mathcal{T}^{\infty}_{2k}\longrightarrow\mathbb{Z}_2\otimes\mathsf{L}'_{k+1}\longrightarrow 0$$

This result is essential to the proof of Theorem 28 above, which is used to translate the computation of the twisted filtration to the framed setting, and sheds light on the role of the higher-order Arf invariants as obstructions to framing a twisted Whitney tower [11, Sec.5].

This paper also exposits the relationship between the first order Whitney tower intersection invariant τ_1 and the Kirby–Siebenmann invariant of a closed 4–manifold, motivated by the idea that a proper algebraic organization of the higher-order τ_n may contribute to a better understanding of 4–manifolds.

[13] "Geometric filtrations of string links and homology cylinders" (with J. Conant and P. Teichner) (To appear in *Quantum Topology*) http://arxiv.org/abs/1202.2482.

This paper applies extensions of the techniques of the above-described computation of the Whitney tower filtrations $\mathbb{L}\supset\mathsf{W}_0\supset\mathsf{W}_1\supset\mathsf{W}_2\supset\cdots$ and $\mathsf{W}^{\infty}_n\supset\mathsf{W}_n$ on the set $\mathbb{L}=\mathbb{L}(m)$ of concordance classes of framed m-component links in the 3–sphere to study the following filtrations of string links and homology cylinders:

$\mathbb{SL}$: – The analogous Whitney tower filtrations $\mathbb{SW}_n$ and $\mathbb{SW}^\infty_n$ on the group $\mathbb{SL} = \mathbb{SL}(m)$ of concordance classes of framed m-component string links (obtained from the usual closure operation from string links to links).
– The Johnson filtration $\mathbb{SJ}_n$ on $\mathbb{SL}$, defined as kernels of nilpotent Artin representations $\mathrm{Artin}_n \colon \mathbb{SL} \to \mathrm{Aut}_0(F/F_{n+2})$, where $F = F(m)$ is a free group on m generators, F_n are the terms in its lower central series and $\mathrm{Aut}_0(F/F_n)$ consists of those automorphisms of F/F_n which are defined by conjugating each generator and which fix the product of generators.
– The Goussarov-Habiro Y-filtration $\mathbb{SY}_n$ on $\mathbb{SL}$ consists of string links obtained from the unlink via surgeries along claspers with n nodes.

$\mathbb{HC}$: The Johnson and Goussarov-Habiro filtrations generalize to filtrations $\mathbb{J}_n$ and $\mathbb{Y}_n$ respectively on the group $\mathbb{HC} = \mathbb{HC}(g, b)$ of homology cobordism classes of homology cylinders over a surface $\Sigma_{g,b}$ of genus g with b boundary circles.

The graded groups associated to these filtrations will be denoted by the sans-serif versions of the above letters, for example $\mathsf{SW}_n := \mathbb{SW}_n/\mathbb{SW}_{n+1}$.

The main results of [13] are as follows:

Theorem 30 ([13] Thm.1)**.** *The sets $\mathbb{SW}_n$ and $\mathbb{SW}^\infty_n$ are normal subgroups of $\mathbb{SL}$ which are central modulo the next order. We obtain nilpotent groups $\mathbb{SL}/\mathbb{SW}_n$ and $\mathbb{SL}/\mathbb{SW}^\infty_n$, with associated graded groups*

$$\mathsf{SW}_n \cong \mathsf{W}_n \quad \text{and} \quad \mathsf{SW}^\infty_n \cong \mathsf{W}^\infty_n$$

Theorem 30 will lead to a connection between the higher-order Arf invariants associated to the Whitney tower filtrations and the computation of the graded groups associated to the $\mathbb{J}_n$ and $\mathbb{Y}_n$ filtrations (Theorem 35 below, and [13, Sec.4]).

The next theorem and subsequent corollary follow from the classification of W^∞_n [11], together with the interpretation of the Artin representation as the "universal" Milnor invariant [32].

Theorem 31 ([13] Thm.2)**.** *We have $\mathbb{SW}^\infty_n \subset \mathbb{SJ}_n$, and the Artin representation Artin_n induces an epimorphism*

$$\mathrm{Artin}^\infty_n \colon \mathbb{SL}/\mathbb{SW}^\infty_n \twoheadrightarrow \mathrm{Aut}_0(F/F_{n+2})$$

The kernel is a finite 2-group, generated by (internal band sums of) iterated Bing-doubles of the figure eight string knot (possibly with some additional trivial strands). In particular, for each n there is an upper bound on the size of this kernel.

