RESEARCH DESCRIPTION

ROB SCHNEIDERMAN

1. Overview

My field of research is *Low-dimensional Topology*: The geometry and topology of 3- and 4-dimensional manifolds, and their knotted and immersed submanifolds. In collaboration with Peter Teichner (UC Berkeley), I have developed a theory of *Whitney towers*, which addresses the question of which maps of 2-dimensional surfaces in 4-manifolds can be deformed to embeddings. In dimensions greater than 4, the solution to this problem is well-understood and has led to many powerful classification theorems for high-dimensional manifolds. The invariants associated to Whitney towers are combinatorially defined, but are intimately connected to elements of Algebra and Analysis, including Lie algebras, iterated commutators, von Neumann signatures and the Kontsevich integral. Some details on the work and work-in-progress described in this brief overview ([19, 20, 21, 22, 24, 25, 26, 27]) will be given in sections 2, 3 and 4 below.

The classification theorems for high-dimensional manifolds exploit a fairly free "flow" from algebra to topology, thanks in large part to the existence of generically embedded *Whitney disks*, which are used to eliminate singularities by guiding motions of submanifolds called *Whitney moves*, ([31]). This flow breaks down in dimensions less than or equal to 4 because of singularities in and among the Whitney disks and the failure of the Whitney move ([15]) is a defining characteristic of low-dimensional topology which is "measured" by the invariants associated to Whitney towers.

Whitney towers are built from layers of Whitney disks (pairing up higher order singularities – see Figure 1) and are parametrized by unitrivalent trees which determine group elements that represent obstructions to embedding and/or separating the underlying immersed surfaces. These higher order *intersection trees* provide a topological interpretation of the (analytically defined) Kontsevich integral ([25]), and the (algebraically defined) Milnor invariants for links in the 3-sphere ([26, 27]). (Work-in-progress suggests that intersection trees are sensitive to certain torsion which is not detected by the Kontsevich integral and Milnor invariants.) Whitney towers also provide a geometric characterization of the Arf invariant of a knot ([21]), and are related to Cochran, Orr and Teichner's recently discovered (n)-*solvable* filtration of the classical knot concordance group ([2]).

A primary goal of the general theory of Whitney towers is to extract invariants from the intersection trees that only depend on the homotopy classes of the underlying immersed surfaces in an arbitrary 4-manifold. This has been accomplished (jointly with P. Teichner) for the special case where the surfaces are 2-spheres gotten by attaching handles to a 4-ball ([25]), and in general for the next higher order after the usual (Wall) intersection invariant ([24]).

A key idea here is that the natural target for the invariants are certain groups of decorated trees (enhanced Feynman diagrams) whose generators actually sit as embedded subsets of the Whitney towers. The group relations correspond to geometric manipulations of Whitney towers, and include the usual Jacobi and antisymmetry relations of a Lie algebra. More subtle relations are determined by Whitney disk framings and lower order intersections, and further decorations come from the fundamental group of the ambient manifold. An important part of my on-going research is to develop computational and theoretical methods for working with these groups.

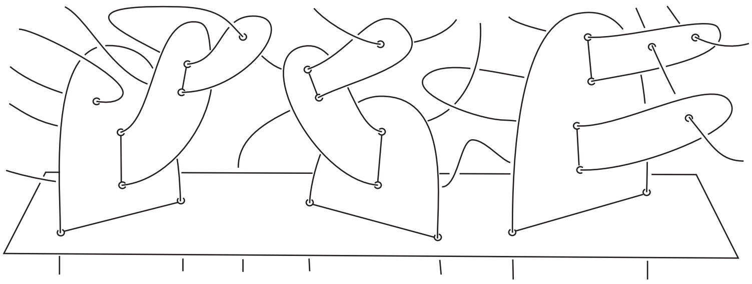

FIGURE 1. Part of a Whitney tower in a 4-dimensional manifold. The apparent arcs in the figure are sheets of surfaces that extend into the 'past and future' of a fourth 'time' coordinate.

2. Summary of published work

Here are abstracts of my published papers, and one accepted paper.

- *Algebraic linking numbers of knots in 3-manifolds*, Alg. and Geom. Topology 3 (2003) 921–968.

