Math 70300

Homework 7

Due: within 96 hours

1. Let u be harmonic in a region G and suppose that the closed disc $\overline{D(a,R)}$ is contained in G. Show that
$$u(a) = \frac{1}{\pi R^2} \int_{D(a,R)} u(x,y)\,dxdy.$$
Hint: Use polar coordinates.

2. Prove Hadamard's three circles theorem: Let $f(z)$ be holomorphic in an open set containing the annulus
$$r_1 \leq |z| \leq r_2, \quad 0 < r_1 < r_2.$$
Show that with the notation $M(r) = \sup_{|z|=r} |f(z)|$
$$M(r) \leq M(r_1)^{\frac{\log r_2 - \log r}{\log r_2 - \log r_1}} M(r_2)^{\frac{\log r - \log r_1}{\log r_2 - \log r_1}}.$$

3. Fix $R > 0$. Show that, if n is large enough, then
$$P_n(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$$
has no zeroes in $\{z : |z| \leq R\}$.

4. Let $|f(z)| \leq 1$ for $|z| < 1$ be a non-constant analytic function. Prove that
(i) If $f(0) > 0$, then
$$\frac{|f(0)| - |z|}{1 - |f(0)z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + |f(0)z|}.$$
Hint: Apply Schwarz lemma to an appropriate composition of functions. Where does the circle of radius r is mapped by the standard linear fractional transformations of the unit disc (assume they have real coefficients)?
(ii) Show that the inequality is true in general, without the assumption $f(0) > 0$, by using an appropriate rotation.

5. (a) Show that $w = \tan(\pi z/4)$ maps the infinite strip $-1 < \Re(z) < 1$ onto the unit disk.
(b) Let $f(z)$ be a holomorphic function on $|z| < 1$ with $|\Re f(z)| < 1$ and $f(0) = 0$. Show that
$$|\Re(f(z))| \leq \frac{4}{\pi} \arctan |z|, \quad |\Im(f(z))| \leq \frac{2}{\pi} \log \frac{1 + |z|}{1 - |z|}$$
Hint: Use Exercise 5 in Homework 6.

6. Let $f(z)$ be holomorphic in $|z| < R$ with Taylor expansion $f(z) = \sum a_n z^n$ and set

$$I_2(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta, \quad 0 \le r < R.$$

Show that

(a) $I_2(r) = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$.

(b) $I_2(r)$ is increasing.

(c) $|f(0)|^2 \le I_2(r) \le M(r)^2$, with $M(r) = \sup_{|z|=r} |f(z)|$.

(d) $\log I_2(r)$ is a convex function of $\log r$, when f is not identically zero. This means

$$\log I_2(r) \le \frac{\log r_2 - \log r}{\log r_2 - \log r_1} \log I_2(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log I_2(r_2).$$

Hint: Set $u = \log r$, $J(u) = I_2(e^u)$, show that $d^2 \log J(u)/du^2 = (JJ_2'' - J'^2)/J^2$ and use the Cauchy-Schwarz inequality.

7. Let $U(\xi)$ be piecewise continuous and bounded for all real ξ. Show that

$$P_U(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} U(\xi) d\xi$$

is a harmonic function in the upper half plane with boundary values $U(\xi)$ at points of continuity. This is the Poisson integral for the half-plane.

8. Let

$$P_r(t) = \Re\left(\frac{1+z}{1-z}\right), \quad z = re^{it}$$

be the Poisson kernel for the unit disc $|z| < 1$. Let $U(\theta)$ be a continuous function of the interval $[0, \pi]$ with $U(0) = U(\pi) = 0$. Show that the function

$$u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{\pi} \{P_r(t-\theta) - P_r(t+\theta)\} U(t)\, dt$$

is harmonic in the half-disc

$$\{re^{i\theta}, 0 \le r < 1, 0 \le \theta \le \pi\}$$

and has the following limiting behavior on the boundary:

$$\lim_{z \to e^{i\theta_0}} u(z) = U(\theta_0), \quad 0 < \theta_0 < \pi$$

$$u(x) = 0, \quad -1 < x < 1.$$