See Figures 1 and 2 of [13] for the definition of Bing-doubling and internal band sums in the setting of string links.

In fact the kernel of Artin^∞_n can be characterized geometrically in several other ways:

Corollary 32 ([13] Cor.3)**.** *The following subsets of $\mathbb{SL}/\mathbb{SW}^\infty_n$ are equal to the kernel of* Artin^∞_n*:*

(i) *The subgroup generated by (internal band sums of) iterated Bing-doubles of a fixed string knot K_0 with nontrivial Arf invariant (possibly with some additional trivial strands).*
(ii) *The subgroup generated by (internal band sums) of iterated Bing-doubles of all string knots with non-trivial Arf invariant (possibly with some additional trivial strands).*
(iii) *The set of equivalence classes of boundary string links.*
(iv) *The set of equivalence classes of π_1-null string links.*

Here a string link σ is a *boundary string link* if the components of the standard closure L_σ bound disjoint surfaces in the 3–ball B; and a π_1*-null string link* is a string link σ whose standard closure L_σ bounds a surface Σ in the 4–ball $B^4 = B\times[0,1]$ such that $\pi_0(L_\sigma)\to\pi_0(\Sigma)$ is a bijection and which for which there is a push-off inducing the trivial homomorphism $\pi_1(\Sigma)\to\pi_1(B^4\setminus\Sigma)$. Note that Bing-doubling preserves boundary links (see Definition 9 of [13]).

Regarding the other filtrations, Proposition 33 of [13] shows that $\mathbb{SY}_n \subseteq \mathbb{SW}_n$. (In fact, an upcoming paper will show that this is an equality, and that the relation of order $k-1$ Whitney tower concordance is equivalent to the notion of C_k-concordance studied by Meilhan and Yasuhara [47].) Summarizing, we see that the filtrations on string links $\mathbb{SL}$ are ordered as follows:

$$\mathbb{SY}_n \subseteq \mathbb{SW}_n \subseteq \mathbb{SW}^\infty_n \subseteq \mathbb{SJ}_n$$

For $n=1$, all these filtrations are equal to the set $\mathbb{SL}_1 := \mathbb{SY}_1 = \mathbb{SW}_1 = \mathbb{SW}^\infty_1 = \mathbb{SJ}_1$ of concordance classes of string links with trivial linking numbers and framings.

Similarly, for $n=1$ the above filtrations of $\mathbb{HC}(g,b)$ give those homology cylinders which induce the identity homomorphism on first homology $H_1(\Sigma_{g,b})$. We write $\mathbb{HC}_1 = \mathbb{Y}_1 = \mathbb{J}_1$ for this subgroup. By taking the complement of a string link in $D^2\times[0,1]$, one gets a well-known group homomorphism

$$\mathrm{C} : \mathbb{SL}(m) \to \mathbb{HC}(0, m+1)$$

which takes $\mathbb{SJ}_n$ to $\mathbb{J}_n$, and takes $\mathbb{SY}_n$ to $\mathbb{Y}_n$. In unpublished work [34], Habegger used the fact that $\Sigma_{g,1}\times[0,1]\cong\Sigma_{0,2g+1}\times[0,1]$ to give a bijection

$$\mathbb{HC}_1(0,2g+1)\longleftrightarrow\mathbb{HC}_1(g,1)$$

which is not a group homomorphism but identifies the filtrations $\mathbb{J}_n$ (respectively $\mathbb{Y}_n$) on the two different types of homology cylinders. In [13, Sec.4] (see Figure 6), the map C is generalized to another geometric map

$$\mathrm{H} : \mathbb{SL}_1(2g) \to \mathbb{HC}_1(g,1)$$

which is not a homomorphism but takes both $\mathbb{SY}_n$ to $\mathbb{Y}_n$ and $\mathbb{SJ}_n$ to $\mathbb{J}_n$ for $n\geq 1$, and it is shown that this map H agrees with Habegger's bijection pre-composed with C. Combining results from [32] and [30] it follows that the induced maps on the associated graded groups $\mathrm{C}_n : \mathsf{SJ}_n \to \mathsf{J}_n$ are group isomorphisms for all $n\geq 1$, and by composing with Habegger's bijection we see that the same is true for $\mathrm{H}_n : \mathsf{SJ}_n \to \mathsf{J}_n$. Here SJ_n and J_n are the quotient groups $\mathbb{SJ}_n/\mathbb{SJ}_{n+1}$ and $\mathbb{J}_n/\mathbb{J}_{n+1}$ of the Johnson filtrations.