  Relative self-linking and linking "numbers" for pairs of oriented knots and 2–component links in oriented 3–manifolds are defined in terms of intersection invariants of immersed surfaces in 4–manifolds. The resulting concordance invariants generalize the usual homological notion of linking by taking into account the fundamental group of the ambient manifold and often map onto infinitely generated groups. The knot invariants generalize the type 1 invariants of Kirk and Livingston ([13, 14]) and when taken with respect to certain preferred knots, called *spherical knots*, relative self-linking numbers are characterized geometrically as the complete obstruction to the existence of a singular concordance which has all singularities paired by Whitney disks. This geometric equivalence relation, called *W–equivalence*, is also related to finite type 1–equivalence (in the sense of Habiro ([10]) and Goussarov ([9])) via the work of Conant and Teichner ([5, 6]) and represents a "first order" improvement to an arbitrary singular concordance. For null-homotopic knots, a slightly weaker equivalence relation is shown to admit a group structure. (This program will be further pursued in [22, 23].)

- *Whitney towers and Gropes in 4–manifolds*, Trans. Amer. Math. Soc. 358 (2006), 4251-4278.

  Many open problems and important theorems in low-dimensional topology have been formulated as statements about certain 2–complexes called *gropes* (eg [8, 28, 29]). This paper describes a precise correspondence between embedded gropes in 4–manifolds and the failure of the Whitney move in terms of iterated 'towers' of Whitney disks. The 'flexibility' of these *Whitney towers* is used to demonstrate some geometric consequences for knot and link concordance connected to n-solvability ([2]), k-cobordism ([11]) and grope concordance. The key observation is that the essential structure of gropes and Whitney towers can be described by embedded unitrivalent trees which can be controlled during surgeries and Whitney moves. It is shown that a Whitney move in a Whitney tower induces an IHX (Jacobi) relation on the embedded trees.

- *Simple Whitney towers, half-gropes and the Arf invariant of a knot.* Pacific Journal of Mathematics, Vol. 222, No. 1, Nov (2005).

  A geometric characterization of the Arf invariant of a knot in the 3–sphere is given in terms of two kinds of 4–dimensional bordisms, *half-gropes* and *Whitney towers*. These types of bordisms have associated complexities *class* and *order* which filter the condition of bordism by an embedded annulus, i.e. knot concordance, and it is shown constructively that the Arf invariant is exactly the obstruction to cobording pairs of knots by half-gropes and Whitney towers of arbitrarily high class and order, respectively. This illustrates geometrically how, in the setting of knot concordance, the Vassiliev (isotopy) invariants 'collapse' to the Arf invariant.

- *Higher order intersection numbers of 2–spheres in 4–manifolds*, (with P. Teichner), Alg. and Geom. Topology 1 (2001) 1–29.

  This is the beginning of an obstruction theory for deciding whether a map $f : S^2 \to X$ is homotopic to a topologically flat embedding, in the presence of fundamental group and in the absence of dual spheres in a topological 4-manifold X. The first obstruction is Wall's well known self-intersection number $\mu(f)$ which tells the whole story in higher dimensions. Our higher order obstruction $\tau(f)$ is defined if $\mu(f)$ vanishes and has formally very similar properties, except that it lies in a quotient of the group ring of two copies of $\pi_1 X$ modulo $\mathcal{S}_3$-symmetry (rather then just one copy modulo $\mathcal{S}_2$-symmetry). It generalizes to the non-simply connected setting the Kervaire-Milnor invariant defined in [8] and [20] which corresponds to the Arf-invariant of knots in 3-space.

We also give necessary and sufficient conditions for homotoping three maps $f_1, f_2, f_3 : S^2 \to X$ to a position in which they have *disjoint* images. The obstruction $\lambda(f_1, f_2, f_3)$ generalizes Wall's intersection number $\lambda(f_1, f_2)$ which answers the same question for two spheres but is not sufficient (in dimension 4) for three spheres. In the same way as intersection numbers correspond to linking numbers in 3-space, our new invariant corresponds to the Milnor invariant $\mu(1, 2, 3)$, generalizing the Matsumoto triple to the non simply-connected setting.

Finally, we explain some simple algebraic properties of these new cubic forms on $\pi_2(X)$. These are straightforward generalizations of the properties of quadratic forms as defined by Wall [30, §5]. A particularly attractive formula is

$$\lambda(f, f, f) = \sum_{\sigma \in \mathcal{S}_3} \tau(f)^\sigma$$

which generalizes the well known fact that Wall's invariants satisfy

$$\lambda(f, f) = \mu(f) + \overline{\mu(f)} = \sum_{\sigma \in \mathcal{S}_2} \mu(f)^\sigma$$

for an immersion f with trivial normal bundle.

