Math 70300

Homework 1

Due: September 12, 2006

The homework consists mostly of a selection of problems from the suggested books.

1. (a) Find the value of $(1+i)^n + (1-i)^n$ for every $n \in \mathbb{N}$.

   (b) Show that
   $$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = 1, \quad \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1.$$

   (c) Find the fourth roots of -1.

2. Find the conditions under which the equation $az + b\bar{z} + c = 0$ has exactly one solution and compute the solution.

3. Describe geometrically the sets of points z in the complex plane defined by the following relations.

   (a) $|z - z_1| = |z - z_2|$, where z_1, z_2 are fixed points in $\mathbb{C}$.

   (b) $1/z = \bar{z}$.

   (c) $|z| = \Re(z) + 1$.

4. Prove Lagrange identity
   $$\left|\sum_{i=1}^{n} a_i b_i\right|^2 = \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{1 \le i < j \le n} |a_i \bar{b}_j - a_j \bar{b}_i|^2.$$

5. Prove that
   $$\left|\frac{a-b}{1-\bar{a}b}\right| < 1$$
   if $|a| < 1$ and $|b| < 1$.

6. Prove that it is impossible to define a total ordering on $\mathbb{C}$. In other words, one cannot find a relation $\gg$ between complex numbers so that:

   (i) For any two complex numbers z and w one and only one of the following is true: $z \gg w$, $w \gg z$ or $z = w$.

   (ii) For all $z_1, z_2, z_3 \in \mathbb{C}$ the relation $z_1 \gg z_2$ implies $z_1 + z_3 \gg z_2 + z_3$.

   (iii) Moreover, for all $z_1, z_2, z_3 \in \mathbb{C}$ with $z_3 \gg 0$
   $$z_1 \gg z_2 \Longrightarrow z_1 z_3 \gg z_2 z_3.$$

   Hint: Is $i \gg 0$?

7. Prove that the points z_1, z_2, z_3 are vertices of an equilateral triangle if $z_1+z_2+z_3=0$ and $|z_1|=|z_2|=|z_3|$.

8. Verify the Cauchy-Riemann equations for z^2 and z^3.

9. Express $\cos(3\phi)$ and $\cos(4\phi)$ in terms of $\cos(\phi)$. Express $\sin(3\phi)$ in terms of $\sin(\phi)$.

10. Simplify $1+\cos(\phi)+\cos(2\phi)+\cdots+\cos(n\phi)$ and $\sin(\phi)+\sin(2\phi)+\cdots+\sin(n\phi)$.

11. Prove that the diagonals of a parallelogram bisect each other and that the diagonals of a rhombus are orthogonal.

12. Prove rigorously that the functions $f(z)$ and $\overline{f(\bar{z})}$ are simultaneously holomorphic.

13. Suppose that U and V are open sets in the complex plane. Prove that if $f : U \to V$ and $g : V \to \mathbb{C}$ are two functions that are differentiable in the real sense (in x and y) and $h = g \circ f$, then the complex version of the chain rule is

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z}, \quad \frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial \bar{z}}.$$

14. Show that in polar coordinates the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta, \quad z = r(\cos\theta + i\sin\theta), \quad -\pi < \theta < \pi$$

is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

15. Consider the function defined by

$$f(x+iy) = \sqrt{|x||y|}, \quad x, y \in \mathbb{R}.$$

Show that f satisfies the Cauchy-Riemann equations at the origin, yet f is not holomorphic at 0.