The analogous induced maps for the Y-filtrations are not yet fully understood but again the statements for C_n and H_n are equivalent:

Theorem 33 ([13] Thm.4)**.** *The induced maps* $\mathrm{C}_n, \mathrm{H}_n : \mathsf{SY}_n \to \mathsf{Y}_n$ *are group isomorphisms for* $n \equiv 0,2,3 \mod 4$. *In the remaining cases,* C_{4n+1}, H_{4n+1} *are epimorphisms with finitely generated 2-torsion kernel.*

The graded group SY_n is defined as the quotient of $\mathbb{SY}_n$ by the equivalence relation generated by simple order $n+1$ clasper surgeries. (So for example two string links representing elements in $\mathbb{SY}_n$ are equivalent if and only if they differ by a sequence of concordances and order $n+1$ simple clasper surgeries.) Similarly Y_n is defined as the quotient of $\mathbb{Y}_n$ by the equivalence relations generated by order $n+1$ simple clasper surgeries.

By a theorem of Nielsen [50, 60], $\mathbb{HC}(g,b)$ contains the mapping class group of $\Sigma_{g,b}$. This is one source of interest in the filtrations $\mathbb{J}_n$ and $\mathbb{Y}_n$.

Levine had already observed that in $\mathbb{HC}(g,1)$ there is an inclusion $\mathbb{Y}_n\subseteq\mathbb{J}_n$ and he started to study the difference in [42, 43]. He conjectured the statements of the next theorem which is proved in the above-described [9].

Theorem 34 ([9])**.** *For $n \geq 1$, the inclusion of filtrations $\mathbb{Y}_n \subseteq \mathbb{J}_n$ of $\mathbb{HC}(g,1)$ induces the following exact sequences of associated graded groups:*

$$0 \to \mathsf{Y}_{2n} \to \mathsf{J}_{2n} \to \mathsf{L}_{n+1} \otimes \mathbb{Z}_2 \to 0$$

$$\mathbb{Z}_2^m \otimes \mathsf{L}_{n+1} \to \mathsf{Y}_{2n+1} \to \mathsf{J}_{2n+1} \to 0$$

Levine did not conjecture that the map $\mathbb{Z}_2^m \otimes \mathsf{L}_{n+1} \to \mathsf{Y}_{2n+1}$ is injective, and in fact it is *not* because the *framing relations* introduced in [11] are also present in this context. Much of [13] is dedicated to extending the controlled constructions on (twisted) Whitney towers to the 3-dimensional clasper calculus pioneered by Habiro, and unravelling the odd order case is the main result of this paper:

Theorem 35 ([13] Thm.6)**.** *For $n \geq 1$, there are exact sequences of associated graded groups:*

$$0 \to \mathsf{L}_{2n+1} \otimes \mathbb{Z}_2 \to \mathsf{Y}_{4n-1} \to \mathsf{J}_{4n-1} \to 0$$

$$0 \to \mathsf{K}^{\mathsf{Y}}_{4n+1} \to \mathsf{Y}_{4n+1} \to \mathsf{J}_{4n+1} \to 0$$

and the kernel $\mathsf{K}^{\mathsf{Y}}_{4n+1}$ fits into the exact sequence $\mathsf{L}_{n+1} \otimes \mathbb{Z}_2 \stackrel{a_{n+1}}{\to} \mathsf{K}^{\mathsf{Y}}_{4n+1} \to \mathsf{L}_{2n+2} \otimes \mathbb{Z}_2 \to 0$.

The calculation of $\mathsf{K}^{\mathsf{Y}}_{4n+1}$ is thus reduced to the calculation of $\mathrm{Ker}(a_{n+1})$. This is the precise analog of the question "how nontrivial are the higher-order Arf invariants?" in the setting of Whitney tower filtrations of classical links (compare the a_{n+1} in [13, Thm.6] with the maps α_{n+1} defined in [8]).

