Math 70300

Homework 4

Due: within 72 hours

1. (a) Let z_1 and z_2 be two points on a circle C. Let z_3 and z_4 be symmetric with respect to the circle. Show that the cross ratio (z_1, z_2, z_3, z_4) has absolute value 1.

   Use a linear fractional transformation that maps the circle to the real axis (such T exists). Then $T(z_3)$ and $T(z_4)$ are symmetric with respect to the real line, i.e., $T(z_3) = \overline{T(z_4)}$. On the other hand $T(z_1)$ and $T(z_2)$ are real, so equal to their conjugate. This gives $\overline{T(z_1) - T(z_3)} = T(z_1) - T(z_4)$, $\overline{T(z_2) - T(z_3)} = T(z_2) - T(z_4)$, and as a consequence they have the same modulus (conjugate numbers have the same absolute value). This gives, using the invariance of the cross ratio under a linear fractional transformation,

$$|(z_1, z_2, z_3, z_4)| = |(Tz_1, Tz_2, Tz_3, Tz_4)| = \left|\frac{Tz_1 - Tz_3}{Tz_1 - Tz_4}\right| \cdot \left|\frac{Tz_2 - Tz_4}{Tz_2 - Tz_3}\right| = 1 \cdot 1 = 1.$$

   (b) Let $ad - bc = 1$, $c \neq 0$ and consider $T(z) = \dfrac{az+b}{cz+d}$. Show that it increases lengths and areas inside the circle $|cz + d| = 1$ and decreases lengths and areas outside the circle $|cz + d| = 1$.

   We have

$$T'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad - bc}{(cz+d)^2} = \frac{1}{(cz+d)^2}.$$

   This means that $|T'(z)| = |cz + d|^{-2}$. The inside of the circle is given by $|cz + d| < 1$, while the outside by $|cz + d| > 1$. This means that inside we have $|T'(z)| > 1$ and outside we have $|T'(z)| < 1$. The infinitesimal magnification factor at z for the length is $|T'(z)| < 1$ and for the area is $|T'(z)|^2$. This essentially proves the result. If one wants to be more precise, let $\gamma(t), a \leq t \leq b$ be a smooth curve and $T(\gamma(t))$ its image under T. Then

$$L(\gamma(t)) = \int_a^b |\gamma'(t)|dt, \quad L(T(\gamma(t))) = \int_a^b |T'(\gamma(t))||\gamma'(t)|dt$$

   If the curve is inside the circle $|T'(\gamma(t)| > 1$, so

$$L(T(\gamma(t))) \geq \int_a^b |\gamma'(t)|dt = L(\gamma(t)).$$

   The opposite inequality holds outside the circle. For the areas we have

$$A(U) = \int_U 1dxdy, \quad A(T(U)) = \int_U |T'(z)|^2 dxdy.$$

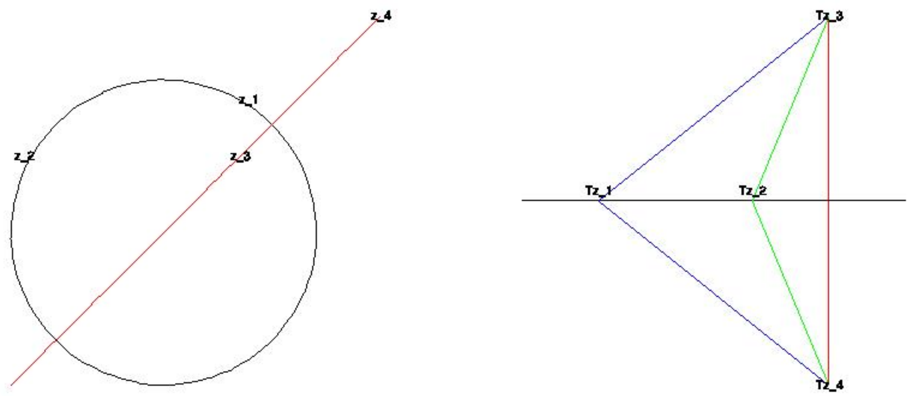

Figure 1: The circle and the real line in 1(a)

If the set U is inside the circle $|T'(\gamma(t)| > 1$, so

$$A(T(U)) \geq \int_U 1 dx dy = A(U).$$

The opposite inequality holds outside the circle.

