Math 70300

Homework 7

Due: within 72 hours

1. Let u be harmonic in a region G and suppose that the closed disc $\overline{D(a,R)}$ is contained in G. Show that
$$u(a) = \frac{1}{\pi R^2} \int_{D(a,R)} u(x,y)\, dxdy.$$

   Hint: Use polar coordinates.

   For every r between 0 and R we have by the mean value property:
$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta.$$

   We integrate this equality multiplied by r, since $dxdy = rdrd\theta$ in polar coordinates. This gives
$$\int_0^R u(a) r\, dr = \frac{1}{2\pi} \int_0^R \int_0^{2\pi} u(a + re^{i\theta})\, d\theta\, r\, dr.$$

   Using polar coordinates centered at a, we cover the whole disc $\overline{D(a,R)}$ when $0 \leq r \leq R$ and $0 \leq \theta \leq 2\pi$. We remark again that $dxdy = rdrd\theta$. So the right-hand side becomes
$$\frac{1}{2\pi} \int_{D(a,R)} u(x,y)\, dxdy. \tag{1}$$

   The left-hand side is the integral of a linear function
$$\int_0^R u(a) rdr = u(a) \int_0^R rdr = u(a) \frac{R^2}{2}. \tag{2}$$

   We just combine (1) and (2) to get the result.

2. Prove Hadamard's three circles theorem: Let $f(z)$ be holomorphic in an open set containing the annulus
$$r_1 \leq |z| \leq r_2, \quad 0 < r_1 < r_2.$$

   Show that with the notation $M(r) = \sup_{|z|=r} |f(z)|$
$$M(r) \leq M(r_1)^{\frac{\log r_2 - \log r}{\log r_2 - \log r_1}} M(r_2)^{\frac{\log r - \log r_1}{\log r_2 - \log r_1}}.$$

   We want to apply the maximum modulus principle to a function of the form $z^\alpha f(z)$. The problem is that for non integer α the function z^α is not holomorphic in a disc or annulus containing 0, as it requires to define first a branch of $\log(z)$ and then define

$z^\alpha = e^{\alpha \log(z)}$. And we know that there is no annulus centered at 0 on which the logarithmic function is defined as a holomorphic function. However, the problem is superficial. The maximum modulus principle can be applied in any small neighborhood off 0, where $z^\alpha f(z)$ can be defined. Moreover, the modulus of z^α is independent of the branch of $\log(z)$ we will use, as $|z^\alpha| = |z|^\alpha$. To understand why the maximum modulus of $z^\alpha f(z)$ is achieved on the boundary of the annulus for any branch of z^α is suffices to assume that it is achieved inside at a point z_0. Then is a small neighborhood of z_0 we violate the standard maximum modulus principle. We get

$$r^\alpha M(r) \leq \max(r_1^\alpha M(r_1), r_2^\alpha M(r_2)). \tag{3}$$

The standard trick is to choose α so that the two expressions on the right become equal. This gives

$$\begin{aligned} r_1^\alpha M(r_1) = r_2^\alpha M(r_2) &\Leftrightarrow \alpha \log r_1 + \log M(r_1) = \alpha \log r_2 + \log M(r_2) \\ &\Leftrightarrow \alpha = \frac{\log M(r_1) - \log M(r_2)}{\log r_2 - \log r_1}. \end{aligned}$$

We take logarithms in (3) and substitute our choice of α to get

$$\begin{aligned} \alpha \log r + \log M(r) &\leq \alpha \log r_1 + \log M(r_1) \Leftrightarrow \\ \log M(r) &\leq \frac{\log M(r_1) - \log M(r_2)}{\log r_2 - \log r_1}(\log r_1 - \log r) + \log M(r_1) \\ \Leftrightarrow \log M(r) &\leq \left(\frac{\log r_1 - \log r}{\log r_2 - \log r_1} + 1\right) \log M(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log M(r_2). \end{aligned}$$

We only now need to add the fractions to get the result in logarithmic form:

$$\frac{\log r_1 - \log r}{\log r_2 - \log r_1} + 1 = \frac{\log r_1 - \log r + \log r_2 - \log r_1}{\log r_2 - \log r_1} = \frac{\log r_2 - \log r}{\log r_2 - \log r_1}.$$

3. Fix $R > 0$. Show that, if n is large enough, then

$$P_n(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$$

has no zeroes in $\{z : |z| \leq R\}$.