Conjecture 36. *The homomorphisms a_{n+1} are injective for all $n \geq 1$.*

The connection between this conjecture and the higher-order Arf invariants defined for links is explained in Section 4 of [13], which derives several commutative diagrams comparing string links and homology cylinders.

[14] **"Pulling apart 2–spheres in 4–manifolds"**
(with P. Teichner)
***Documenta Mathematica* Vol. 19/31 (2014) 941–992.**

As a first step towards applying the theory of Whitney towers to study 4–manifolds, this paper attacks the problem of representing homotopy classes of 2–spheres by disjoint maps using an obstruction theory for *non-repeating* Whitney towers. Although much of the material in [14] is developed for non-simply connected 4–manifolds, this summary will stick to the simply connected setting, presenting only selected results.

Definition 37. A tree (as in Definition 2) is called *non-repeating* if all of its univalent labels are distinct, and *repeating* otherwise. Whitney disks and intersection points are called *non-repeating* if their associated trees are non-repeating, and *repeating* otherwise. A Whitney tower $\mathcal{W}$ is an order n *non-repeating Whitney tower* if all non-repeating intersections of order (strictly) less than n are paired by Whitney disks. In particular, if $\mathcal{W}$ is an order n Whitney tower then $\mathcal{W}$ is also an order n non-repeating Whitney tower. In a non-repeating Whitney tower repeating intersections of any order are not required to be paired by Whitney disks.

Since all the relations in Definition 3 are homogeneous in the univalent labels, restricting the generators to be non-repeating order n trees defines a subgroup $\Lambda_n(m) < \mathcal{T}_n(m)$.

Definition 38. If $\mathcal{W}$ is an order n non-repeating Whitney tower, the order n *non-repeating intersection invariant* $\lambda_n(\mathcal{W})$ is defined by summing the non-repeating trees $\pm t_p$ over all order n non-repeating intersections $p \in \mathcal{W}$:

$$\lambda_n(\mathcal{W}) := \sum \operatorname{sign}(p) \cdot t_p \in \Lambda_n(m)$$

The obstruction theory works just as in the repeating setting:

Theorem 39. *If $A_1, \ldots, A_m$ admit a non-repeating Whitney tower $\mathcal{W}$ of order n with $\lambda_n(\mathcal{W}) = 0 \in \Lambda_n(m)$, then the A_i are homotopic (rel boundary) to $A'_1, \ldots, A'_m$ admitting an order $n+1$ non-repeating Whitney tower.*

For a collection of order zero surfaces $A_1, A_2, \ldots, A_m \looparrowright X$, if the A_i are homotopic (rel boundary) to pair-wise disjoint immersions, then we say that the A_i can be *pulled apart.* As a first step towards determining whether or not any given A_i can be pulled apart, we have the following translation of the problem into the language of Whitney towers:

Proposition 40. *If $A_1, \ldots, A_m$ admit a non-repeating Whitney tower of order $m-1$, then the A_i can be pulled apart.*

The existence of a non-repeating Whitney tower of sufficient order encodes "pushing down" homotopies and Whitney moves which lead to disjointness (see [14, Prop.1]).

Combining Theorem 39 with Proposition 40 above yields the following result, which was announced in [3, Thm.3]:

Theorem 41. *If $A_1, \ldots, A_m$ admit a non-repeating Whitney tower $\mathcal{W}$ of order $(m-2)$ such that $\lambda_{(m-2)}(\mathcal{W})$ vanishes in $\Lambda_{(m-2)}(m)$, then the A_i can be pulled apart.*

Thus, the problem of deciding whether or not a given collection of order zero surfaces A_i can be pulled apart can be attacked inductively by determining the extent to which $\lambda_n(\mathcal{W})$ only depends on the homotopy classes of the A_i.

A setting where $\lambda_n(\mathcal{W}) \in \Lambda_n(m)$ does indeed tell the whole story is described next.

Some simply connected 4-manifolds. Denote by X_L the 4–manifold which is gotten by attaching 0-framed 2–handles to the 4–ball along a link L in the 3–sphere.