2. (a) By considering the contour integral

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}$$

prove that

$$\int_0^{2\pi} \cos^{2n}\theta \, d\theta = 2\pi \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

We use the binomial theorem for $(z + 1/z)^n$ to get

$$\int_{|z|=1} \sum_{j=0}^{2n} \left(\begin{array}{c} 2n \\ j \end{array} \right) z^j (z^{-1})^{2n-j} z^{-1} dz = \sum_{j=0}^{2n} \left(\begin{array}{c} 2n \\ j \end{array} \right) \int_{|z|=1} z^{2j-2n-1} dz = 2\pi i \left(\begin{array}{c} 2n \\ n \end{array} \right),$$

since $\int_{|z|=1} z^k dz = 2\pi i$ if $k = -1$ and 0 otherwise. Now we substitute $z = e^{i\theta}$ to get (use $2\cos\theta = e^{i\theta} + e^{-i\theta}$)

$$\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z} = \int_0^{2\pi} (2\cos\theta)^{2n} \frac{ie^{i\theta}}{e^{i\theta}} d\theta = 2^{2n} i \int_0^{2\pi} \cos^{2n}\theta \, d\theta.$$

Comparing the two results we get

$$\int_0^{2\pi} \cos^{2n}\theta \, d\theta = \frac{2\pi}{2^{2n}} \left(\begin{array}{c} 2n \\ n \end{array} \right) = \frac{2\pi}{2^{2n}} \frac{(2n)!}{n!n!} = \frac{2\pi 1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n-1) \cdot (2n)}{2^{2n} 1 \cdot 2 \cdots n \cdot 1 \cdot 2 \cdots n}.$$

We cancel the even numbers from the numerator with a factor of 2 and an integer from 1 to n. This gives

$$\int_0^{2\pi} \cos^{2n}\theta\, d\theta = \frac{2\pi \cdot 1 \cdot 3 \cdots (2n-1)}{2^n \cdot 1 \cdot 2 \cdots n}$$

and this gives the result by doubling every integer from 1 to n to get the even integers from 2 to $2n$ in the denominator.

(b) Prove that

$$\int_0^{\pi/2} \sin^{2n}\theta\, d\theta = \frac{\pi}{2}\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

The substitution $\theta = x + \pi/2$ gives in (a) (use that $\cos\theta = \cos(x+\pi/2) = -\sin x$)

$$\int_0^{2\pi} \cos^{2n}\theta\, d\theta = \int_{-\pi/2}^{3\pi/2} \sin^{2n} x\, dx = \int_0^{2\pi} \sin^{2n} x\, dx$$

using the periodicity of $\sin x$. Now we remark that the values of $\sin^2 x$ on $[0, \pi/2]$ are the same on the other three intervals $[\pi/2, \pi]$, $[\pi, 3\pi/2]$ and $[3\pi/2, 2\pi]$, so the same is true for $\sin^{2n} x$. This means that the integral we want is $1/4$ the result from (a).

3. Map conformally the region inside both circles $|z-1| < 1$ and $|z+i| < 1$ to the upper-half plane $\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\}$.

   Hint: Use first the map $w = z^{-1}$ and at some later stage use $w = z^2$.

   Let $C_1 = \{z : |z-1| = 1\}$ and $C_2 = \{z : |z+i| < 1\}$. The map $w = T(z) = z^{-1}$ maps the two circles to two lines, as $T(0) = \infty$ and 0 belongs to both circles. The second common point of the two circles is $1-i$ (check this!). This is mapped to $T(1-i) = 1/2 + i/2$. The two circles are orthogonal at 0, as the tangent lines there are the real and the imaginary axis. This is also true at $1-i$ (the tangent lines are horizontal and vertical). So the images of the two circles are two lines meeting perpendicularly at $1/2 + i/2$ and ∞. Since $T(2) = 1/2$, $T(C_1) = \{w : \Re w = 1/2\}$ and, in fact, as we traverse C_1 in the positive sense, the image is traversed from up to down (preservation of the orientation). Since $T(-2i) = i/2$, $T(C_2) = \{w : \Im w = 1/2\}$ and, in fact, as we traverse C_2 in the positive sense, the image is traversed from left to right. This implies that the region between the two circles is mapped to the set $\Omega = \{w : \Re w > 1/2, \Im w > 1/2\}$. Now translate Ω to the first quadrant by the map $S(w) = w - (1/2 + i/2)$. The first quadrant is mapped conformally to the upper half plane by the squaring function $H(z) = z^2$. The final map is the composition $H \circ S \circ T$.

