Math 434/734

Midterm 2: Solutions

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1. Use Green's theorem to evaluate

$$\int_C ydx - x^2dy$$

where C is the square with vertices at $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$ oriented counter-clockwise.

We have

$$d(ydx - x^2dy) = \frac{\partial y}{\partial y}\,dydx - \frac{\partial x^2}{\partial x}\,dxdy = 1\,dydx - 2x\,dxdy = -\,dxdy - 2x\,dxdy = -(1+2x)dxdy.$$

If the square is denoted by S then

$$\int_C ydx - x^2dy = \int_S -(1+2x)dxdy = -\int_0^2\int_0^2 (1+2x)dxdy = -\int_0^2 [x+x^2]_{x=0}^{x=2}dy$$

$$= -\int_0^2 2+2^2\,dy = -\int_0^2 6dy = -6\cdot 2 = -12.$$

Remark: We can solve this problem by calculating the four line integrals along the four sides of the square: $C_1 = [0,2]\times\{0\}$, $C_2 = \{2\}\times[0,2]$, $C_3 = [0,2]\times\{2\}$, $C_4 = \{0\}\times[0,2]$.

On C_1 we have $y = 0$ and $dy = 0$, so that $\int_{C_1} ydx - x^2dy = 0$.

On C_4 (oriented from bottom to top) we have $x = 0$ and $dx = 0$, so that $\int_{C_4} ydx - x^2dy = 0$.

On C_2 we have $x = 2$, $dx = 0$, so that

$$\int_{C_2} ydx - x^2dy = y\cdot 0 - 4dy = \int_0^2 -4dy = -4\cdot 2 = -8.$$

On C_3 (oriented from left to right) we have $y = 2$ and $dy = 0$. So

$$\int_{C_3} ydx - x^2dy = \int_0^2 2dx = 2\cdot 2 = 4.$$

Finally

$$\int_C ydx - x^2dy = \int_{C_1} + \int_{C_2} - \int_{C_3} - \int_{C_4} = -8 - 4 = -12.$$

2. (a) Calculate the divergence of the following vector field:

$$\mathbf{F} = x^2\mathbf{i} + 2y\mathbf{j} + z\mathbf{k}$$

$$\text{div } \mathbf{F} = \frac{\partial x^2}{\partial x} + \frac{\partial(2y)}{\partial y} + \frac{\partial z}{\partial z} = 2x + 2 + 1 = 2x + 3.$$

(b) Calculate $\int_S \mathbf{F} \cdot \mathbf{n}\, dS$, where S is the surface of the cube $[0,2] \times [0,2] \times [0,2]$ with $\mathbf{n}$ the outer unit normal vector on it.

Set T to be the solid encloded by the surface S, i.e. the solid cube $[0,2] \times [0,2] \times [0,2]$. Since we calculated the divergence of the vector field in (a), we use the divergence (Gauss') theorem:

$$\int_S \mathbf{F} \cdot \mathbf{n}\, dS = \int_T \text{div } \mathbf{F}\, dxdydz = \int_0^2 \int_0^2 \int_0^2 (2x+3) dxdydz = \int_0^2 \int_0^2 [x^2 + 3x]_{x=0}^{x=2} dydz$$

$$= \int_0^2 \int_0^2 (4+6) dydz = 10 \cdot 2 \cdot 2 = 40.$$

Remark: One can try to compute the flux of $\mathbf{F}$ over the surface of the cube directly by computing the six surfaces integrals along the six faces of the cube. This is not too difficult in this case:

On $F_1 = [0,2] \times [0,2] \times \{0\}$ (face on the xy-plane) we have $\mathbf{n} = -\mathbf{k}$ and $z = 0$, so that $\mathbf{F} \cdot \mathbf{n} = 0$, $\int_{F_1} \mathbf{F} \cdot \mathbf{n}\, dS = 0$.

On $F_2 = [0,2] \times [0,2] \times \{2\}$ (face parallel to the xy-plane) we have $\mathbf{n} = \mathbf{k}$ and $z = 2$, so that $\mathbf{F} \cdot \mathbf{n} = 2$, $\int_{F_2} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{F_2} 2\, dS = 2$ Area $(F_2) = 2 \cdot 4 = 8$.

On $F_3 = \{0\} \times [0,2] \times [0,2]$ (face on the yz-plane) we have $\mathbf{n} = -\mathbf{i}$ and $x = 0$, so that $\mathbf{F} \cdot \mathbf{n} = 0$, $\int_{F_3} \mathbf{F} \cdot \mathbf{n}\, dS = 0$.

On $F_4 = \{2\} \times [0,2] \times [0,2]$ (face parallel to the yz-plane) we have $\mathbf{n} = \mathbf{i}$ and $x = 2$, so that $\mathbf{F} \cdot \mathbf{n} = 2^2 \cdot 1 = 4$, $\int_{F_4} \mathbf{F} \cdot \mathbf{n}\, dS = 4 \cdot$ Area $(F_4) = 4 \cdot 4 = 16$.

On $F_5 = [0,2] \times \{0\} \times [0,2]$ (face on the xz-plane) we have $\mathbf{n} = -\mathbf{j}$ and $y = 0$, so that $\mathbf{F} \cdot \mathbf{n} = 0$, $\int_{F_5} \mathbf{F} \cdot \mathbf{n}\, dS = 0$.

