Math 70300

Homework 5

Due: November 14

1. Calculate the integrals using contour integration. Complete explanations are required.

$$\text{(i)} \int_0^\infty \frac{dx}{x^3+1} \quad \text{(ii)} \int_0^\infty \frac{\cos x}{x^2+1} dx, \quad \text{(iii)} \int_0^\infty \frac{(\log x)^2}{1+x^2} dx$$

$$\text{(iv)} \int_0^\infty \frac{\cos(ax)}{(x^2+b^2)^2} dx \quad (a,b>0), \quad \text{(v)} \int_0^{\pi/2} \frac{d\theta}{a+\sin^2\theta} \quad (a>0), \quad \text{(vi)} \int_0^\infty \frac{x^2 dx}{x^4+5x^2+6}$$

$$\text{(vii)} \int_{-\infty}^\infty \frac{dx}{(1+x^2)^{n+1}} \quad \text{(viii)} \int_0^\infty \frac{\log x}{(1+x^2)^2} dx \quad \text{(ix)} \int_0^{2\pi} \frac{d\theta}{(a+\cos\theta)^2} \quad (a>1).$$

Answers: (i) $2\pi/(3\sqrt{3})$, (ii) $\pi e^{-1}/2$, (iii) $\pi^3/8$, (iv) $\pi(1+ab)e^{-ab}/(4b^3)$, (v) $\pi/(2\sqrt{a^2+a})$, (vi) $(\sqrt{3}-\sqrt{2})\pi/2$, (vii) $1{\cdot}3{\cdot}5\cdots(2n-1)\pi/(2{\cdot}4{\cdot}6\cdots(2n))$, (viii) $-\pi/4$, (ix) $2\pi a/(a^2-1)^{3/2}$.

2. Prove that

$$\sum_{n=-\infty}^\infty \frac{1}{(u+n)^2} = \frac{\pi^2}{\sin^2(\pi u)} \quad (u \notin \mathbb{Z}), \quad \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6},$$

using the function $f(z) = \dfrac{\pi\cot(\pi z)}{(u+z)^2}$ integrated over the boundary of the square $[-(N+1/2), N+1/2] \times [-(N+1/2), N+1/2]$, $N \geq |u|$, $N \in \mathbb{N}$. This is one of the many derivations of the value $\sum 1/n^2$, due originally to Euler.

3. In this problem $\int_{c-i\infty}^{c+i\infty}$ denotes a contour integral along the vertical line $\Re(s) = c$ traversed upwards.

(a) Prove that for $c>0$ we have $\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s^2} ds = \begin{cases} \log x, & x>1, \\ 0, & 0<x\leq 1. \end{cases}$

(b) Prove that, for $c>0$, $\dfrac{1}{2\pi i}\displaystyle\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} ds = \begin{cases} 1, & x>1, \\ 1/2, & x=1, \\ 0, & 0<x<1. \end{cases}$ (Perron formula)

(c) Let the function $f(s)$ be defined by the absolutely convergent series

$$f(s) = \sum_{n=1}^\infty \frac{a_n}{n^s}, \quad \Re(s) > a \geq 0.$$

Show that for $x \notin \mathbb{Z}$

$$\sum_{n\leq x} a_n = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{x^s}{s} ds, \quad c>a.$$