4. Let $f(z)$ be holomorphic on the unit disc and $f(0) = 1$. By working with

$$\frac{1}{2\pi i}\int_{|z|=1} \left[2 \pm (z + \frac{1}{z})\right] f(z)\frac{dz}{z}$$

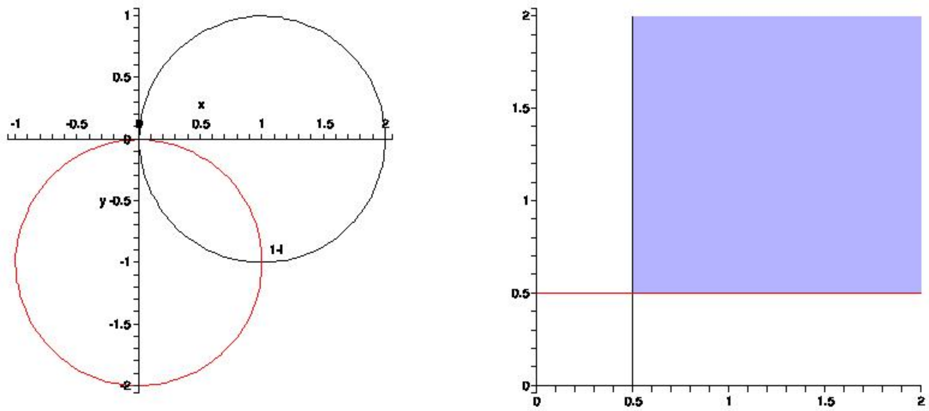

Figure 2: The region in problem 3 and its image by $w = z^{-1}$

prove that

$$\frac{2}{\pi}\int_0^{2\pi} f(e^{i\theta})\cos^2\frac{\theta}{2}\,d\theta = 2 + f'(0), \quad \frac{2}{\pi}\int_0^{2\pi} f(e^{i\theta})\sin^2\frac{\theta}{2}\,d\theta = 2 - f'(0).$$

We have $2\sin^2(\theta/2) = 1 - \cos\theta$ and $2\cos^2(\theta/2) = 1 + \cos\theta$. Parameterizing the circle as $z = e^{i\theta}$ we get

$$\frac{1}{2\pi i}\int_{|z|=1}\left[2 \pm (z + \frac{1}{z})\right] f(z)\frac{dz}{z} = \frac{1}{2\pi i}\int_0^{2\pi}(2 \pm 2\cos\theta) f(e^{i\theta}) i d\theta$$

$$= \frac{1}{\pi}\int_0^{2\pi}(1 \pm \cos\theta) f(e^{i\theta}) d\theta = \frac{2}{\pi}\int_0^{2\pi}\left\{ \begin{array}{c} \cos^2(\theta/2) \\ \sin^2(\theta/2) \end{array} \right\} f(e^{i\theta}) d\theta.$$

This is how we get the two integrals on the left-hand side of the result. For the right-hand sides we use the Cauchy Integral formulas:

$$\frac{1}{2\pi i}\int_{|z|=1} f(z)\frac{dz}{z} = f(0) = 1, \quad \frac{1}{2\pi i}\int_{|z|=1|}\frac{f(z)}{z^2}dz = f'(0),$$

while Cauchy's theorem gives directly

$$\frac{1}{2\pi i}\int_{|z|=1} f(z)\,dz = 0.$$

As a result

$$\frac{1}{2\pi i}\int_{|z|=1}\left[2 \pm (z + \frac{1}{z})\right] f(z)\frac{dz}{z} = \frac{1}{2\pi i}\int_{|z|=1}\left(2\frac{f(z)}{z} \pm (f(z) + \frac{f(z)}{z^2})\right) dz = 2 \pm f'(0).$$

(b) If $f(z)$ is holomorphic on $|z| \le 1$, $f(0) = 1$, and for all $|z| \le 1$ we have $\Re(f(z)) \ge 0$, then $-2 \le \Re(f'(0)) \le 2$.