On $F_6 = [0,2] \times \{2\} \times [0,2]$ (face parallel to the xz-plane) we have $\mathbf{n} = \mathbf{j}$ and $y = 2$, so that $\mathbf{F} \cdot \mathbf{n} = 2 \cdot 2 = 4$, $\int_{F_6} \mathbf{F} \cdot \mathbf{n}\, dS = 4 \times$ Area $(F_6) = 4 \cdot 4 = 16$.

The total flux is $8+16+16 = 40$, which agrees with the use of the divergence theorem.

3. (a) Calculate the curl of the following vector field:

$$\mathbf{F} = z^2\mathbf{i} + 2x\mathbf{j} - y^2\mathbf{k}$$

$$\text{curl }\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 & 2x & -y^2 \end{vmatrix} = \left(\frac{\partial(-y^2)}{\partial y} - \frac{\partial(2x)}{\partial z}\right)\mathbf{i}$$

$$-\left(\frac{\partial(-y^2)}{\partial x} - \frac{\partial z^2}{\partial z}\right)\mathbf{j} + \left(\frac{\partial(2x)}{\partial x} - \frac{\partial z^2}{\partial y}\right)\mathbf{k} = -2y\mathbf{i} + 2z\mathbf{j} + 2\mathbf{k}.$$

(b) Calculate the (line integral)

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

where C is the ellipse on the xy-plane with equation $x^2 + 4y^2 = 4$ oriented counterclockwise.

According to Stokes' theorem we need to compute the flux through the surface for the curl of the vector field. In this case the unit normal to the surface is $\mathbf{k}$, so that the boundary of the elliptical region S is oriented counterclockwise.

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_S \text{curl }\mathbf{F}\cdot\mathbf{n}\,dS = \int_S (-2y\mathbf{i}+2z\mathbf{j}+2\mathbf{k})\cdot\mathbf{k}\,dS = \int_S 2\,dS = 2\text{Area }(S) = 2\cdot\pi\cdot 1\cdot 2 = 4\pi,$$

since the area of the elliptic region with semiaxes $a = 1$ and $b = 2$ is $\pi ab = \pi\cdot 1\cdot 2$.

Remark: We can calculate directly the line integral: we parametrize the ellipse as $x = 2\cos t$, $y = \sin t$, $z = 0$, $0 \le t \le 2\pi$. Then

$$\frac{dx}{dt} = -2\sin t \quad \frac{dy}{dt} = \cos t, \quad \frac{dz}{dt} = 0$$

and calculate the pullback of the one-form $z^2dx + 2xdy - y^2dz$ on the t-space to get:

$$0^2(-2\sin t)dt + 2\cdot 2\cos t\cos t\,dt - \sin^2 t\,0\,dt = 4\cos^2 t dt.$$

We integrate the pullback to get

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{2\pi} 4\cos^2 t\,dt = 4\int_0^{2\pi}\frac{1+\cos(2t)}{2}dt = 4\frac{1}{2}2\pi = 4\pi.$$

(c) (*Extra credit*) Reformulate (a) and (b) in terms of forms and redo the calculations.

We want to calculate the exterior derivative of the one-form

$$\omega = z^2dx + 2xdy - y^2dz,$$

which gives

$$d\omega = 2z\,dzdx + 2\,dxdy - 2y\,dydz = -2y\,dydz + 2z\,dzdx + 2\,dxdy.$$

This follows since the coefficients of the one-form are functions only of one variable. This two-form corresponds to the curl of the given vector field. We use the general form of Stokes' theorem to get

$$\int_C \omega = \int_S d\omega = \int_S -2y\, dydz + 2z\, dzdx + 2\, dxdy = \int_S 2\, dxdy$$

as the region S is on the xy-plane. The constant two-form $2dxdy$ measures the (oriented) area of the region multiplied by 2. This is $2\pi \cdot 1 \cdot 2 = 4\pi$.

4. Assume that the mixed partial derivatives of the given function are equal. Show that div (curl $\mathbf{F}$) $= 0$, which in alternative notation is $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

   See Homework 5.

5. Consider the differential form $\omega = \cos y\, dx + (3y^2 - x \sin y)\, dy$. Is it closed? Is it exact? Why?

   We calculate

$$d\omega = \left(\frac{\partial \cos y}{\partial x} dx + \frac{\partial \cos y}{\partial y} dy \right) dx + \left(\frac{\partial (3y^2 - x \sin y)}{\partial x} dx + \frac{\partial (3y^2 - x \sin y)}{\partial y} dy \right) dy$$

$$= -\sin y\, dydx + (-\sin y\, dx + 6y - x \cos y\, dy)dy = -\sin y\, dxdy - \sin y\, dxdy = 0dxdy,$$

   since $dydy = 0$. So the form is closed.

   We examine for exactness. We look for a function $F(x, y)$ such that

$$dF = \omega \Leftrightarrow \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = \cos y\, dx + (3y^2 - x \sin y)\, dy.$$

   This gives

$$\frac{\partial F}{\partial x} = \cos y, \quad \frac{\partial F}{\partial y} = 3y^2 - x \sin y.$$

   We solve the first equation by integrating to get

$$F(x, y) = \int \cos y dx = x \cos y + k(y),$$

   where the constant of integration may depend on y but not on x. We get as a result

$$\frac{\partial F}{\partial y} = -x \sin y + \frac{dk}{dy},$$

   which should equal $3y^2 - x \sin y$. This is possible only if

$$\frac{dk}{dy} = 3y^2 \Longrightarrow k(y) = \int 3y^2 dy = y^3 + c.$$

   This gives the potential function $F(x, y) = x \cos y + y^3 + c$ for any constant c. So the form is exact.

   Remark: If we prove first that the form is exact, then we also know that it is closed.