- *Whitney towers and the Kontsevich integral* (with P. Teichner), Proceedings of a conference in honor of Andrew Casson, UT Austin 2003, Geometry and Topology Monograph Series, Vol. 7 (2004), 101-134.

  An obstruction theory for embedding 2–spheres into 4–manifolds is developed in terms of Whitney towers. The proposed intersection invariants take values in certain graded abelian groups generated by labelled trivalent trees, and with relations well known from the 3–dimensional theory of finite type invariants. Surprisingly, the same exact relations arise in 4 dimensions, for example the Jacobi (or IHX) relation comes in our context from the freedom of choosing Whitney arcs. We use the finite type theory to show that our invariants agree with the (leading term of the tree part of the) Kontsevich integral in the case where the 4–manifold is obtained from the 4–ball by attaching handles along a link in the 3–sphere.

- *Jacobi relations in low-dimensional topology*, (with J. Conant and P. Teichner), to appear in Compositio Mathematica.

  The Jacobi identity is the key relation in the definition of a Lie algebra. In the last decade, it also appeared at the heart of the theory of finite type invariants of knots, links and 3-manifolds (and is there called the IHX-relation). In addition, this relation was recently found to arise naturally in a theory of embedding obstructions for 2-spheres in 4-manifolds [25]. We expose the underlying topological unity between the 3- and 4-dimensional IHX-relations, deriving from a picture, Figure 2, of the Borromean rings embedded on the boundary of an unknotted genus 3 handlebody in 3-space. This is most naturally related to knot and 3-manifold invariants via the theory of grope cobordisms [5].

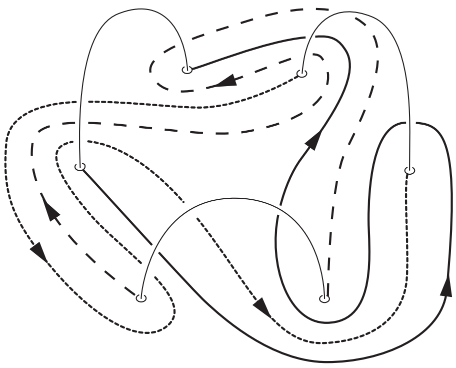

FIGURE 2. The geometric origin of the Jacobi identity in Dimension 4.

3. Research goals

Here are some goals of my research:

- **Classify class n grope concordance of classical links.** Class n gropes are geometric embodiments of iterated degree n commutators in a group. Preliminary work with P. Teichner ([27]) suggests that Whitney towers should provide the key invariants necessary to completely describe this fundamental notion of "topological non-commutativity". The relevant algebra is closely related to work of J. Levine ([16, 17]), who was led from a completely different point of view to study *quasi* Lie algebras (in which the bracket of an element with itself only vanishes modulo 2).
- **Provide topological interpretations of 3-dimensional quantum invariants (eg the Kontsevich integral) in terms of Whitney towers.** The Kontsevich integral is a powerful and mysterious invariant of knots and links (in 3-manifolds) which takes values in power series of unitrivalent graphs and is conjectured to completely detect knotting ([1, 12]). Originally motivated by physics, the geometric implications of the Kontsevich integral have been a source of intense study by topologists. I hope to use the Whitney tower interpretation of the tree part of the Kontsevich invariant ([21, 25]) to gain an understanding of the graphs as obstructions to "pushing down" 4-dimensional constructions into 3-dimensions.
- **Define intersection invariants which detect obstructions to (n)-solvability of knots.** The recently discovered (n)-solvable filtration of the classical knot concordance group by Cochran, Orr and Teichner ([2, 3, 4]) has been a major

breakthrough in knot theory. The geometry of the filtration is given by certain gropes and Whitney towers but the associated invariants are von Neumann signatures. It would be very interesting to interpret these analytically defined invariants directly in terms of the intersection trees of the Whitney towers.