If $\Re f(z) \geq 0$, then $\Re f(e^{i\theta})\cos^2(\theta/2) \geq 0$ and $\Re f(e^{i\theta})\sin^2(\theta/2) \geq 0$. We take real parts in the two integrals of (a) to get

$$\int_0^{2\pi} \Re(f(e^{i\theta}))\cos^2(\theta/2)d\theta = 2 + \Re f'(0), \quad \int_0^{2\pi} \Re(f(e^{i\theta}))\sin^2(\theta/2)d\theta = 2 - \Re f'(0).$$

This gives (as the integral of a nonnegative function is nonnegative)

$$2 + \Re f'(0) \geq 0, \quad 2 - \Re f'(0) \geq 0,$$

which implies the result.

5. Let $f(z)$ be holomorphic in the region $|z| \leq R$ with power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Let the partial sum of the series be defined as

$$s_N(z) = \sum_{n=0}^{N} a_n z^n.$$

Show that for $|z| < R$ we have

$$s_N(z) = \frac{1}{2\pi i}\int_{|w|=R} f(w)\frac{w^{N+1} - z^{N+1}}{w - z}\frac{dw}{w^{N+1}}.$$

We use the identity $A^N - B^N = (A - B)(A^{N-1} + A^{N-2}B + \cdots AB^{N-1} + B^N)$. We, therefore, get

$$\frac{1}{2\pi i}\int_{|w|=R} f(w)\frac{w^{N+1} - z^{N+1}}{w - z}\frac{dw}{w^{N+1}} = \frac{1}{2\pi i}\int_{|w|=R} f(w)(w^N + w^{N-1}z + \cdots + wz^{N-1} + z^N)\frac{dw}{w^{N+1}}$$

$$= \frac{1}{2\pi i}\int_{|w|=R} f(w)(w^{-1} + w^{-2}z + w^{-3}z^2 \cdots + w^{-N}z^{N-1} + w^{-N-1}z^N)dw$$

$$= f(0) + f'(0)z + \frac{f''(0)}{2}z^2 + \cdots + \frac{f^{(N-1)}(0)}{(N-1)!}z^{N-1} + \frac{f^{(N)}(0)}{N!}z^N$$

by Cauchy's formula for the derivatives. Since $a_n = f^{(n)}(0)/n!$ we get that the result is exactly $s_N(z)$.

6. Let C be a circle enclosing the distinct points $z_1, z_2, \ldots z_n$. Let

$$p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$$

be (the) polynomial of degree n with roots at these points. Let $f(z)$ be holomorphic in a disc that includes C. Show that

$$P(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{p(w)}\frac{p(w) - p(z)}{w - z}dw$$

is a polynomial of degree $n-1$, with the property

$$P(z_i) = f(z_i), \quad i = 1, 2, \dots n.$$

We plug z_i to get

$$P(z_i) = \frac{1}{2\pi i}\int_C \frac{f(w)}{p(w)}\frac{p(w)-p(z_i)}{w-z_i}dw = \frac{1}{2\pi i}\int_C \frac{f(w)}{p(w)}\frac{p(w)}{w-z_i}dw = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_i}dw = f(z_i)$$

by Cauchy's formula. It suffices to prove that it is a polynomial of degree $\le n-1$. We expand $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ and get

$$\frac{p(w)-p(z)}{w-z} = \frac{w^n - z^n + a_{n-1}(w^{n-1}-z^{n-1}) + \cdots + a_1(w-z)}{w-z}$$

$$= w^{n-1}+w^{n-2}z+\cdots+wz^{n-2}+z^{n-1}+a_{n-1}(w^{n-2}+w^{n-3}z+\cdots+wz^{n-3}+z^{n-2})+\cdots+a_1,$$

where we used that $A^N - B^N = (A-B)(A^{N-1} + A^{N-2}B + \cdots AB^{N-1} + B^N)$. This is clearly a polynomial in z of degree $n-1$. When we plug it into our integral in the variable w we get by linearity a polynomial of degree $\le n-1$ in z with coefficients the values of certain contour integrals in w.