Theorem 42. *If L bounds an order n Whitney tower in the 4–ball, then the following both hold:*

(i) *Any collection collection $A = A_1, A_2, \ldots, A_m$ of immersed 2–spheres in X_L admits an order n Whitney tower $\mathcal{W}$.*
(ii) *The non-repeating intersection invariant $\lambda_n(A) := \lambda_n(\mathcal{W}) \in \Lambda_n(m)$ only depends on the homotopy class of A.*

Recall that an order n Whitney tower can also be considered to be an order n non-repeating Whitney tower. Using the realization techniques for Whitney towers in the 4–ball described in [11, Sec.3], examples of such A realizing any value in $\Lambda_n(m)$ can be constructed.

Corollary 43. *For L bounding an order n Whitney tower and $A = A_1, A_2, \ldots, A_m \looparrowright X_L$ as in Theorem 42:*

(i) *$\lambda_n(A) = 0 \in \Lambda_n(m)$ if and only if A admits an order $n+1$ non-repeating Whitney tower.*
(ii) *In the case $m = n+2$, $\lambda_n(A) = 0 \in \Lambda_n(n+2)$ if and only if the A_i can be pulled apart.*

The "only if" parts of the statements in Corollary 43 follow from Theorem 39 and Theorem 41 above; the "if" statements follow from Theorem 42.

Note that Theorem 42 and Corollary 43 provide a complete answer to the question of whether or not $A_1, A_2, \ldots, A_m \looparrowright X_L$ can be pulled apart for the cases $m \leq n+2$.

Pulling apart parallel surfaces. The next theorem generalizes Milnor's surprising result that the components of any link of 0-parallel knots in the 3–sphere bound disjoint immersed disks into the 4–ball (Theorem 4 of [48]).

Theorem 44. *If $A \looparrowright X$ is an immersed 2–sphere in a simply connected 4–manifold with $[A]\cdot[A]=0$, then any number of parallel copies of A can be pulled apart.*

Here $[A]\cdot[A] \in \mathbb{Z}$ is the usual self-intersection number of the homology class $[A] \in H_2(X;\mathbb{Z})$, and "parallel copies" of A are normal sections. Note that each transverse self-intersection of A gives rise to m^2-m non-repeating order zero intersections among m parallel copies of A. The proof of Theorem 44 proceeds by building a non-repeating Whitney tower of order $m-1$ and then applying Proposition 40. The same proof works for properly immersed disks, and is completely geometric, in contrast to Milnor's algebraic proof of the above mentioned result in [48]. The statement of Theorem 44 is *not* generally true in 4–manifolds with arbitrary fundamental group, as illustrated in Example 7.2 of [14].

Indeterminacies from lower-order intersections. The sufficiency results of Theorem 39 and Theorem 41 show that the groups $\Lambda_n(m)$ provide upper bounds on the invariants needed for a complete obstruction theoretic answer to the question of whether or not surfaces $A_1,\ldots,A_m$ in a 4–manifold X can be pulled apart. And as illustrated by Theorem 42 above, there are settings in which $\lambda_n(\mathcal{W}) \in \Lambda_n(m)$ only depends on the homotopy classes of the underlying order 0 surfaces A_i, sometimes giving the complete obstruction to pulling them apart.

In general however, more relations are needed in the target space to account for indeterminacies in the choices of possible Whitney towers on a given collection of order 0 surfaces. In particular, for Whitney towers in a 4–manifold X with non-trivial second homotopy group $\pi_2 X$, there can be indeterminacies which correspond to tubing the interiors of Whitney disks into immersed 2–spheres. Such INT *intersection relations* are, in principle, inductively manageable in the sense that they are determined by strictly lower-order intersection invariants on generators of $\pi_2 X$. For instance, the INT relations in the target groups of the order 1 invariants of [2, 45] are determined by the order zero intersection form on $\pi_2 X$. However, as described in Section 9 of [14], higher-order INT relations can be non-linear and if one wants the resulting target space to carry exactly the obstruction to the existence of a higher order tower then interesting subtleties already arise in the order 2 setting.

Homotopy invariance. The proposed program for pulling apart 2–spheres in 4–manifolds via non-repeating Whitney towers involves refining Theorems 39 and 41 by formulating (and computing) the relations $\mathrm{INT}_n(A) \subset \Lambda_n(m)$ so that the vanishing of $\lambda_n(A) := \lambda_n(\mathcal{W}) \in \Lambda_n(m)/\mathrm{INT}_n(A)$ is both necessary and sufficient for the existence of an order $n+1$ non-repeating tower supported by $A = A_1,\ldots,A_m$.