We apply Rouché's theorem to $f(z) = e^z$ and $g(z) = P_n(z) - e^z$. We fix R. We consider on $|z| = R$ the modulus of f and g.

$$|f(z)| = |e^z| = e^{\Re(z)} \geq e^{-R}, \quad \text{since } -R \leq \Re(z) \leq R.$$

Moreover, using the Taylor series of e^z, we get

$$|g(z)| = |P_n(z) - e^z| = \left| \sum_{k=n+1}^{\infty} \frac{z^k}{k!} \right| \leq \sum_{k=n+1}^{\infty} \frac{|z|^k}{k!} = \sum_{k=n+1}^{\infty} \frac{R^k}{k!} \to 0, \quad n \to \infty$$

as this is the remainder in the (convergent) Taylor series of e^R. In particular, for n sufficiently large, $|g(z)| \leq e^{-R}$. So we can apply Rouché's theorem to get that for $|z| \leq R$ the functions e^z and $f(z) + g(z) = P_n(z)$ have the same number of zeros. Since e^z never has zeros, the result follows.

4. Let $|f(z)| \leq 1$ for $|z| < 1$ be a non-constant analytic function. Prove that

(i) If $f(0) > 0$, then

$$\frac{|f(0)| - |z|}{1 - |f(0)z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)z|}.$$

Hint: Apply Schwarz lemma to an appropriate composition of functions. Where is the circle of radius r mapped by the standard linear fractional transformations of the unit disc (assume they have real coefficients)?

Consider $T : \mathbb{D} \to \mathbb{D}$ given by

$$T(w) = \frac{w - f(0)}{1 - \overline{f(0)}w}$$

the linear fractional transformation of $\mathbb{D}$ such that $T(f(0)) = 0$. We consider $T \circ f : \mathbb{D} \to \mathbb{D}$ which has $(T \circ f)(0) = 0$. We apply the Schwarz lemma to it:

$$|T(f(z))| \leq |z|, \quad |z| < 1.$$

We have

$$T^{-1}(z) = \frac{z + f(0)}{1 + \overline{f(0)}z}.$$

We fix $|z| = r$. For real $f(0)$ we determine the image of the circle $|z| = r$ by T^{-1}. Since T^{-1} is a linear fractional transformation, the image has to be a circle (inside the unit disc). Since T^{-1} has real coefficients, it maps the real line to the real line. Moreover,

$$T^{-1}(0) = f(0), \quad T^{-1}(r) = \frac{r + f(0)}{1 + f(0)r}, \quad T^{-1}(-r) = \frac{-r + f(0)}{1 - rf(0)}.$$

Since $T^{-1}(0) = f(0)$ is real and $T^{-1}(\infty) = 1/f(0)$ is real, the images have to be symmetric with respect to the image circle. Points symmetric with respect to a circle lie on the ray from the center. This proves that the center of the image circle is real. (Notice the center of $T^{-1}(\{z, |z| = r\})$ is not $T^{-1}(0)$).

We remark that $1 - rf(0) > 0$. Assume first that $f(0) > r$. The point closest to the origin for the image circle is $T^{-1}(-r)$ and the point further is $T^{-1}(r)$ Check:

$$T^{-1}(r) = \frac{r + f(0)}{1 + f(0)r} > T^{-1}(-r) = \frac{-r + f(0)}{1 - rf(0)}$$

$$\Leftrightarrow r + f(0) - r^2 f(0) - rf(0)^2 > -r + f(0) - r^2 f(0) + rf(0)^2 \Leftrightarrow 2r > 2rf(0)^2,$$

i.e. $f(0) < 1$, which is true. Now by $|T(f(z))| \leq |z|$ we see that $f(z)$ belongs to this image circle. So we get the inequalities

$$\frac{f(0) - r}{1 - rf(0)} \leq |f(z)| \leq \frac{r + f(0)}{1 + f(0)r},$$

which is exactly what we want to prove.