- **Prove that the intersection invariants associated to non-repeating Whitney towers are the complete obstruction to separating a collection of immersed surfaces in a 4-manifold.** The notion of a *non-repeating* Whitney tower introduced in [25, 26] gives a sufficient condition for pulling apart immersed surfaces in a 4-manifold. I am optimistic about being able to show that the associated invariants only depend on the homotopy classes of the surfaces, thus giving a complete obstruction.
- **Show in general that the invariants associated to general Whitney towers only depend on the homotopy classes of the underlying surfaces.** This is a key (perhaps long-term) goal in the theory of Whitney towers. Part of the challenge here is to come up with an efficient algebraic point of view, which should be a higher order analogue of the description of the invariants in [24] as a "cubic intersection form with quadratic indeterminacies."

4. Introduction to Whitney towers

This section outlines an introduction to Whitney towers, indicating the application of the theory to the question of separating homotopy classes of 2-spheres in simply-connected 4-manifolds. After giving the precise definition of an *order n Whitney tower* on immersed surfaces in a 4-manifold, which filters the condition that the immersions are homotopic to pairwise disjoint embeddings, the constructions of the associated intersection invariants are sketched. The general idea is that *paired* intersection points (paired by Whitney disks) correspond to *rooted* trees (having a preferred univalent vertex – which can be thought of as lying in the interior of the Whitney disk) which bifurcate down through lower order Whitney disks, whereas *unpaired* intersection points (between Whitney disks) correspond to *unrooted* trees which are the result of gluing together the roots of the trees associated to the intersecting Whitney disks. The *order* of a unitrivalent tree is defined to be the number of trivalent vertices, and the orders of the Whitney disks and intersection points in a Whitney tower correspond to the orders of the corresponding unitrivalent trees. The lowest order unpaired intersections in an order n Whitney tower are of order n, and their associated trees represent obstructions to the existence of a Whitney tower of higher order. In the interest of brevity and simplicity, details regarding orientations will mostly be suppressed, and we will concentrate on simply connected 4-manifolds.

Details on immersed surfaces in 4–manifolds can be found in [8], including *Whitney moves* and (Casson) *finger moves*. For more on Whitney towers see [7, 20, 21, 24, 25].

4.1. **Whitney towers.**

Definition 1.

- A *surface of order 0* in a 4–manifold X is a properly immersed surface (boundary embedded in the boundary of X and interior immersed in the interior of X). A *Whitney tower of order 0* in X is a collection of order 0 surfaces.
- The *order of a (transverse) intersection point* between a surface of order n_1 and a surface of order n_2 is $n_1 + n_2$.
- The *order of a Whitney disk* is $n + 1$ if it pairs intersection points of order n.
- For $n \geq 0$, a *Whitney tower of order* $n + 1$ is a Whitney tower $\mathcal{W}$ of order n together with Whitney disks pairing all order n intersection points of $\mathcal{W}$. These top order disks are allowed to intersect each other as well as lower order surfaces.

The Whitney disks in a Whitney tower are required to be *framed* ([8]) and have disjointly embedded boundaries. Intersections in surface interiors are assumed to be transverse.

Some further terminology: If $\mathcal{W}$ is an order n Whitney tower containing A_i as its order 0 surfaces then the A_i are said to *admit* an order n Whitney tower and we say that $\mathcal{W}$ is a Whitney tower *on* the A_i.

Note that if A_i admit a Whitney tower $\mathcal{W}$ that contains no unpaired intersection points, then the A_i are homotopic to pairwise disjoint embeddings – with the homotopy guided by the Whitney disks in $\mathcal{W}$.

4.2. **Intersection trees for Whitney towers in simply-connected 4-manifolds.**

Consider a Whitney tower $\mathcal{W}$ on order 0 surfaces $A_1, A_2, \ldots, A_m$. The *order* of a uni-trivalent tree is the number of trivalent vertices. To each surface (both Whitney disks and order 0 surfaces) in a Whitney tower $\mathcal{W}$ we associate an *m-labelled* rooted tree of the same order as follows. To each order 0 surface A_i is associated the (order 0) tree $t(A_i)$ which consists of a single edge with one vertex designated as a root and the other vertex labelled i. To a Whitney disk $W_{(I,J)}$ pairing intersections between surfaces W_I and W_J we associate the tree $t(W_{(I,J)})$ which is gotten by gluing together the roots of the trees $t(W_I)$ and $t(W_J)$ associated to W_I and W_J, and then sprouting a new rooted edge from the vertex which corresponds to the glued roots. ($t(W_{(I,J)})$ is the *rooted product* of $t(W_I)$ and W_J), also denoted $t(W_I) * t(W_J)$.) For instance, to a Whitney disk $W_{(i,j)}$ pairing intersections between order 0 surfaces A_i and A_j is associated the order 1 "Y" tree $t(W_{(i,j)})$ which has two of its three univalent vertices labelled by i and j, and the other univalent vertex designated as the root. The bracket subscripts on the Whitney disks in the notation come from the usual correspondence between rooted binary trees and non-associative bracketings of the univalent labels (recall that we are suppressing orientations which would correspond to skew symmetry of the bracket). The understanding here is that if a bracket I is just a singleton $I = i$, then the surface W_I is not a Whitney disk but just the order 0 surface $W_I = W_i = A_i$.