Note that if $\lambda_n(\mathcal{W}) \in \Lambda_n(m)/\mathrm{INT}_n(A)$ does not depend the choice of order n non-repeating Whitney tower $\mathcal{W}$ for a fixed immersion A, then $\lambda_n(A) := \lambda_n(\mathcal{W}) \in \Lambda_n(m)/\mathrm{INT}_n(A)$ only depends on the homotopy class of A: Up to isotopy, any generic homotopy from A to A' can be realized as a sequence of finitely many finger moves followed by finitely many Whitney moves. Since any

Whitney move has a finger move as an "inverse", there exists A'' which differs from each of A and A' by only finger moves (up to isotopy). But a finger move is supported near an arc, which can be assumed to be disjoint from the Whitney disks in a Whitney tower, and the pair of intersections created by a finger move admit a local Whitney disk; so any Whitney tower on A or A' gives rise to a Whitney tower on A'' with the same intersection invariant.

Thus, the problem is to find $\mathrm{INT}_n(A)$ relations which give independence of the choice of $\mathcal{W}$, and can be realized geometrically so that $\lambda_n(\mathcal{W}) \in \mathrm{INT}_n(A)$ implies that A bounds an order $n+1$ non-repeating Whitney tower. We conjecture that all these needed relations do indeed correspond to lower-order intersections between 2–spheres, and hence deserve to be called "intersection" relations. Although such $\mathrm{INT}_n(A)$ relations are completely understood for $n = 1$ [3], a precise formulation for the $n = 2$ case already presents interesting subtleties.

Useful necessary and sufficient conditions for pulling apart four or more 2–spheres in an arbitrary 4–manifold are not currently known. In [14, Sec.9] the intersection indeterminacies for an order 2 non-repeating intersection invariant in the simply connected setting are examined, and shown to be computable as the image in $\Lambda_2(4) \cong \mathbb{Z}^2$ of a map whose non-linear part is determined by certain Diophantine quadratic forms which are coupled by the intersection form on $\pi_2 X$. Carrying out this computation in general raises interesting number theoretic questions, and has motivated work of Konyagin and Nathanson in [39].

References

[1] **R Schneiderman**, *Algebraic linking numbers of knots in 3–manifolds*, Alg. and Geom. Topology 3 (2003) 921–968.

[2] **R Schneiderman**, **P Teichner**, *Higher order intersection numbers of 2–spheres in 4–manifolds*, Alg. and Geom. Topology 1 (2001) 1–29.

[3] **R Schneiderman**, **P Teichner**, *Whitney towers and the Kontsevich integral*, Proceedings of the Casson Fest, Geometry and Topology Monograph Series (2004).

[4] **R Schneiderman**, *Simple Whitney towers, half-gropes and the Arf invariant of a knot*, Pacific Journal of Mathematics, Vol. 222, No. 1, Nov (2005) 169–184.

[5] **R Schneiderman**, *Whitney towers and Gropes in 4–manifolds*, Trans. Amer. Math. Soc. 358 (2006) 4251–4278.

[6] **J Conant**, **R Schneiderman**, **P Teichner**, *Jacobi identities in low-dimensional topology*, Compositio Mathematica 143 Part 3 (2007) 780–810.

[7] **R Schneiderman**, *Stable concordance of knots in 3–manifolds*, Alg. and Geom. Topology 10 (2010) 373–432.

[8] **J Conant**, **R Schneiderman**, **P Teichner**, *Higher-order intersections in low-dimensional topology*, Proc. Natl. Acad. Sci. USA 2011 108 (20) 8081–8084.

[9] **J Conant**, **R Schneiderman**, **P Teichner**, *Tree homology and a conjecture of Levine*, Geometry and Topology 16 (2012) 55–600.

[10] **J Conant**, **R Schneiderman**, **P Teichner**, *Milnor Invariants and Twisted Whitney Towers*, Journal of Topology 7 no. 1 (2014) 187–224.