In the case $f(0) < r$ we only need to prove the right-hand inequality. In this case $T^{-1}(-r) < 0$ and we see that

$$T^{-1}(r) = \frac{r + f(0)}{1 + f(0)r} > |T^{-1}(-r)| = \frac{r - f(0)}{1 - rf(0)}$$

$$\Leftrightarrow r + f(0) - r^2 f(0) - rf(0)^2 > r - f(0) + r^2 f(0) - rf(0)^2 \Leftrightarrow 2f(0) > 2r^2 f(0),$$

which holds as $f(0) > 0$ and $r < 1$. So the point on the circle further away from 0 is $T^{-1}(r)$. This gives the desired inequality.

(ii) Show that the inequality is true in general, without the assumption $f(0) > 0$, by using an appropriate rotation.

Define $g(z) = f(z)e^{-i \arg f(0)}$. Then g maps the unit disc to the unit disc and $g(0) = |f(0)| > 0$. Moreover, $|g(z)| = |f(z)|$ and this suffices to prove the result. If $f(0) = 0$, the result is just the Schwarz lemma for the right-hand inequality and obvious for the left-hand inequality.

5. (a) Show that $w = \tan(\pi z/4)$ maps the infinite strip $-1 < \Re(z) < 1$ onto the unit disk.

We consider the map

$$w = g(z) = e^{i\pi z/2}.$$

We explain why g maps the strip $|\Re(z)| \leq 1$ to the right hand-plane $\Re(w) \geq 0$. It maps the line $\Re(z) = 1$ to the ray $\{ix, x \geq 0\}$, as, if we set $z = 1 + iy$, we get

$$e^{i\pi(1+iy)/2} = e^{i\pi/2}e^{-\pi y/2} = ie^{-\pi y/2}.$$

Similarly, if $z = -1 + iy$ we have $e^{i\pi z/2} = -ie^{-\pi y/2}$. So g maps $\Re(z) = -1$ to $\{ix, x \leq 0\}$. Moreover, $g(0) = 1$. The fact that the strip has width 2 and we multiply by $i\pi/2$, makes the function g one-to-one as e^z has period $2\pi i$. Now we need to compose with the standard map from the right-half plane $\Re(w) \geq 0$ to the unit disc given by

$$T(w) = \frac{w-1}{i(w+1)} \Longrightarrow T(e^{i\pi z/2}) = \frac{e^{i\pi z/2} - 1}{i(e^{i\pi z/2} + 1)} = \frac{e^{i\pi z/4} - e^{-i\pi z/4}}{i(e^{i\pi z/4} + e^{-i\pi z/4})} = \tan(\pi z/4).$$

Remark: The i in the mapping T just gives the correct rotation (=automorphism of $\mathbb{D}$) to get exactly $\tan(\pi z/4)$. We also see that indeed on the imaginary axis ix the numerator has the same modulus as the denominator: $|T(ix)| = |(ix-1)/(i(ix+1)| = 1$. Moreover, $T(1) = 0$. This explains why T maps the right-half plane to the unit disc.

(b) Let $f(z)$ be a holomorphic function on $|z| < 1$ with $|\Re f(z)| < 1$ and $f(0) = 0$. Show that

$$|\Re(f(z))| \leq \frac{4}{\pi} \arctan|z|, \quad |\Im(f(z))| \leq \frac{2}{\pi} \log \frac{1+|z|}{1-|z|}$$

Hint: Use Exercise 5 in Homework 6.

