Math 70300

Homework 6

Due: December 5

1. Let $f(z)$ be a holomorphic function in the disc $|z| < R_1$ and set

$$M(r) = \sup_{|z|=r} |f(z)|, \quad A(r) = \sup_{|z|=r} \Re(f(z)), \quad 0 \le r < R_1.$$

(a) Show that $M(r)$ is monotonic and, in fact, strictly increasing, unless f is a constant.

(b) Show that $A(r)$ is monotonic and, in fact, strictly increasing, unless f is constant.

2. Assume $f(z)$ is a holomorphic function on $|z| \le 1$ with $|f(z)| \le 1$. Show that

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}.$$

When does equality hold for a point z_0 inside $|z| < 1$?

3. Let $\mathbb{D} = \{z : |z| < 1\}$. Suppose that $f : \mathbb{D} \to \mathbb{D}$ is analytic, $f(1/3) = 0$ and $f'(1/3) = 0$. Show that $|f(0)| \le 1/9$.

4. Prove that if $f(z) : \mathbb{H} \to \mathbb{H}$ is an analytic function from the upper-half plane to itself, then

$$\frac{|f(z) - f(z_0|}{|f(z) - \overline{f(z_0)}|} \le \frac{|z - z_0|}{|z - \overline{z_0}|}, \quad z, z_0 \in \mathbb{H}$$

and

$$\frac{|f'(z)|}{\Im f(z)} \le \frac{1}{\Im z}, \quad z \in \mathbb{H}.$$

When does equality hold?

5. Suppose $z = \phi(\zeta)$ and $w = \psi(\zeta)$ are one-to-one analytic maps from the unit disc $D(0,1)$ onto the regions G_1 and G_2. Set $\phi(0) = z_0$ and $\psi(0) = w_0$. Let $0 < r < 1$ and $\Omega_1(r) = \phi(D(0,r))$, $\Omega_2(r) = \psi(D(0,r))$. Assume $f : G_1 \to G_2$ be a holomorphic map with $f(z_0) = w_0$. Show that

$$f(\Omega_1(r)) \subset \Omega_2(r).$$

6. Show that if an entire function f maps the real axis into itself and the imaginary axis into itself, then f is an odd function, i.e., $f(-z) = -f(z)$ for any z.

Give two proofs, which are really different.

7. (a) Consider two rectangles $R = [0, a] \times [0, b]$ and $R' = [0, a'] \times [0, b']$. Suppose $f : R \to R'$ is a homeomorphism which is holomorphic in the interior of R and maps a-sides to a'-sides and b-sides to b'-sides (i.e., $f([0, a] \times \{0\}) = [0, a'] \times \{0\}$, $f([0, a] \times \{b\}) = [0, a'] \times \{b'\}$, $f(\{0\} \times [0, b]) = \{0\} \times [0, b']$, and $f(\{a\} \times [0, b]) = \{a'\} \times [0, b']$). Show that $a/b = a'/b'$.

   (b) Let $A = \{z, R_1 \leq |z| \leq R_2\}$ and $B = \{w, r_1 \leq |w| \leq r_2\}$ be two annuli and $f : A \to B$ be a holomorphic one-to-one and onto map that maps $|z| = R_i$ to $|w| = r_i$, $i = 1, 2$. Show that there exists a $c \in \mathbb{C}$, $|c| = 1$ such that

$$\frac{R_1}{R_2} = \frac{r_1}{r_2}$$

   and

$$f(z) = c\frac{r_1}{R_1}z.$$

8. (a) Let f be analytic in a bounded region D and its boundary C, such that $|f(z)| = 1$ on C. Show that f has at least one zero inside D, unless f is a constant.

   (b) Let $f(z)$ be an analytic function in a region D except for one simple pole and assume $|f(z)| = 1$ on the boundary of D. Prove that every value a with $|a| > 1$ is taken by $f(z)$ inside D once and once only.

9. (a) How many roots of the equation $z^4 - 6z + 3 = 0$ have their modulus between 1 and 2?

   (b) Find the number of the roots of the equation

$$z^6 - 5z^4 + 8z - 1 = 0$$

   in the annulus $\{z : 1 < |z| < 2\}$.

10. Let λ be real and $\lambda > 1$, Show that the equation

$$ze^{\lambda - z} = 1$$

   has exactly one solution in the disc $|z| = 1$, which is real and positive.