Math 434/734

Midterm 1: Solutions

March 14, 2007

1. Evaluate the 1-form $dx + 2dy + 3dz$ on the oriented line segment PQ where $P(3, 4, 5)$ and $Q(6, -1, 7)$.

   Since $\vec{PQ} = (6-3)\mathbf{i} + (-1-4)\mathbf{j} + (7-5)\mathbf{k} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$, i.e. $dx = 3$, $dy = -5$, $dz = 2$ we get for the evaluation of the 1-form

$$3 + 2(-5) + 3 \cdot 2 = -1.$$

2. Find the pull-back of the 1-form $\dfrac{xdx + ydy}{x^2 + y^2}$ under the map $x = r\cos\theta$ and $y = r\sin\theta$ (i.e. express the form in polar coordinates). Simplify as much as you can.

   We have

$$dx = \cos\theta dr - r\sin\theta\, d\theta,$$
$$dy = \sin\theta dr + r\cos\theta\, d\theta.$$

   Since $x^2 + y^2 = r^2$ we get

$$\frac{xdx + ydy}{x^2 + y^2} = \frac{r\cos\theta(\cos\theta dr - r\sin\theta\, d\theta) + r\sin\theta(\sin\theta dr + r\cos\theta\, d\theta)}{r^2}$$

$$= \frac{(r\cos^2\theta + r\sin^2\theta)dr + (-r^2\sin\theta\cos\theta + r^2\sin\theta\cos\theta)d\theta}{r^2} = \frac{rdr}{r^2} = \frac{dr}{r}$$

3. Evaluate **one** of the integrals.

$$\int_0^\pi \int_y^\pi \frac{\sin x}{x}\, dx\, dy, \qquad \int\int_R e^{-(x^2+y^2)}\, dA, \qquad R = \left\{(x, y), x^2 + y^2 \le 4\right\}.$$

   For the first integral we interchange the order of integration. It is given as the integral over a triangle written as a horizontally simple region. We will compute it as a vertically simple region.

$$\int_0^\pi \int_y^\pi \frac{\sin x}{x}\, dx\, dy = \int_0^\pi \int_0^x \frac{\sin x}{x} dy\, dx = \int_0^\pi \left[\frac{\sin x}{x} y\right]_{y=0}^{y=x} dx = \int_0^\pi \sin x dx$$

$$= [-\cos x]_0^\pi = -(-1) + 1 = 2.$$

For the second integral we use polar coordinates to get

$$\int\int_R e^{-(x^2+y^2)}\,dA = \int_0^{2\pi}\int_0^2 e^{-r^2} r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{-1}{2}e^{-r^2}\right]_0^2 d\theta = \int_0^{2\pi} -\frac{1}{2}e^{-2^2}+\frac{1}{2}e^{-0}d\theta$$

$$= 2\pi\frac{1}{2}\left(1-e^{-4}\right) = \pi(1-e^{-4}).$$

4. Calculate the integral of the 1-form $y^2dx + x^2dy$ along the square with vertices $(0,0)$, $(3,0)$, $(3,3)$, $(0,3)$ oriented counterclockwise.

Along the horizontal side from $(0,0)$ to $(3,0)$ the parametrization is $x = t$, $y = 0$, $0 \le t \le 3$. So $dy/dt = 0$ and $dx/dt = 1$ and the pullback of the 1-form is

$$y^2dx + x^2dy = (0^2 \cdot 1 + t^2 \cdot 0)dt = 0dt.$$

So the line integral along this side is 0.

Along the vertical side from $(3,0)$ to $(3,3)$ the parametrization is $x = 3$, $y = t$, $0 \le t \le 3$. So $dy/dt = 1$ and $dx/dt = 0$ and the pullback of the 1-form is

$$y^2dx + x^2dy = (t^2 \cdot 0 + 3^2 \cdot 1)dt = 9dt.$$

So the line integral along this side is

$$\int_0^3 9dt = 9 \cdot 3 = 27.$$

Along the horizontal side from $(3,3)$ to $(0,3)$ the parametrization is $x = 3-t$, $y = 3$, $0 \le t \le 3$. So $dy/dt = 0$ and $dx/dt = -1$ and the pullback of the 1-form is

$$y^2dx + x^2dy = (3^2 \cdot (-1) + (3-t)^2 \cdot 0)dt = -9dt.$$

So the line integral along this side is

$$\int_0^3 -9dt = -9 \cdot 3 = -27.$$

Along the vertical side from $(0,3)$ to $(0,0)$ the parametrization is $x = 0$, $y = 3-t$, $0 \le t \le 3$. So $dy/dt = -1$ and $dx/dt = 0$ and the pullback of the 1-form is

$$y^2dx + x^2dy = ((3-t)^2 \cdot 0 + 0 \cdot (-1))dt = 0dt.$$

So the line integral along this side is 0.

The total integral is 0.

5. Do either:

(a) Find the 2-form describing flow in the three dimensional space from a source at $(0, 0, 0)$, assuming that the flow is outward at all points with a magnitude depending only on r (radial flow) and assuming there are no sources between spheres around the origin.

Hint: The surface area of the sphere of radius r is $4\pi r^2$.

See homework 2.

(b) Evaluate the integral $\int_S \mathbf{F} \cdot \mathbf{n} \, dS$ (surface integral or flow) of the vector field

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$

where S is the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$ oriented counterclockwise as seen from the outside.

See homework 3.