We use $\phi(z) = z$ and $\psi(z) = S^{-1}(z)$, where $S(z) = T \circ g(z) = \tan(\pi z/4)$. We get that $f(D(0,r)) \subset S^{-1}(D(0,r)) = g^{-1}(T^{-1}(D(0,r)))$ for $0 < r < 1$. Finding the preimage of the circle $C(0,r)$ under S will give bounds for the $\Re f(z)$ and $\Im f(z)$. We first determine the image of the circle $C(0,r)$ under T^{-1}. We solve to get $T^{-1}(z) = (iz+1)/(1-iz)$. It maps circles into circles or lines. Since the only point mapped to infinity is $-i \notin C(0,r)$, $T^{-1}(C(0,r))$ is a circle inside $T^{-1}(\mathbb{D}) = \{z, \Re(z) > 0\}$. We get

$$T^{-1}(ir) = \frac{-r+1}{1+r}, \qquad T^{-1}(-ir) = \frac{r+1}{1-r}, \qquad T^{-1}(0) = 1, \qquad T^{-1}(\infty) = -1.$$

As in problem 4, we get that the points 0 and ∞ are mapped by T^{-1} to points symmetric with respect to $T^{-1}(C(0,r))$ and their line contains the center C of this circle. So the center is on the real axis. It is located at the midpoint of $T^{-1}(ir)$ and $T^{-1}(-ir)$ and the radius R is half the distance between them. We calculate:

$$C = \frac{1}{2}\frac{(1-r)^2+(r+1)^2}{1-r^2} = \frac{1+r^2}{1-r^2}, \qquad R = \frac{1}{2}\frac{-(1-r)^2+(r+1)^2}{1-r^2} = \frac{2r}{1-r^2}.$$

We also have

$$g^{-1}(w) = \frac{2}{i\pi}\log w, \quad \Re g^{-1}(w) = \frac{2}{\pi}\arg w, \quad \Im g^{-1}(w) = -\frac{2}{\pi}\log|w|.$$

The point of maximum modulus on the circle $T^{-1}(C(0,r))$ is the right-most point $T^{-1}(-ir) = (r+1)/(1-r)$. This gives

$$|\Im f(z)|| \le \frac{2}{\pi}\log\frac{r+1}{1-r},$$

which gives the one required inequality with $|z| = r$. For the other, the situation is a bit more complicated because we need to give bounds for the argument on circle $T^{-1}(C(0,r))$. We draw the tangent line from the origin to the circle and use trigonometry. If the maximum argument (=angle) in Figure 5 is a then

$$\sin a = \frac{2r/(1-r^2)}{(1+r^2)/(1-r^2)} = \frac{2r}{1+r^2} \Longrightarrow \cos a = \frac{1-r^2}{1+r^2} \Longrightarrow \tan a = \frac{2r}{1-r^2},$$

(solve $\cos^a + \sin^2 a = 1$, for instance). This gives the inequality

$$|\Re f(z)| \le \frac{2}{\pi}\arctan\frac{2r}{1-r^2} = \frac{2}{\pi}\arctan\frac{2|z|}{1-|z|^2}.$$

This is not the inequality in the form given, but the discrepancy can be explained using trigonometric identities. We have

$$\tan(a) = \frac{2\tan(a/2)}{1-\tan^2(a/2)} \Longrightarrow \tan(a/2) = r \Longrightarrow a = 2\arctan r = 2\arctan|z|.$$

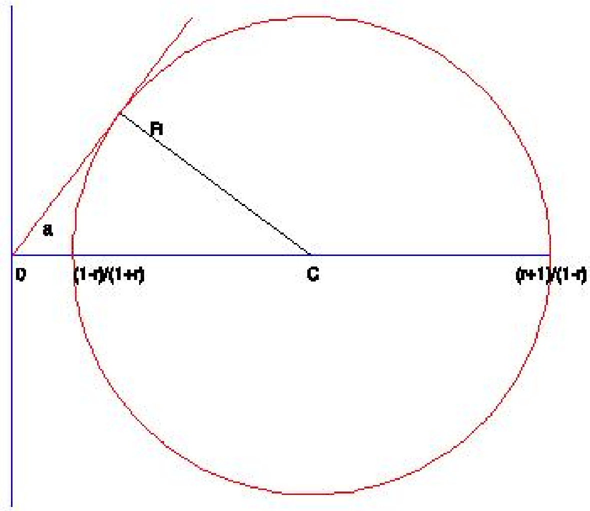

Figure 1: $T^{-1}(C(0,r))$

6. Let $f(z)$ be holomorphic in $|z| < R$ with Taylor expansion $f(z) = \sum a_n z^n$ and set

$$I_2(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta, \quad 0 \le r < R.$$

Show that

(a) $I_2(r) = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$.