7. Let $f(z)$ be holomorphic on $|z| < 1$ and $|f(z)| < \dfrac{1}{1-|z|}$ for $|z| < 1$. Show that the Taylor coefficients a_n of $f(z)$ satisfy

$$|a_n| \le (n+1)\left(1+\frac{1}{n}\right)^n < e(n+1).$$

We use the Cauchy's estimate $|a_n| \le M(R)/R^n$, where $M(R) = \max\{|f(Re^{i\theta})|, \theta \in [0, 2\pi]\}$. Using $|f(z)| < 1/(1-|z|)$ we get

$$|a_n| \le \frac{1/(1-R)}{R^n} = \frac{1}{(1-R)R^n}.$$

We need to make a choice of R in $[0,1)$ to minimize this expression. This is equivalent in maximizing $g(R) = (1-R)R^n$. We do this using calculus

$$g'(R) = (R^n - R^{n+1})' = nR^{n-1} - (n+1)R^n = R^{n-1}(n-(n+1)R).$$

So $g'(R) = 0$ if $R = n/(n+1) < 1$. This is our choice for R. Then

$$g(R) = \left(\frac{n}{n+1}\right)^n\left(1-\frac{n}{n+1}\right) = \left(\frac{n}{n+1}\right)^n\frac{1}{n+1}.$$

We now get

$$|a_n| \le \left(\frac{n+1}{n}\right)^n(n+1) = (n+1)\left(1+\frac{1}{n}\right)^n.$$

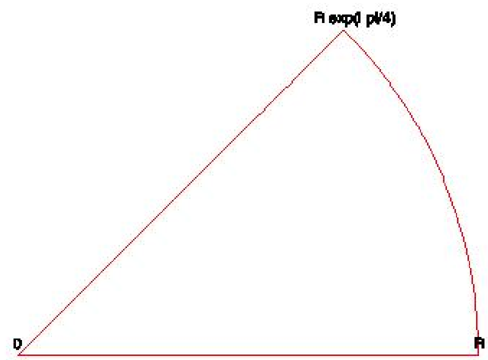

Figure 3: The contour for the Fresnel integrals

The last inequality follows from the well-known fact that

$$\lim \left(1 + \frac{1}{n}\right)^n = e$$

and, in fact, the sequence $(1 + n^{-1})^n$ increases to e. If you are not familiar with this, consider the function

$$h(x) = \log(1 + x^{-1})^x = x \log(1 + x^{-1}) = \frac{\log(1 + x^{-1})}{x^{-1}}.$$

The Taylor series of $\log(1 + y)$ close to 0 is $\log(1 + y) = y - y^2/2 + y^3/3 - \cdots$, and this gives

$$h(x) = 1 - x^{-1}/2 + x^{-2}/3 - \cdots \to 1, \quad x \to \infty.$$

This implies that $e^{h(x)} \to e$, $x \to \infty$. The function $h(x)$ is increasing iff $h(y^{-1}) = y^{-1} \log(1+y)$ is a decreasing function. The function $\log(1+y)$ is concave downwards, so the slopes with fixed left endpoint are decreasing: fix $(0, 0)$ on its graph and joint it to $(y, \log(1 + y))$ to get a slope $\log(1 + y)/y$.

8. Calculate the Fresnel integrals

$$\int_0^\infty \cos(x^2)\, dx = \int_0^\infty \sin(x^2)\, dx = \sqrt{2\pi}/4.$$

Hint: Consider the function $f(z) = e^{iz^2}$ and a contour that includes the line segment from 0 to $Re^{i\pi/4}$ and the circular arc from R to $Re^{i\pi/4}$.