To each unpaired intersection point $p \in W_I \cap W_J$ is associated the unrooted labelled tree t_p which is the *inner product* $t(W_I) \cdot t(W_J)$ formed by gluing together the roots of the trees $t(W_I)$ and $t(W_J)$ associated to W_I and W_J. The *(unoriented) geometric intersection tree* $t(\mathcal{W})$ of a Whitney tower $\mathcal{W}$ is defined to be the disjoint union $t(\mathcal{W}) := \amalg t_p$ over all unpaired intersection points $p \in \mathcal{W}$.

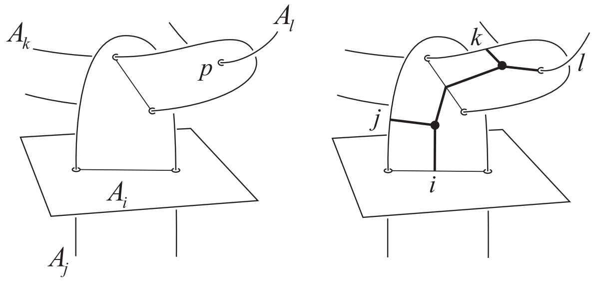

FIGURE 3. Part of an order 2 Whitney tower on order 0 surfaces A_i, A_j, A_k, and A_l and the labeled tree t_p associated to the order 2 intersection point p.

Although the combinatorics of the singular set of a Whitney tower can in general be quite complicated (with multiple self intersections and intersections among Whitney disks), it is in fact not hard to "clean up in a controlled way" (by homotopy) an arbitrary Whitney tower so that all singularities are contained in local standard *sub*-Whitney towers which are contained in disjoint thickenings of *embeddings* of the trees in $t(\mathcal{W})$ into the ambient 4-manifold (Figure 3). Thus, $t(\mathcal{W})$ also encodes the singularities of $\mathcal{W}$ geometrically, and certain modifications of $\mathcal{W}$ correspond to relations among trees.

By arbitrarily orienting the Whitney disks in $\mathcal{W}$, (as described in [25]) it is possible to associate a sign ϵ_p to each p and trivalent (cyclic) orientations to each t_p which only depend on the orientations of the underlying order 0 surfaces, up to the *antisymmetry* relation which changes sign with a change of trivalent orientation. In an order n Whitney tower $\mathcal{W}$, the disjoint union of signed oriented order n trees $t_n(\mathcal{W}) := \amalg \epsilon_p \cdot t_p$ is the *order n (oriented) geometric intersection tree* of $\mathcal{W}$, and determines the *nth order intersection tree*

$$\tau_n(\mathcal{W}) := \sum \epsilon_p \cdot t_p \in \mathcal{T}_n(m)$$

which is an obstruction to the existence of a higher order Whitney tower on the underlying order 0 surfaces A_i. Here $\mathcal{T}_n(m)$ denotes the abelian group on m-labelled, oriented, order n unitrivalent trees, modulo the AS (antisymmetry) and IHX (Jacobi) relations. (The usual Jacobi relation for brackets can be directly expressed in terms of rooted oriented labelled trees, and the corresponding local relation for unrooted trees which appears in the 3-dimensional finite type theories is called the IHX relation due to the shapes of the three terms I-H+X.)

It is important to note that all relations in the target group are "realizable" by controlled geometric manipulations of Whitney towers. For instance the IHX relations can be realized by modifying the boundary arcs of three trivial Whitney disks as illustrated in Figure 2 and then extending the construction to the interiors of the disks (as described in [7]).