We substitute the Taylor series of $f(z)$ to get

$$I_2(r) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \sum_{m=0}^{\infty} \bar{a}_m r^m e^{-im\theta} d\theta = \frac{1}{2\pi}\sum_{n,m=0}^{\infty} a_n \bar{a}_m r^{n+m} \int_0^{2\pi} e^{i(n-m)\theta} d\theta,$$

by uniform convergence of the Taylor series inside the radius of convergence. Since

$$\int_0^{2\pi} e^{ij\theta} d\theta = \begin{cases} 2\pi, & j = 0, \\ 0, & j \neq 0, \end{cases}$$

we get

$$I_2(r) = \frac{1}{2\pi}\sum_{n=0}^{\infty} |a_n|^2 r^{2n} 2\pi.$$

(b) $I_2(r)$ is increasing.

This follows from (a), since each term increases for larger r.

(c) $|f(0)|^2 \le I_2(r) \le M(r)^2$, with $M(r) = \sup_{|z|=r} |f(z)|$.

We have $a_0 = f(0)$, so that $|f(0)|^2 \le \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$, as $|a_0|^2$ is the first term in the series of positive terms. For the second inequality we resort to the definition of $I_2(r)$: since $|f(re^{i\theta})| \le M(r)$, we get

$$I_2(r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \le \frac{1}{2\pi}\int_0^{2\pi} M(r)^2 d\theta = M(r)^2.$$

(d) $\log I_2(r)$ is a convex function of $\log r$, when f is not identically zero. This means

$$\log I_2(r) \le \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log I_2(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log I_2(r_2).$$

Hint: Set $u = \log r$, $J(u) = I_2(e^u)$, show that $d^2 \log J(u)/du^2 = (JJ_2'' - J'^2)/J^2$ and use the Cauchy-Schwarz inequality.

With this definition of $J(u)$ and the quotient rule we get

$$(\log J(u))' = \frac{J'(u)}{J(u)}, \qquad (\log J(u))'' = \frac{J''(u)J(u) - J'(u)J'(u)}{J^2(u)}.$$

Now using (a) and $r = e^u$ we get

$$J(u) = \sum_{n=0}^{\infty} |a_n|^2 e^{2nu}, \quad J'(u) = \sum_{n=0}^{\infty} |a_n|^2 2n e^{2nu}, \quad J''(u) = \sum_{n=0}^{\infty} |a_n|^2 (2n)^2 e^{2nu}.$$

Now we apply Cauchy-Schwarz inequality for sequences in the form

$$\left|\sum_n c_n d_n\right|^2 \le \sum_n |c_n|^2 \sum_n |d_n|^2.$$

We take $c_n = a_n e^{nu}$ and $d_n = 2n\bar{a}_n e^{nu}$ to get

$$J'(u)^2 \le \sum_{n=0}^{\infty} |a_n|^2 e^{2nu} \sum_{n=0}^{\infty} |a_n|^2 (2n)^2 e^{2nu} = J(u)J''(u).$$

This implies that $(\log J(u))'' \ge 0$, i.e. $\log J(u)$ is a convex function of u. This means that the graph is less than any secant segment on its graph. We fix $u_1 = \log r_1$ and $u_2 = \log r_2$. The segment between them is $au_1 + (1-a)u_2$. To get u we choose $a = (u_2 - u)/(u_2 - u_1)$ i.e.

