The homework consists mostly of a selection of problems from the suggested books.

1. (a) Find the value of $(1 + i)^n + (1 - i)^n$ for every $n \in \mathbb{N}$.
   (b) Show that
   $$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = 1, \quad \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1.$$ 
   (c) Find the fourth roots of $-1$.

2. Find the conditions under which the equation $az + b\bar{z} + c = 0$ has exactly one solution and compute the solution.

3. Describe geometrically the sets of points $z$ in the complex plane defined by the following relations.
   (a) $|z - z_1| = |z - z_2|$, where $z_1, z_2$ are fixed points in $\mathbb{C}$.
   (b) $1/z = \bar{z}$.
   (c) $|z| = \Re(z) + 1$.

4. Prove Lagrange identity
   $$\left|\sum_{i=1}^{n} a_i b_i \right|^2 = \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{1 \leq i < j \leq n} |a_i\bar{b}_j - a_j\bar{b}_i|^2.$$ 

5. Prove that
   $$\left|\frac{a - b}{1 - \bar{a}b}\right| < 1$$ 
   if $|a| < 1$ and $|b| < 1$.

6. Prove that it is impossible to define a total ordering on $\mathbb{C}$. In other words, one cannot find a relation $\gg$ between complex numbers so that:
   (i) For any two complex numbers $z$ and $w$ one and only one of the following is true: $z \gg w$, $w \gg z$ or $z = w$.
   (ii) For all $z_1, z_2, z_3 \in \mathbb{C}$ the relation $z_1 \gg z_2$ implies $z_1 + z_3 \gg z_2 + z_3$.
   (iii) Moreover, for all $z_1, z_2, z_3 \in \mathbb{C}$ with $z_3 \gg 0$
   $$z_1 \gg z_2 \implies z_1 z_3 \gg z_2 z_3.$$ 
   
   *Hint:* Is $i \gg 0$?
7. Prove that the points \( z_1, z_2, z_3 \) are vertices of an equilateral triangle if \( z_1 + z_2 + z_3 = 0 \) and \(|z_1| = |z_2| = |z_3|\).

8. Verify the Cauchy-Riemann equations for \( z^2 \) and \( z^3 \).

9. Express \( \cos(3\phi) \) and \( \cos(4\phi) \) in terms of \( \cos(\phi) \). Express \( \sin(3\phi) \) in terms of \( \sin(\phi) \).

10. Simplify \( 1 + \cos(\phi) + \cos(2\phi) + \cdots + \cos(n\phi) \) and \( \sin(\phi) + \sin(2\phi) + \cdots + \sin(n\phi) \).

11. Prove that the diagonals of a parallelogram bisect each other and that the diagonals of a rhombus are orthogonal.

12. Prove rigorously that the functions \( f(z) \) and \( f(\bar{z}) \) are simultaneously holomorphic.

13. Suppose that \( U \) and \( V \) are open sets in the complex plane. Prove that if \( f : U \to V \) and \( g : V \to \mathbb{C} \) are two functions that are differentiable in the real sense (in \( x \) and \( y \)) and \( h = g \circ f \), then the complex version of the chain rule is

\[
\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z} \frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}} \frac{\partial \bar{f}}{\partial z}, \quad \frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial z} \frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}} \frac{\partial \bar{f}}{\partial \bar{z}}.
\]

14. Show that in polar coordinates the Cauchy-Riemann equations take the form

\[
\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{1}{r} \frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.
\]

Use these equations to show that the logarithm function defined by \( \log z = \log r + i\theta \), \( z = r(\cos \theta + i \sin \theta) \), \( -\pi < \theta < \pi \) is holomorphic in the region \( r > 0 \) and \( -\pi < \theta < \pi \).

15. Consider the function defined by

\[
f(x + iy) = \sqrt{|x||y|}, \quad x, y \in \mathbb{R}.
\]

Show that \( f \) satisfies the Cauchy-Riemann equations at the origin, yet \( f \) is not holomorphic at 0.