Using realization results from [20] and [7], the following theorem was proved in [25], which recovers an "nth order" approximation of the flow from algebraic cancellation to geometric cancellation that successful Whitney moves would provide:

Theorem 2 ([25])**.** *If A_i admit an order n Whitney tower $\mathcal{W}$ with $\tau_n(\mathcal{W}) = 0 \in \mathcal{T}_n(m)$, then the A_i are homotopic (rel ∂) to A_i' which admit a Whitney tower of order $(n+1)$.*

This illustrates the obstruction theoretic nature of Whitney towers and their intersection trees as well as giving a geometric interpretation of the groups $\mathcal{T}_n(m)$ which frequently appear in 3-dimensional settings (often tensored with $\mathbb{Q}$).

4.3. **Separating homotopy classes of 2-spheres.** A *non-repeating* Whitney tower on order 0 surfaces A_i is almost the same as a Whitney tower except that only intersection points whose associated trees have distinctly labelled univalent vertices are paired up by Whitney disks. Such (*non-repeating*) intersection points are the higher order analogues of *self*-intersections among surfaces. Thus, all the trees associated to the Whitney disks (as well as to the intersection points) in an order n non-repeating Whitney tower $\mathcal{W}$ have distinctly labelled vertices. The geometric intersection tree $t(\mathcal{W})$ is defined as before, and determines the order n *non-repeated intersection tree* $\lambda_n(\mathcal{W})$ in the subgroup $\Lambda_n(m) < \mathcal{T}_n(m)$ generated by non-repeating order n trees. The following theorem shows that, at least from an obstruction theoretic point of view, the non-repeating intersection tree is the relevant algebraic object to consider when attempting to separate the underlying order 0 surfaces by a homotopy (rel boundary).

Theorem 3 ([25, 26])**.** *If surfaces $A_1, A_2, \ldots, A_{n+2}$ admit an order n non-repeating Whitney tower $\mathcal{W}$ with $\lambda_n(\mathcal{W}) = 0 \in \Lambda_n(n+2)$, then the A_i are homotopic (rel ∂) to pairwise disjoint immersions A_i'.*

The proof involves two main steps: First of all, the vanishing of $\lambda_n\mathcal{W}$ in $\Lambda_n(n+2)$ yields an order $n+1$ non-repeated Whitney tower, just as in Theorem 2 for general Whitney towers. All of the unpaired intersection points in such an order $n+1$ non-repeating Whitney tower are necessarily *repeating* intersection points, because order $n+1$ trees have $n+3$ univalent vertices and there are only $n+2$ order 0 surfaces. A homotopy (supported near the Whitney tower) which separates the A_i can now be described explicitly. The idea is roughly that Whitney disks which contain unpaired repeating intersection points are good enough to guide singular Whitney moves to pull apart the A_i (after a bit of cleaning up by finger moves); alternatively, one can first push all unpaired intersections down in such a way that they eventually become self-intersections among the A_i and then do (non-singular) Whitney moves on all the clean Whitney disks.

The fundamental question in this setting is: "To what extent does $\lambda_n(\mathcal{W}) \in \Lambda_n(n+2)$ only depend on the homotopy classes (rel ∂) of the A_i?" In the case is where the A_i generate $\pi_2 X$ there should not be any further relations:

Conjecture 4. *If the A_i generate $\pi_2 X$, then for any non-repeating Whitney tower $\mathcal{W}$,*

$$\lambda(A_1, A_2, \ldots, A_{n+2}) := \lambda_n(\mathcal{W}) \in \Lambda_n(n+2)$$

is a well defined homotopy invariant which vanishes if and only if the A_i are homotopic (rel ∂) to pairwise disjoint immersions A'_i in X.

For general A_i (not generating $\pi_2 X$) more relations are needed in the target in order to get homotopy invariance. These *intersection relations* are described in [26], and are in principle inductively computable – since they are determined by lower order intersections – but in general can lead to some interesting systems of quadratic diophantine equations.

References

[1] **D Bar-Natan**, *On the Vassiliev knot invariants,* Topology 34 (1995) 423–472.

[2] **T Cochran**, **K Orr**, **P Teichner**, *Knot concordance, Whitney towers and L^2-signatures*, Annals of Math., Volume 157 (2003) 433–519.

[3] **T Cochran**, **K Orr**, **P Teichner**, *Strucure in the classical knot concordance group*, Preprint 2002 (to appear in Comment. Math. Helv.).

[4] **T Cochran**, **P Teichner**, *Knot concordance and von Neumann ρ-invariants*, Preprint 2004.

[5] **J Conant**, **P Teichner**, *Grope cobordism of classical knots*, Topology 43 (2004) 119–156.