Let the contour be $\gamma = \gamma_1 + \gamma_R - \gamma_2$, where $\gamma_1(t) = t + i0$, $0 \le t \le R$, $\gamma_R(\theta) = Re^{i\theta}$, $0 \le \theta \le \pi/4$, and, finally, $\gamma_2(t) = te^{i\pi/4}$, $0 \le t \le R$. By Cauchy's theorem

$$0 = \int_\gamma e^{iz^2} dz = \int_{\gamma_1} e^{iz^2} dz + \int_{\gamma_R} e^{iz^2} dz - \int_{\gamma_2} e^{iz^2} dz.$$

On γ_1 we have

$$\int_{\gamma_1} e^{iz^2} dz = \int_0^R e^{ix^2} dx = \int_0^R (\cos(x^2)+i\sin(x^2))dx = \int_0^R \cos(x^2)dx + i\int_0^R \sin(x^2)dx.$$

The last integral tends to

$$\int_0^\infty \cos(x^2)\,dx + i\int_0^\infty \sin(x^2)\,dx, \quad R \to \infty.$$

On γ_2 we have

$$\int_{\gamma_2} e^{iz^2} dz = \int_0^R e^{i(te^{i\pi/4})^2} e^{i\pi/4} dt = \int_0^R e^{it^2 i} e^{i\pi/4} dt = e^{i\pi/4} \int_0^R e^{-t^2}\, dt$$

$$\to e^{i\pi/4} \int_0^\infty e^{-t^2} dt = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\frac{\sqrt{\pi}}{2}, \quad R \to \infty.$$

The result for the two Fresnel integrals will follow, if we show that the contour integral over γ_R tends to 0, $R \to \infty$. We have

$$\int_{\gamma_R} e^{iz^2} dz = \int_0^{\pi/4} e^{i(Re^{i\theta})^2} Rie^{i\theta}\, d\theta = \int_0^{\pi/4} e^{iR^2(\cos 2\theta + i\sin 2\theta)} Rie^{i\theta} d\theta = \int_0^{\pi/4} e^{iR^2\cos 2\theta - R^2 \sin 2\theta} iRe^{i\theta} d\theta.$$

We now use the concavity of $\sin\theta$ on $[0, \pi/2]$ to get $\sin\theta \ge (2/\pi)\theta$. This implies that

$$\left|\int_{\gamma_R} e^{iz^2} dz\right| \le \int_0^{\pi/4} e^{-R^2 \sin 2\theta} R d\theta \le \int_0^{\pi/4} e^{-R^2(4/\pi)\theta} R d\theta = \left[\frac{e^{-R^2(4/\pi)\theta}}{-R^2(4/\pi)} R\right]_0^{\pi/4} = \frac{\pi}{4R}(1 - e^{-R^2}),$$

which clearly tends to 0, as $R \to \infty$.

9. Let $f(z) = u + iv$ be an analytic function, $\psi(u, v)$ any function with second order partial derivatives and $g(u, v)$ any function with first partial derivatives.

(a) Let $\Delta_{x,y}$ be the Laplace operator in x, y coordinates, i.e. $\Delta_{x,y} = \partial_x^2 + \partial_y^2$, and $\Delta_{u,v}$ be the Laplace operator in u, v coordinates, i.e. $\Delta_{u,v} = \partial_u^2 + \partial_v^2$. Show that

$$\Delta_{x,y}(\psi \circ f) = \Delta_{u,v}\psi \cdot |f'(z)|^2.$$

If $w = u + iv$, the chain rule for the composition of derivatives in z and $\bar{z}$ (homework 1) gives (use also that f is analytic, i.e., $f_{\bar{z}} = 0$ and $\bar{f}_{\bar{z}} = \overline{f_z}$)

$$\Delta_{x,y}(\psi \circ f) = 4\partial_z \partial_{\bar{z}}(\psi \circ f) = 4\partial_z(\psi_w f_{\bar{z}} + \psi_{\bar{w}} \bar{f}_{\bar{z}}) = 4\partial_z(\psi_{\bar{w}} \overline{f_z}) = 4\partial_z(\psi_{\bar{w}})\overline{f_z} + 4\psi_{\bar{w}}\partial_z(\overline{f_z})$$

by the product rule for partial derivatives. As f is analytic, f_z is analytic and its conjugate does not depend on z. So $\partial_z \overline{f_z} = 0$. This gives

$$\Delta_{x,y}(\psi \circ f) = 4\partial_z(\psi_{\bar{w}})\overline{f_z} = 4(\partial_w(\psi_{\bar{w}})f_z + \partial_{\bar{w}}(\psi_{\bar{w}})\cdot(\overline{f})_z)\overline{f_z} = 4\partial_w(\psi_{\bar{w}})\cdot f_z\overline{f_z} = \Delta_{u,v}\psi|f'(z)|^2,$$

since $\overline{f}$ does not depend on z.