$$u \frac{u_2 - u}{u_2 - u_1} u_1 + \frac{u - u_1}{u_2 - u_1} u_2.$$

We join the points $(u_1, \log J(u_1))$ and $(u_2, \log J(u_2))$ on the graph of $\log J(u)$ with a line segment of slope

$$m = \frac{\log J(u_2) - \log J(u_1)}{u_2 - u_1} = \frac{\log I_2(r_2) - \log I_2(r_1)}{\log r_2 - \log r_1}$$

with equation

$$y = m(u - u_1) + \log J(u_1) = \frac{\log I_2(r_2) - \log I_2(r_1)}{\log r_2 - \log r_1}(u - \log r_1) + \log I_2(r_1).$$

At a given $u = \log r$ this line segment is higher in the plane than the point on the graph $(u, \log J(u))$:

$$\log J(u) = \log I_2(r) \le \frac{\log I_2(r_2) - \log I_2(r_1)}{\log r_2 - \log r_1}(\log r - \log r_1) + \log I_2(r_1)$$

$$= \left(1 + \frac{\log r_1 - \log r}{\log r_2 - \log r_1}\right) \log I_2(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log I_2(r_2).$$

Now we add the fraction and 1 in parentheses to get the required inequality.

7. Let $U(\xi)$ be piecewise continuous and bounded for all real ξ. Show that

$$P_U(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} U(\xi) d\xi$$

is a harmonic function in the upper half plane with boundary values $U(\xi)$ at points of continuity. This is the Poisson integral for the half-plane.

This is really an exercise in changing variables in the integral using

$$T : \mathbb{H} \to \mathbb{D}, \quad w = T(z) = \frac{z-i}{z+i}, \qquad T'(z) = \frac{2i}{(z+i)^2}.$$

We set $T(\xi) = e^{it}$. We define $\mathbf{V}(t) = V(e^{it}) = U(\xi)$, i.e. we set $V \circ T = U$ and we consider the circle to be parametrized with $t \in [0, 2\pi]$. Since $ie^{it}dt = T'(\xi)d\xi$, we get that

$$dt = \frac{T'(\xi)d\xi}{iT(\xi)} = \frac{2i/(\xi+i)^2}{i(\xi-i)/(\xi+1)} d\xi = \frac{2d\xi}{\xi^1 + 1}.$$

We know that the Poisson integral $P_{\mathbf{V}}(w)$ is a harmonic function in the unit disc with boundary values $V(e^{it})$ at points of continuity. Since the composition of a harmonic function with a holomorphic function is still harmonic, we need to show that $P_{\mathbf{V}}(w)$ can be written as $P_U(z)$ in the z, ξ variables. The boundary behavior is obvious since $T(\mathbb{R})$ is the unit circle. We have with $w = re^{i\theta} = T(z)$

$$P_{\mathbf{V}}(w) = \frac{1}{2\pi} \int_0^{2\pi} \Re\left(\frac{1 + re^{i(\theta-t)}}{1 - re^{i(\theta-t)}}\right) V(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} \Re\left(\frac{1 + we^{-it}}{1 - we^{-it}}\right) V(e^{it}) dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |w|^2}{|1 - we^{-it}|^2} V(e^{it}) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - |T(z|^2}{|1 - T(z)\overline{T(\xi)}|^2} U(\xi) \frac{2}{\xi^2+1} d\xi$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - |(z-i)/(z+i)|^2}{|1 - \overline{T(\xi)}(z-i)/(z+i)|^2} U(\xi) \frac{2}{\xi^2+1} d\xi = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|z+i|^2 - |z-i|^2}{|z+i - \overline{T(\xi)}(z-i)|^2} U(\xi) \frac{d\xi}{\xi^2+1}$$

We calculate the numerator and denominator in the fraction with $z = x + iy$:

$$|z+i|^2 - |z-i|^2 = x^2 + (y+i)^2 - (x^2 + (y-1)^2) = 4y,$$

$$|(z+i)-(z-i)(\xi+i)/(\xi-i)|^2 = \frac{1}{\xi^2+1}|(\xi-i)(z+i)-(\xi+i)(z-i)|^2 = \frac{1}{\xi^2+1}|2i\xi-2iz|^2$$
$$= \frac{4}{\xi^2+1}((x-\xi)^2+y^2).$$

This gives finally

$$P_{\mathbf{V}}(T(z)) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{4y}{4((x-\xi)^2+y^2)} U(\xi) d\xi,$$

which is exactly the given formula.