[11] **J Conant**, **R Schneiderman**, **P Teichner**, *Whitney tower concordance of classical links*, Geometry and Topology 16 (2012) 1419–1479.

[12] **J Conant**, **R Schneiderman**, **P Teichner**, *Universal quadratic forms and Whitney tower intersection invariants*, Proceedings of the Freedman Fest, Geometry and Topology Monographs 18 (2012) 35–60.

[13] **J Conant**, **R Schneiderman**, **P Teichner**, *Geometric filtrations of string links and homology cylinders*, To appear in Quantum Topology.

[14] **R Schneiderman**, **P Teichner**, *Pulling apart 2–spheres in 4–manifolds*, Documenta Mathematica Vol. 19/31 (2014) 941–992.

[15] **H Baues**, *Quadratic functors and metastable homotopy*, J. Pure Appl. Algebra, vol. 91 (1994) 49–107.

[16] **H Baues**, *On the group of homotopy equivalences of a manifold.* Trans. A. M. S. 348 (1996) 4737–4773.

[17] **J C Cha**, *Link concordance, homology cobordism, and Hirzebruch-type defects from iterated p-covers*, Journal of the European Mathematical Society 12 (2010), 555–610.

[18] **T Cochran**, *Derivatives of links, Milnor's concordance invariants and Massey products*, Mem. Amer. Math. Soc. Vol. 84 No. 427 (1990).

[19] **T Cochran**, *k-cobordism for links in S^3*, Trans. Amer. Math. Soc. 327 no 2 (1991) 641–654.

[20] **T Cochran**, **S Harvey**, **C Leidy**, *Knot concordance and higher-order Blanchfield duality*, Geom. Topol. 13 (2009), no. 3, 1419–1482.

[21] **T Cochran**, **K Orr**, **P Teichner**, *Knot concordance, Whitney towers and L^2-signatures*, Annals of Math., Volume 157 (2003) 433–519.

[22] **T Cochran**, **K Orr**, **P Teichner**, *Structure in the classical knot concordance group*, Commentarii Mathematici Helvetici, Volume 79, Number 1, January (2004) 105–123.

[23] **T Cochran**, **P Teichner**, *Knot concordance and von Neumann ρ-invariants*, Duke Math. J. Volume 137, Number 2 (2007), 337–379.

[24] **J Conant**, **P Teichner**, *Grope cobordism of classical knots*, Topology 43 (2004) 119–156.

[25] **J Conant**, **P Teichner**, *Grope Cobordism and Feynman Diagrams*, Math. Annalen 328 (2004) 135–171.

[26] **R Forman**, *Morse theory for cell complexes*, Adv. Math. 134 (1998), no. 1, 90–145.

[27] **M Freedman**, *The disk theorem for four-dimensional manifolds*, Proc. ICM Warsaw (1983) 647–663.

[28] **M Freedman**, **R Kirby** *A geometric proof of Rochlin's theorem*, Algebraic and geometric topology Part 2, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc. (1978) 85–97.

[29] **M Freedman**, **F Quinn**, *The topology of 4–manifolds*, Princeton Math. Series 39 Princeton, NJ, (1990).

[30] **S Garoufalidis**, **J Levine**, *Homology surgery and invariants of 3–manifolds*, Geom. Topol. Vol. 5 (2001) 75–108.

[31] **M N Gussarov**, *On n-equivalence of knots and invariants of finite degree*, Topology of manifolds and varieties, (O. Viro Ed.) Amer. Math Soc. Providence RI 173–192.

[32] **N Habegger**, **X Lin**, *On link concordance and Milnor's μ-invariants*, Bull. London Math. Soc. 30 (1998) 419–428.

[33] **N Habegger**, **G Masbaum**, *The Kontsevich integral and Milnor's invariants*, Topology 39 (2000) 1253–1289.

[34] **K Habiro**, *Claspers and finite type invariants of links*, Geom. Topol. Vol. 4 (2000) 1–83.

[35] **K Igusa**, **K Orr**, *Links, pictures and the homology of nilpotent groups*, Topology 40 (2001) 1125–1166.

[36] **M Kervaire**, **J Milnor**, *On 2–spheres in 4–manifolds*, Proc. Nat. Acad. Sci. Vol. 47 (1961) 1651–1657.