(b) Let $\nabla_{u,v} = \partial_u \mathbf{i} + \partial_v \mathbf{j}$ be the gradient vector in u, v, and $\nabla_{x,y} = \partial_x \mathbf{i} + \partial_y \mathbf{j}$ the gradient vector in x, y coordinates. Show that

$$|\nabla_{x,y}(g \circ f)|^2 = |\nabla_{u,v} g|^2 \cdot |f'(z)|^2,$$

where $|\cdot|$ is the euclidean norm in $\mathbb{C}$.

We first express $\nabla_{x,y}$ in terms of the ∂_z and $\partial_{\bar{z}}$ operators.

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

We sum and subtract to get

$$\partial_x = \partial_z + \partial_{\bar{z}}, \quad \partial_y = -i(\partial_{\bar{z}} - \partial_z).$$

This gives

$$\nabla_{x,y} = \partial_x \mathbf{i} + \partial_y \mathbf{j} = \mathbf{i}(\partial_z + \partial_{\bar{z}}) - i\mathbf{j}(\partial_{\bar{z}} - \partial_z).$$

Similarly we have

$$\nabla_{u,v} = \partial_u \mathbf{i} + \partial_v \mathbf{j} = \mathbf{i}(\partial_w + \partial_{\bar{w}}) - i\mathbf{j}(\partial_{\bar{w}} - \partial_w).$$

We get

$$\nabla_{x,y}(g \circ f) = \mathbf{i}(\partial_z + \partial_{\bar{z}})(g \circ f) - i\mathbf{j}(\partial_{\bar{z}} - \partial_z)(g \circ f) = \mathbf{i}(g_w f_z + g_{\bar{w}}\bar{f}_z + g_w f_{\bar{z}} + g_{\bar{w}}\bar{f}_{\bar{z}})$$
$$-i\mathbf{j}(g_w f_{\bar{z}} + g_{\bar{w}}\bar{f}_{\bar{z}} - g_w f_z - g_{\bar{w}}\bar{f}_z) = \mathbf{i}(g_w f_z + g_{\bar{w}}\bar{f}_{\bar{z}}) - i\mathbf{j}(g_{\bar{w}}\bar{f}_{\bar{z}} - g_w f_z),$$

since $f_{\bar{z}} = \bar{f}_z = 0$. This gives

$$|\nabla_{x,y}(g \circ f)|^2 = |g_w f_z + g_{\bar{w}}\bar{f}_{\bar{z}}|^2 + |-i(g_{\bar{w}}\bar{f}_{\bar{z}} - g_w f_z)|^2 = |g_w f_z + g_{\bar{w}}\bar{f}_{\bar{z}}|^2 + |g_{\bar{w}}\bar{f}_{\bar{z}} - g_w f_z|^2.$$

Since for any complex numbers a and b we have $|a+b|^2 + |a-b|^2 = 2|a|^2 + 2|b|^2$ we get

$$|\nabla_{x,y}(g \circ f)|^2 = 2|g_w f_z|^2 + |g_{\bar{z}}\bar{f}_{\bar{z}}|^2 = 2|g_w|^2|f_z|^2 + 2|g_{\bar{w}}|^2\left|\overline{f_z}\right|^2 = (2|g_w|^2 + 2|g_{\bar{w}}|^2)|f_z|^2.$$

On the other hand

$$|\nabla_{u,v} g|^2 = |g_u|^2 + |g_v|^2 = |g_w + g_{\bar{w}}|^2 + |-i(g_{\bar{w}} - g_w)|^2 = |g_w + g_{\bar{w}}|^2 + |g_{\bar{w}} - g_w|^2 = 2|g_w|^2 + 2|g_{\bar{w}}|^2.$$

This completes the proof.