[6] **J Conant**, **P Teichner**, *Grope Cobordism and Feynman Diagrams*, to appear in Math. Annalen.

[7] **J Conant**, **R Schneiderman**, **P Teichner**, *Jacobi relations in low-dimensional topology*, to appear in Compositio Mathematica.

[8] **M Freedman**, **F Quinn**, *The topology of 4–manifolds*, Princeton Math. Series 39 Princeton, NJ, (1990).

[9] **M N Gussarov**, *On n-equivalence of knots and invariants of finite degree*, Topology of manifolds and varieties, (O. Viro Ed.) Amer. Math Soc. Providence RI 173–192.

[10] **K Habiro**, *Claspers and finite type invariants of links*, Geometry and Topology Vol. 4 (2000) 1–83.

[11] **K Igusa**, **K Orr**, *Links, pictures and the homology of nilpotent groups*, Topology 40 (2001) 1125–1166.

[12] **M Kontsevich**, *Vassiliev's knot invariants*, Adv. Soviet Math. 16 (2) (1993) pp 137–1500.

[13] **P Kirk**, **C Livingston**, *Type 1 knot invariants in 3–manifolds*, Pacific Journal of Mathematics Vol. 183 No. 2, (1998) 305–331.

[14] **P Kirk**, **C Livingston**, *Knot invariants in 3–manifolds and essential tori*, Pacific Journal of Mathematics Vol. 197 No. 1, (2001) 73–96.

[15] **M Kervaire**, **J Milnor**, *On 2–spheres in 4–manifolds*, Proc. Nat. Acad. Sci. Vol. 47 (1961) 1651–1657.

[16] **J Levine**, *Homology cylinders: an enlargement of the mapping class group*, Alg. and Geom. Topology 1 (2001) 243–270.

[17] **J Levine**, *Addendum and correction to: Homology cylinders: an enlargement of the mapping class group*, Alg. and Geom. Topology 2 (2002) 1197–1204.

[18] **Y. Matsumoto**, *Secondary intersectional properties of 4–manifolds and Whitney's trick*, Proceedings of Symposia in Pure mathematics Vol. 32 Part 2 (1978) 99–107.

[19] **R Schneiderman**, *Algebraic linking numbers of knots in 3-manifolds*, Alg. and Geom. Topology 3 (2003) 921–968.

[20] **R Schneiderman**, *Whitney towers and Gropes in 4–manifolds*, Trans. Amer. Math. Soc. 358 (2006), 4251-4278.

[21] **R Schneiderman**, *Simple Whitney towers, half-gropes and the Arf invariant of a knot.* Pacific Journal of Mathematics, Vol. 222, No. 1, Nov (2005).

[22] **R Schneiderman**, *Stably slicing knots in 3–manifolds.* (In preparation).

[23] **R Schneiderman**, *Linking invariants and essential tori in 3–manifolds.* (In preparation).

[24] **R Schneiderman**, **P Teichner**, *Higher order intersection numbers of 2–spheres in 4–manifolds*, Alg. and Geom. Topology 1 (2001) 1–29.

[25] **R Schneiderman**, **P Teichner**, *Whitney towers and the Kontsevich integral*, Proceedings of a conference in honor of Andrew Casson, UT Austin 2003, Geometry and Topology Monograph Series, Vol. 7 (2004), 101-134.

[26] **R Schneiderman**, **P Teichner**, *Pulling apart 2–spheres in 4–manifolds*, In preparation.

[27] **R Schneiderman**, **P Teichner**, *Grope concordance of classical links*, In preparation.

[28] **P Teichner**, *Knots, von Neumann Signatures, and Grope Cobordism*, Proceedings of the International Congress of Math. Vol II: Invited Lectures (2002) 437–446.

[29] **P Teichner**, *What is ... a grope?*
To appear in the Notices of the A.M.S.

[30] **T Wall**, *Surgery on Compact Manifolds*, London Math. Soc. Monographs 1, Academic Press 1970 or Second Edition, edited by A. Ranicki, Math. Surveys and Monographs 69, A.M.S.

[31] **H Whitney**, *The self intersections of a smooth n-manifold in $2n$-space*, Annals of Math. 45 (1944) 220–246.

Dept. of Mathematics and Computer Science, Lehman College, City University of New York, Bronx NY, USA

E-mail address: robert.schneiderman@lehman.cuny.edu