Math 70300

Homework 7

Due: within 96 hours

1. Let $u$ be harmonic in a region $G$ and suppose that the closed disc $D(a, R)$ is contained in $G$. Show that

$$u(a) = \frac{1}{\pi R^2} \int_{D(a, R)} u(x, y) \, dx \, dy.$$ 

*Hint:* Use polar coordinates.

2. Prove Hadamard’s three circles theorem: Let $f(z)$ be holomorphic in an open set containing the annulus

$$r_1 \leq |z| \leq r_2, \quad 0 < r_1 < r_2.$$ 

Show that with the notation $M(r) = \sup_{|z|=r} |f(z)|$

$$M(r) \leq M(r_1) \frac{\log r_2 - \log r}{\log r_2 - \log r_1} M(r_2) \frac{\log r - \log r_1}{\log r_2 - \log r_1}.$$ 

3. Fix $R > 0$. Show that, if $n$ is large enough, then

$$P_n(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}$$ 

has no zeroes in $\{z : |z| \leq R\}$.

4. Let $|f(z)| \leq 1$ for $|z| < 1$ be a non-constant analytic function. Prove that

(i) If $f(0) > 0$, then

$$\frac{|f(0)| - |z|}{1 - \frac{|f(0)|}{1}} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 + \frac{|f(0)|}{1}}.$$ 

*Hint:* Apply Schwarz lemma to an appropriate composition of functions. Where does the circle of radius $r$ is mapped by the standard linear fractional transformations of the unit disc (assume they have real coefficients)?

(ii) Show that the inequality is true in general, without the assumption $f(0) > 0$, by using an appropriate rotation.

5. (a) Show that $w = \tan(\pi z/4)$ maps the infinite strip $-1 < \Re(z) < 1$ onto the unit disk.

(b) Let $f(z)$ be a holomorphic function on $|z| < 1$ with $|\Re f(z)| < 1$ and $f(0) = 0$. Show that

$$|\Re(f(z))| \leq \frac{4}{\pi} \arctan |z|, \quad |\Im(f(z))| \leq \frac{2}{\pi} \log \frac{1 + |z|}{1 - |z|}.$$ 

*Hint:* Use Exercise 5 in Homework 6.
6. Let \( f(z) \) be holomorphic in \( |z| < R \) with Taylor expansion \( f(z) = \sum a_n z^n \) and set
\[
I_2(r) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta, \quad 0 \leq r < R.
\]
Show that
(a) \( I_2(r) = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \).
(b) \( I_2(r) \) is increasing.
(c) \( |f(0)|^2 \leq I_2(r) \leq M(r)^2 \), with \( M(r) = \sup_{|z|=r} |f(z)| \).
(d) \( \log I_2(r) \) is a convex function of \( \log r \), when \( f \) is not identically zero. This means
\[
\log I_2(r) \leq \log r_2 - \log r \log I_2(r_1) + \frac{\log r - \log r_1}{\log r_2 - \log r_1} \log I_2(r_2).
\]
*Hint:* Set \( u = \log r, \ J(u) = I_2(e^u) \), show that \( d^2 \log J(u)/du^2 = (JJ'' - J'^2)/J^2 \) and use the Cauchy-Schwarz inequality.

7. Let \( U(\xi) \) be piecewise continuous and bounded for all real \( \xi \). Show that
\[
P_U(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2 + y^2} U(\xi) d\xi
\]
is a harmonic function in the upper half plane with boundary values \( U(\xi) \) at points of continuity. This is the Poisson integral for the half-plane.

8. Let
\[
P_r(t) = \Re \left( \frac{1+z}{1-z} \right), \quad z = re^{it}
\]
be the Poisson kernel for the unit disc \( |z| < 1 \). Let \( U(\theta) \) be a continuous function of the interval \([0, \pi]\) with \( U(0) = U(\pi) = 0 \). Show that the function
\[
u(re^{i\theta}) = \frac{1}{2\pi} \int_0^{\pi} \{P_r(t-\theta) - P_r(t+\theta)\} U(t) \, dt
\]
is harmonic in the half-disc
\[
\{re^{i\theta}, 0 \leq r < 1, 0 \leq \theta \leq \pi\}
\]
and has the following limiting behavior on the boundary:
\[
\lim_{z \rightarrow e^{i\theta_0}} u(z) = U(\theta_0), \quad 0 < \theta_0 < \pi
\]
\[
u(x) = 0, \quad -1 < x < 1.
\]