MAT 237/CMP 232: Problem Sheet #2

Topic: Functions, Sequences, and Summations

Instructions. The following is a collection of questions pertaining to the topic indicated above. Please bring this worksheet to class for each day we discuss this topic. Though some problems will be assigned to solve for homework, others will be discussed in class.

Problems.

1. Why is f not a function from $\mathbb{R}$ to $\mathbb{R}$?
   (a) $f(x) = 1/x$?
   (b) $f(x) = \sqrt{x}$?
   (c) $f(x) = \pm\sqrt{x^2+1}$?
2. Determine whether f is a function from $\mathbb{Z}$ to $\mathbb{R}$.
   (a) $f(n) = \pm n$
   (b) $f(n) = \sqrt{n^2+1}$
   (c) $f(n) = 1/(n^2-4)$
3. Find these values.
   (a) $\lfloor 1.1 \rfloor$
   (b) $\lceil 1.1 \rceil$
   (c) $\lfloor -0.1 \rfloor$
   (d) $\lceil -0.1 \rceil$
   (e) $\lceil 2.99 \rceil$
   (f) $\lceil -2.99 \rceil$
   (g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$
   (h) $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil$
4. Find these values.
   (a) $\lceil \frac{3}{4} \rceil$
   (b) $\lfloor \frac{7}{8} \rfloor$
   (c) $\lceil -\frac{3}{4} \rceil$
   (d) $\lfloor -\frac{7}{8} \rfloor$
   (e) $\lceil 3 \rceil$
   (f) $\lfloor -1 \rfloor$
   (g) $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor$
   (h) $\lfloor \frac{1}{2} \cdot \lfloor \frac{5}{2} \rfloor \rfloor$
5. Determine whether each of these functions from $\{a,b,c,d\}$ to itself is one-to-one?
   (a) $f(a) = b$, $f(b) = a$, $f(c) = c$, $f(d) = d$
   (b) $f(a) = b$, $f(b) = b$, $f(c) = d$, $f(d) = c$
   (c) $f(a) = d$, $f(b) = b$, $f(c) = c$, $f(d) = d$
6. Which of the functions in the exercise above are onto?
7. Determine whether each function from $\mathbb{Z}$ to $\mathbb{Z}$ is one-to-one.
   (a) $f(n) = n - 1$
   (b) $f(n) = n^3$
   (c) $f(n) = n^2 + 1$
   (d) $f(n) = \lceil n/2 \rceil$
8. Which of the functions in the exercise above are onto?
9. Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ that is:
   (a) one-to-one but not onto
   (b) onto but not one-to-one.
   (c) both onto and one-to-one (but different from the identity function).
   (d) neither one-to-one nor onto.
10. Determine whether each of the functions are a bijection from $\mathbb{R}$ to $\mathbb{R}$.
    (a) $f(x) = 2x + 1$
    (b) $f(x) = x^2 + 1$
    (c) $f(x) = x^3$
    (d) $f(x) = (x^2+1)/(x^2+2)$
    (e) $f(x) = x^5 + 1$
11. Give an example of an increasing function with the set of real numbers as its domain and codomain that is not one-to-one.
12. Let $f(x) = \lfloor x^2/3 \rfloor$. Find $f(S)$ if
    (a) $S = \{-2,-1,0,1,2,3\}$.
    (b) $S = \{0,1,2,3,4,5\}$.
    (c) $S = \{1,5,7,11\}$.
    (d) $S = \{2,6,10,14\}$.

13. Suppose that g is a function from A to B and f is a function from B to C.

    (a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

    (b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

14. Find $f \circ g$ and $g \circ f$, where $f(x) = x^2 + 1$ and $g(x) = x + 2$, are functions from $\mathbb{R}$ to $\mathbb{R}$.

15. Find $f + g$ and fg for the functions given above.

16. Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that:

    (a) $f(S \cup T) = f(S) \cup f(T)$.

    (b) $f(S \cap T) \subseteq f(S) \cap f(T)$.

17. Find these terms of the sequence $\{a_n\}$, where $a_n = 2 \cdot (-3)^n + 5^n$.

    (a) a_0 (b) a_1 (c) a_4 (d) a_5

18. Find the term a_8 of the sequence $\{a_n\}$ if a_n equals

    (a) 2^{n-1}

    (b) $1 + (-1)^n$

    (c) 7

    (d) $-(-2)^n$

19. What are the values of these sums?

    (a) $\displaystyle\sum_{k=1}^{5}(k+1)$

    (b) $\displaystyle\sum_{j=1}^{4}(-2)^j$

    (c) $\displaystyle\sum_{i=1}^{10} 3$

    (d) $\displaystyle\sum_{j=0}^{8}(2^{j+1} - 2^j)$

20. What are the values of these sums, where $S = \{1, 3, 5, 7\}$?

    (a) $\displaystyle\sum_{j \in S} j$

    (b) $\displaystyle\sum_{j \in S} j^2$

    (c) $\displaystyle\sum_{j \in S} (1/j)$

    (d) $\displaystyle\sum_{j \in S} 1$

21. Compute each of these double sums.

    (a) $\displaystyle\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j)$

    (b) $\displaystyle\sum_{i=0}^{2}\sum_{j=0}^{3}(2i+3j)$

    (c) $\displaystyle\sum_{i=1}^{3}\sum_{j=0}^{2} j$

    (d) $\displaystyle\sum_{i=0}^{2}\sum_{j=1}^{3} ij$

22. Determine whether each of these sets are countable or uncountable. For those that are countable, exhibit a one-to-one correspondence between the set of natural numbers and the set.

    (a) the integers greater than 10

    (b) the odd negative integers

    (c) the real numbers between 0 and 2

    (d) the integer multiples of 10

23. Show that the union of countable sets is countable.

24. Explain why the set of real number solutions of the quadratic equations $ax^2 + bx + c = 0$, where a, b, and c are integers is countable.