[37] **P Kirk**, **C Livingston**, *Type 1 knot invariants in 3–manifolds*, Pacific Journal of Mathematics Vol. 183 No. 2, (1998) 305–331.

[38] **P Kirk**, **C Livingston**, *Knot invariants in 3–manifolds and essential tori*, Pacific Journal of Mathematics Vol. 197 No. 1, (2001) 73–96.

[39] **S Konyagin**, **M Nathanson**, *Sums of products of congruence classes and of arithmetic progressions*, Intern. J. Number Theory Vol. 5, Issue: 4 (2009) 625–634.

[40] **D Kozlov**, *Discrete Morse theory for free chain complexes.*, C. R. Math. Acad. Sci. Paris 340 (2005), no. 12, 867–872.

[41] **S. Krushkal**, **P. Teichner**, *Alexander duality, Gropes and link homotopy*, Geom. Topol. Vol. 1 (1997) 51–69.

[42] **J Levine**, *Homology cylinders: an enlargement of the mapping class group*, Alg. and Geom. Topology 1 (2001) 243–270.

[43] **J Levine**, *Addendum and correction to: Homology cylinders: an enlargement of the mapping class group*, Alg. and Geom. Topology 2 (2002) 1197–1204.

[44] **J Levine**, *Labeled binary planar trees and quasi-Lie algebras*, Alg. and Geom. Topology 6 (2006) 935–948.

[45] **Y Matsumoto**, *Secondary intersectional properties of 4–manifolds and Whitney's trick*, Proceedings of Symposia in Pure mathematics Vol. 32 Part 2 (1978) 99–107.

[46] **W Magnus**, **A Karass**, **D Solitar**, *Combinatorial group theory*, Dover Publications, Inc. (1976).

[47] **Meilhan, J-B**, **Yasuhara, A**, *Characterization of finite type string link invariants of degree < 5*, Math. Proc. Cambridge Philos. Soc. 148 (2010), no. 3, 439–472.

[48] **J Milnor**, *Link groups*, Annals of Math. 59 (1954) 177–195.

[49] **J Milnor**, *Isotopy of links*, Algebraic geometry and topology, Princeton Univ. Press (1957).

[50] **J Nielsen**, *Untersuchungen zur Topologie der Geschlossenen Zweiseitigen Flachen I*, Acta Mathematica 50 (1927) 189–358.

[51] **T Ohtsuki**, *Quantum invariants. A study of knots, 3-manifolds, and their sets*, Series on Knots and Everything, 29, World Scientific Publishing Co., Inc., River Edge NJ (2002).

[52] **K Orr**, *Homotopy invariants of links*, Invent. Math. 95 17 (1989) 379–394.

[53] **A Ranicki**, *Algebraic Poincaré cobordism*, Topology, geometry, and algebra: interactions and new directions (Stanford, CA, 1999), Contemp. Math., 279, A.M.S., 213–255.

[54] **R Stong**, *Existence of π_1-negligible embeddings in 4-manifolds: A correction to Theorem 10.5 of Freedman and Quinn* , Proc. of the A.M.S. 120 (4) (1994) 1309-1314.

[55] **P Teichner**, *Knots, von Neumann Signatures, and Grope Cobordism*, Proceedings of the International Congress of Math. Vol II: Invited Lectures (2002) 437–446.

[56] **P Teichner**, *What is ... a grope?*, Notices of the AMS vol. 51, number 8 (2004).

[57] **T Wall**, *Surgery on Compact Manifolds*, London Math.Soc.Monographs 1, Academic Press 1970 or Second Edition, edited by A. Ranicki, Math. Surveys and Monographs 69, A.M.S.

[58] **H Whitney**, *The self intersections of a smooth n-manifold in $2n$-space*, Annals of Math. 45 (1944) 220–246.

[59] **M Yamasaki**, *Whitney's trick for three 2–dimensional homology classes of 4–manifolds*, Proc. Amer. Math. Soc. 75 (1979) 365–371.

[60] **H Zieschang**, **E Vogt**, **H D Coldewey**, *Surfaces and planar discontinuous groups.* Translated from the German by John Stillwell, Lecture Notes in Mathematics 835 Springer, Berlin, (1980).

E-mail address: `robert.schneiderman@lehman.cuny.edu`

Dept. of Mathematics and Computer Science, Lehman College, City University of New York, Bronx, NY