MAT 237/CMP 232: Problem Sheet #3

Topic: Mathematical Induction

Instructions. The following is a collection of questions pertaining to the topic indicated above. Please bring this worksheet to class for each day we discuss this topic. Though some problems will be assigned to solve for homework, others will be discussed in class.

Problems

1. Let $P(n)$ be the statement that $1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$ for the positive integer n.
   (a) What is the statement $P(1)$?
   (b) Show that $P(1)$ is true, completing the basis step of the proof.
   (c) What is the inductive hypothesis?
   (d) What do you need to prove in this inductive step?
   (e) Complete the inductive step, identifying where you use the inductive hypothesis.
   (f) Explain why these steps show that this formula is true whenever n is a positive integer.
2. Let $P(n)$ be the statement that $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ for the positive integer n.
   (a) What is the statement $P(1)$?
   (b) Show that $P(1)$ is true, completing the basis step of the proof.
   (c) What is the inductive hypothesis?
   (d) What do you need to prove in this inductive step?
   (e) Complete the inductive step, identifying where you use the inductive hypothesis.
   (f) Explain why these steps show that this formula is true whenever n is a positive integer.
3. Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n+1)^2 = (n+1)(2n+1)(2n+3)/3$ for $n \geq 0$.
4. Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ whenever n is a positive integer.
5. Prove that for every positive integer n,

$$\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2.$$

6. Let $P(n)$ be the statement that $n! < n^n$, where n is an integer greater than 1.
   (a) What is the statement $P(2)$?
   (b) Show that $P(2)$ is true, completing the basis step of the proof.
   (c) What is the inductive hypothesis?
   (d) What do you need to do to prove the inductive step?
   (e) Complete the inductive step.
   (f) Explain why these steps show that this inequality is true whenever n is an integer greater than 1.

7. Prove that $3^n < n!$ if n is an integer greater than 6.

8. Prove that $n^2 - 7n + 12$ is nonnegative whenever n is an integer with $n \geq 3$.

9. Prove that 2 divides $n^2 + n$ whenever n is a positive integer.

10. Prove that 3 divides $n^3 + 2n$ whenever n s a positive integer.

11. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cap A_2 \cap \cdots \cap A_n) \cup B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B).$$

12. Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$(A_1 \cup A_2 \cup \cdots \cup A_n) \cap B = (A_1 \cap B) \cup (A_2 \cap B) \cup \cdots \cup (A_n \cap B).$$

13. Let $P(n)$ be the statement that a postage of n cents can be formed using just 3-cent and 5-cent stamps. The parts of this exercise outline a strong induction argument that $P(n)$ is true for $n \geq 8$.
   (a) Show that the statements $P(8)$, $P(9)$, and $P(10)$ are true, completing the basis step of this proof.
   (b) What is the inductive hypothesis of the proof?
   (c) What do you need to prove in the inductive step?
   (d) Complete the inductive step for $k \geq 10$.
   (e) Explain why these steps show that this statement is true whenever $n \geq 8$.

14. Which amounts of money can be formed using just two-dollar bills and five-dollar bills? Prove your answer using strong induction.