8. Let

$$P_r(t) = \Re\left(\frac{1+z}{1-z}\right), \quad z = re^{it}$$

be the Poisson kernel for the unit disc $|z| < 1$. Let $U(\theta)$ be a continuous function of the interval $[0, \pi]$ with $U(0) = U(\pi) = 0$. Show that the function

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{\pi} \{P_r(t-\theta) - P_r(t+\theta)\} U(t)\, dt$$

is harmonic in the half-disc

$$\{re^{i\theta}, 0 \le r < 1, 0 \le \theta \le \pi\}$$

and has the following limiting behavior on the boundary:

$$\lim_{z \to e^{i\theta_0}} u(z) = U(\theta_0), \quad 0 < \theta_0 < \pi$$

$$u(x) = 0, \quad -1 < x < 1.$$

We define the following function as an extension of U from $[0, \pi]$ to the interval $[-\pi, \pi]$:

$$\mathbf{U}(\theta) = \begin{cases} U(\theta), & \theta \in [0, \pi] \\ -U(-\theta), & \theta \in [-\pi, 0] \end{cases}$$

We recall that the Poisson kernel is an even function of the angle. We consider $P_{\mathbf{U}}(z)$, the convolution integral of $\mathbf{U}$ with the Poisson kernel ($z = re^{i\theta}$):

$$P_{\mathbf{U}}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\mathbf{U}(t)\, dt = \frac{1}{2\pi}\int_{-\pi}^{0} P_r(\theta-t)(-U(-t))dt + \frac{1}{2\pi}\int_0^{\pi} P_r(\theta-t)U(t)dt$$

$$\overset{t=-s}{=} -\frac{1}{2\pi}\int_0^{\pi} P_r(\theta+s)U(s)ds + \frac{1}{2\pi}\int_0^{\pi} P_r(\theta-t)U(t)dt = \frac{1}{2\pi}\int_0^{2\pi} \{P_r(\theta-t) - P_r(\theta+t)\} U(t)dt.$$

This explains are choice for extending U as an odd function: we get the formula given. Moreover, we know that $P_{\mathbf{U}}(z)$ is harmonic on the whole unit disc, so, in particular, it is harmonic in the upper half of it. We examine the limiting behavior. Since $U(\theta)$ is continuous, its extension is continuous on the whole circle, so Schwarz' theorem gives

$$\lim_{r\to 1} P_{\mathbf{U}}(re^{it}) = \mathbf{U}(e^{it}) = U(e^{it}), \qquad t \in [0,\pi].$$

It remains to show the behavior on the segment $(-1,1)$. These points are interior to the unit disc. For $0 < x < 1$ we have $\theta = 0$, which gives

$$u(x) = \frac{1}{2\pi}\int_0^\pi \{P_x(t-0) - P_x(t+0)\}\, U(t)dt = \frac{1}{2\pi}\int_0^\pi 0U(t)dt = 0.$$

For $-1 < x < 0$, we have $\theta = \pi$. Since $P_r(\pi + t) = P_r(-\pi - t)$ we get

$$2\pi u(x) = \int_0^\pi \{P_{-x}(\pi - t) - P_{-x}(\pi + t)\}\, U(t)dt =$$

$$\int_0^\pi \left\{\Re\left(\frac{1+re^{i(\pi-t)}}{1-re^{i(\pi-t)}}\right) - \Re\left(\frac{1+re^{i(-\pi-t)}}{1-re^{i(-\pi-t)}}\right)\right\} U(t)dt = \int_0^\pi \Re\left(\frac{1-re^{-it}}{1+re^{-it}} - \frac{1-re^{-it}}{1+re^{-it}}\right) U(t)dt = 0$$

since $e^{i\pi} = -1 = e^{-i\pi}$.